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# NOTES ON THE RIEMANN ZETA-FUNCTION -V

BY

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*Dedicated with deepest regards to* **PROFESSOR ALAN BAKER, F.R.S,** *on his sixtieth birth day.*

**§1.INTRODUCTION.** This is a continuation of the paper [KR, AS] with the same title and this paper is also dedicated to him. Enough details are there in [RB.KR], [KR.AS] and [RB. KR, AS, KS]-III already and for this reason presentation in the present paper is some what sketchy. We begin by introducing a meromorphic function $F_0(s)$ as follows. Let $k$ and $l$ be any fixed integers subject to $0 \leq k \leq l$ and $l \geq 1$. Let $\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_l$ be any $k+l$ complex constants. Let $P(s)$ be any Dirichlet polynomial (i.e. a terminating Dirichlet series). Put

$$F_0(s) = P(s)(\prod \zeta(s+\alpha)) \, (\prod \zeta(s+\beta))^{-1}$$

where the first product runs over $\alpha_1, ..., \alpha_k$ and the second over $\beta_1, .., \beta_l$. The main theorem in [RB. KR, AS, KS]-IV runs as follows:

**THEOREM 1.** *Let $F_0(s)$ be not identically zero. Then $F_0(s)$ has infinitely many poles in $t \geq t_0$ (for every fixed $t_0 > 0$). There exist poles with ordinates in $[T, 2T]$, for all $T \geq T_0$ (for some $T_0 > 0$). If we arrange these ordinates in the non decreasing order then the successive gaps are majorised by*

$$d \log T$$

*where $d > 0$ is a constant independent of $T$.*

It is our object in this paper to generalise this theorem to functions of the type

$$F(s) = \sum F_0(s)$$

where the sum is a finite one and is over terms $F_0(s)$ with varying $P(s); k, l; \alpha_1, ..., \alpha_k, \beta_1, ..., \beta_l$. We prove

**THEOREM 2.** *The statement of Theorem 1 is true word for word if we replace $F_0(s)$ by $F(s)$ and assume that ( in some far off right off plane ) $F(s)$ has a non-terminating Dirichlet series expansion.*

**REMARK 1.** Note that we have completely knocked off the condition $\sum^* < 1$ of [KR, AS] which was necessary there since we have to deal with "Explicit formula" for the partial sums of the coefficients of the Dirichlet series expansions of $F(s)$ in some far off right half plane. This needed some delicate estimates for $F(s)$.

**REMARK 2.** Our proof of theorem 2 allows as to consider terms $F_0(s)$ (of the sum $F(s)$) with $\zeta(s+\alpha)$ being replaced by (any derivative of bounded orders of $\zeta(s+\alpha)$) plus a Dirichlet polynomial which may depend on $\alpha$'s. Similarly we may as the same thing for $\zeta(s+\beta)$. But we need the condition that none of the denominators in the terms $F_0(s)$ have a zero constant term. (Here some are all the $\zeta$ can be replaced by $\sum_{n=1}^{\infty}(an+b)^{-s}(a > 0, b > 0$ integer constants ) or L-functions).

**REMARK 3.** In remark 3 we may replace $\zeta(s+\alpha)$ by $\zeta_K(s+\alpha, R)$ the zeta function of a ray class in an algebraic number field $K$ of degree $n(K)$. Also in each of the terms $F_0(s)$ during replacements it is understood that the triplet $(K, \alpha, R)$ may vary. Similarly for the denominators but with the condition that none of the denominators of the terms $F_0(s)$ (of the sum) have a zero constant term. But now we need also the condition

$$\sum n(K) \leq \sum n(K')$$

for each term $F_0(s)$ (of the sum $F(s)$) $n(K')$ being the degree of the fields $K'$ in $\zeta_{K'}(s+\beta, R')$.

**REMARK 4.** When $k \leq l$ and all the $\alpha_1, \alpha_2 \cdots, \alpha_k$ and $\beta_1, \beta_2, \cdots, \beta_l$ ( involved in each term $F_0(s)$) are purely imaginary we get : The gaps between the ordinates of the successive poles of $F(s)$ (of theorem 2) in $[T, 2T](T \geq T_0)$ are majorised in absolute value by the quantity $C \log\log T$. This is still true for number fields provided $\sum n(K) \leq \sum n(K')$. In [RB.KR.AS,KS]-IV , we proved the following result. If $k < l$ or ($k = l$ and $\Re(\sum(\beta-\alpha)) \leq 0$, then for $F_0(s)$ the gaps between the ordinates of the successive poles ( in the sense of ordinates being arranged in the non-decreasing order ) in absolute value are bounded by a positive constant times $\log\log T$.

**REMARK 5.** Our method has lots of applications. We state only a few below.

Our method enables us to prove

**THEOREM 3.** *Let $P(s)$ be any Dirichlet polynomial ($\neq 0$) for which $F(s)$ is defined for complex constants $a$ and $b$ (with $a \neq 1$ and $b \neq 1$ ) by*

$$F(s) = \frac{P(s)(\zeta'(s))^2}{(\zeta(s)-a)(\zeta(s)-b)}.$$

*Then $F(s)$ has infinitely many poles in $t \geq t_0$ and the gaps between the ordinates in $[T, 2T], (T \geq T_0)$, of the successive poles are majorised in absolute value by*

$$d \log\, \log\, T$$

*where $d > 0$ is a constant independent of $T$.*

**REMARK 1.** Note that the poles of $F(s)$ are precisely simple zeros of $(P(s))^{-1}$ $(\zeta(s) - a)(\zeta(s) - b)$ whenever $a \neq b$.

**REMARK 2.** Theorem 3 is true when $\zeta(s)$ is replaced by many other functions such as the zeta function of a number field.

**REMARK 3.** When $a = b = 0$ , in fact one can say much more: The gaps between the ordinates in $[T, 2T](T \geq T_0)$ of these poles are majorised in absolute value by $d_1(\log\log\log T)^{-1}$ which follows from a result of J.E.Littlewood (see [ECT]). When $a = 1$ or $b = 1, F(s)$ can not be expanded as a Dirichlet series and hence it is an exclusion.

We state two more theorems.

**THEOREM 4.** *Let $P(s)$ and $Q(s)$ be any two Dirichlet polynomials of which $Q(s)$ has non-zero constant term. Let $F(s)$ defined by $F(s) = P(s)(Q(s))^{-1}$ have a non-terminating Dirichlet series expansion in some far off right half plane. Then $F(s)$ has infintely many poles and the gaps between successive poles are bounded.*

**REMARK .** Successive in the sense of poles with ordinates being arranged in the non-decreasing order.

**THEOREM 5.** *Let $P(s)$ and $Q(s)$ be two Dirichlet polynomials of which $Q(s)$ has a non zero constant term. Let $F(s)$ defined by*

$$F(s) = P(s)(Q'(s))^2 \, (Q(s))^{-1}$$

*have a non-terminating Dirichlet series expansion in some far off right half plane. Then $F(s)$ has infinitely many poles and the gaps between successive poles are bounded.*

**REMARK** Note that the poles of $F(s)$ are precisely the simple zeros of $Q(s)(P(s))^{-1}$.

## §2. SOME REMARKS ABOUT THE METHOD OF PROOFS.

In [KR, AS] we dealt with 'Explicit formula' for $\sum_{n\leq x} a_n$ where $F(s) = \sum_{n=1}^{\infty} a_n \, n^{-s}$ in some far off right half plane. In our new method we consider "Explicit formula" for E defined by $E = \sum_{n\leq x} a_n(x-n)^q$, where $q$ is a large positive integer constant. This enables us to cope up with "large growth of $F(s)$" and to knock off the condition $\sum^* < 1$ completely. Note that

$$E = E(x) = \frac{1}{2\pi i} \int_{L_1} \frac{F(s) \, x^{s+q}}{s(s+1)....(s+q)} ds$$

over a suitable vertical line $L_1$ ( far off to the right). It is not hard to prove that $E(x)$ is $q-1$ times continuously differentiable and that

$$\frac{d^{q-1}}{dx^{q-1}} \, E(x) = \sum_{n\leq x} a_n(x-n).$$

This helps us to observe that $E(x)$ does not have a continuous derivative of order $q$, provided that $a_n \neq 0$ for infinitely many $n$. On the other hand the explicit formula for $E(x)$ (which is easy to derive), shows that if there are no poles of $F(s)$ in $|t| \geq t_0(>0)$, a large constant then $E(x)$ is $q$ times continuously differentiable. To verify this all that we have to verify is that each of the functions $E_1(x)$ defined by

$$E_1(x) = \frac{1}{2\pi i} \int_{L} \frac{F(s)x^{s+q}ds}{s(s+1)....(s+q)}$$

(where $L$ is the (anti-clockwise) boundary of the rectangles

$$\{|t| = 2t_0, \; -N \leq \sigma \leq -q_1\}$$

for suitable sequence $N = N_\nu(\nu = 1, 2, 3..)$ even in the limit as $N$ tends to $\infty$) are $q$ times continuously differentiable provided $x$ exceeds a large positive constant. Here $q_1$ is a large positive constant. This contradiction proves that $F(s)$ has infinitely many poles in $|t| \geq t_0$, since the poles in $\sigma > -q_1$ contribute a $C^\infty$ function to the sum $E(x)$. Once we have the infinitude of poles in $|t| \geq t_0$ we can apply the results of [RB, KR] (see remark 1 on page 194). We take this opportunity to correct a misprint on page 193, last line; $(F(s)s)$ should read $(F(s)))$. According to this result it follows that $F(s)$ is a good Dirichlet series to be defined presently.

**DEFINITION.** $F(s)$ is said to be good if there exist positive constants $C_1, C_2, C_3$ such that for all $X \geq C_1$ there holds

$$\max_{X \leq n \leq X^{C_2}} |a_n| > X^{-C_3}.$$

Next the paper [RB, KR, AS,KS]-III comes to our rescue to prove all that we want. For the convenience of readers we will give a complete statement of the main reslt of [RB.KR, AS,KS]-III in §3. Apart from goodness the other conditions needed therein are easy to verify.

**§3. MAIN RESULT OF [RB, KR, AS, KS]III.** In what follows $T$ will exceed a large positive constant. The letters $\psi$, $\varphi$ and $H$ will denote positive functions of $T$ bounded below by large positive constants. They are assumed to satisfy

$$H = o(T), \ \log\log\varphi = O(H)$$

and

$$\log\psi = O(H)$$

where the two $O$-constants are assumed to be sufficently small.

(A) Let $F_1(s)$ and $F_2(s)(s = \sigma + it)$ be two Dirichlet series (which may depend on a parameter $T$ and we consider only the interval $T-H \leq t \leq T+H$) convergent absolutely in $\sigma \geq C_0(\geq 100)$ and bounded there. The letter $g > 0$ will denote a large absolute constant and we assume that $F_1(s)$ and $F_2(s)$ can be continued analytically in

$$\sigma \geq -g(\log\psi)(\log\log\psi)^{-1}.$$

Also we assume that $F_2(\sigma) \to 1$ as $\sigma \to \infty$ and that in $\sigma \geq C_0$ the function $\log F_2(s)$ is bounded

(B) Let

$$|F_2(s)| < \varphi^g \ in \ \sigma \geq -20g.$$

(C) Let $F_2(s) \neq 0$ in $-g \geq \sigma \geq -g(\log H)(\log\log H)^{-1}$ and also in the same region we have

$$|F_1(s)(F_2(s))^{-1}| \leq C_4^{\sigma}\psi(C_4 > 0),$$

where $C_4$ is a constant. For convenience we assume that the constant $C_4$ is bounded below and also above.

(D) Let $|F_1(s)| \leq exp((g^2 \log \varphi)^3)$ in $\sigma \geq -g$. Under these conditions we have the following main theorem.

**THEOREM.** *We have a pole $p_1 + ip_2$ with*

$$T - H \leq p_2 \leq T + H,$$

*provided $F(s)$ defined as $F_1(s)(F_2(s))^{-1}$ is good.*

**REMARK 1.** For theorem 3 , it is not difficult to show that $F(s)$ is a non-terminating Dirichlet series whenever $P(s) \neq 0$. If we let $P(s) = \sum_{\nu \leq M} a_\nu \nu^{-s}$ and $\nu_0$ is the least $\nu$ for which $a_\nu \neq 0$, then we find that for all large $p$ , the coefficient of $p^{-2s}\nu_0{}^{-s}$ in $F(s)$ is $\frac{a_{\nu_0}(\log p)^2}{(1-a)(1-b)}$ which in fact gives the "goodness" of $F(s)$. These remarks go through for number fields case etc with the additional condition $\sum n(K) \leq \sum n(K')$. For theorem 3, we find that $\varphi \ll T^A$ and $\psi \ll (\log T)^A$ and hence the proof of theorem 3 follows from the above theorem of [RB, KR, AS, KS]III. Of course we have to use the functional equation of $\zeta(s)$ appropriately here.

**REMARK 2.** For theorems 4 and 5 , we find that $\varphi$ and $\psi$ of the above theorem satisfy $\varphi \ll B$ and $\psi \ll B$ for a large positive constant $B$. Hence the proofs follow from the above theorem of the paper [RB, KR, AS, KS]III.

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