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NOTES ON THE RIEMANN ZETA-FUNCTION -V

BY

R.BALASUBRAMANIAN, K.RAMACHANDRA, A.SANKARANARAYANAN AND
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Dedicated with deepest regards to PROFESSOR ALAN BAKER, FRS,
on his sixtieth birth day.

§1.INTRODUCTION. This is a continuation of the paper [KR, AS] with the same title
and this paper is also dedicated to him. Enough details are there in [RB.KR], [KR.AS]
and [RB. KR. AS, KS]-III already and for this reason presentation in the present paper
is somewhat sketchy. We begin by introducing a meromorphic function $F_0(s)$ as follows.
Let $k$ and $l$ be any fixed integers subject to $0 \leq k \leq l$ and $l \geq 1$. Let $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l$
be any $k + l$ complex constants. Let $P(s)$ be any Dirichlet polynomial (i.e. a terminating
Dirichlet series). Put

$$F_0(s) = P(s)(\prod \zeta(s + \alpha)) (\prod \zeta(s + \beta))^{-1}$$

where the first product runs over $\alpha_1, \ldots, \alpha_k$ and the second over $\beta_1, \ldots, \beta_l$. The main theorem
in [RB. KR. AS, KS]-IV runs as follows:

THEOREM 1. Let $F_0(s)$ be not identically zero. Then $F_0(s)$ has infinitely many
poles in $t \geq t_0$ (for every fixed $t_0 > 0$). There exist poles with ordinates in $[T, 2T]$, for
all $T \geq T_0$ (for some $T_0 > 0$). If we arrange these ordinates in the non decreasing
order then the successive gaps are majorised by

$$d \log T$$

where $d > 0$ is a constant independent of $T$. 
It is our object in this paper to generalise this theorem to functions of the type

\[ F(s) = \sum F_0(s) \]

where the sum is a finite one and is over terms \( F_0(s) \) with varying \( P(s); k, l; \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l \).

We prove

**THEOREM 2.** The statement of Theorem 1 is true word for word if we replace \( F_0(s) \) by \( F(s) \) and assume that (in some far off right off plane) \( F(s) \) has a non-terminating Dirichlet series expansion.

**REMARK 1.** Note that we have completely knocked off the condition \( \sum^* < 1 \) of [KR, AS] which was necessary there since we have to deal with "Explicit formula" for the partial sums of the coefficients of the Dirichlet series expansions of \( F(s) \) in some far off right half plane. This needed some delicate estimates for \( F(s) \).

**REMARK 2.** Our proof of theorem 2 allows us to consider terms \( F_0(s) \) (of the sum \( F(s) \)) with \( \zeta(s + \alpha) \) being replaced by (any derivative of bounded orders of \( \zeta(s + \alpha) \)) plus a Dirichlet polynomial which may depend on \( \alpha \)'s. Similarly we may as the same thing for \( \zeta(s + \beta) \). But we need the condition that none of the denominators in the terms \( F_0(s) \) have a zero constant term. (Here some are all the \( \zeta \) can be replaced by \( \sum_{n=1}^{\infty} (an + b)^{-s} (a > 0, b > 0 \) integer constants ) or L-functions).

**REMARK 3.** In remark 3 we may replace \( \zeta(s + \alpha) \) by \( \zeta_K(s + \alpha, R) \) the zeta function of a ray class in an algebraic number field \( K \) of degree \( n(K) \). Also in each of the terms \( F_0(s) \) during replacements it is understood that the triplet \( (K, \alpha, R) \) may vary. Similarly for the denominators but with the condition that none of the denominators of the terms \( F_0(s) \) (of the sum) have a zero constant term. But now we need also the condition

\[ \sum n(K) \leq \sum n(K') \]

for each term \( F_0(s) \) (of the sum \( F(s) \)) \( n(K') \) being the degree of the fields \( K' \) in \( \zeta_{K'}(s + \beta, R') \).

**REMARK 4.** When \( k \leq l \) and all the \( \alpha_1, \alpha_2, \ldots, \alpha_k \) and \( \beta_1, \beta_2, \ldots, \beta_l \) (involved in each term \( F_0(s) \)) are purely imaginary we get: The gaps between the ordinates of the successive poles of \( F(s) \) (of theorem 2) in \( [T, 2T] (T \geq T_0) \) are majorised in absolute value by the quantity \( C \log \log T \). This is still true for number fields provided \( \sum n(K) \leq \sum n(K') \). In [RB.KR,AS,KS]-IV, we proved the following result. If \( k < l \) or \( k = l \) and \( \Re(\beta - \alpha) \leq 0 \), then for \( F_0(s) \) the gaps between the ordinates of the successive poles (in the sense of
ordinates being arranged in the non-decreasing order) in absolute value are bounded by a positive constant times \( \log \log T \).

**REMARK 5.** Our method has lots of applications. We state only a few below.

Our method enables us to prove

**THEOREM 3.** Let \( P(s) \) be any Dirichlet polynomial \(( \neq 0) \) for which \( F(s) \) is defined for complex constants \( a \) and \( b \) (with \( a \neq 1 \) and \( b \neq 1 \)) by

\[
F(s) = \frac{P(s)(\zeta'(s))^2}{(\zeta(s) - a)(\zeta(s) - b)}.
\]

Then \( F(s) \) has infinitely many poles in \( t \geq t_0 \) and the gaps between the ordinates in \([T, 2T], (T \geq T_0)\), of the successive poles are majorised in absolute value by

\[
d \log \log T
\]

where \( d > 0 \) is a constant independent of \( T \).

**REMARK 1.** Note that the poles of \( F(s) \) are precisely simple zeros of \((P(s))^{-1}(\zeta(s) - a)(\zeta(s) - b)\) whenever \( a \neq b \).

**REMARK 2.** Theorem 3 is true when \( \zeta(s) \) is replaced by many other functions such as the zeta function of a number field.

**REMARK 3.** When \( a = b = 0 \), in fact one can say much more: The gaps between the ordinates in \([T, 2T], (T \geq T_0)\) of these poles are majorised in absolute value by \( d_1(\log \log \log T)^{-1} \) which follows from a result of J.E. Littlewood (see [ECT]). When \( a = 1 \) or \( b = 1 \), \( F(s) \) can not be expanded as a Dirichlet series and hence it is an exclusion.

We state two more theorems.

**THEOREM 4.** Let \( P(s) \) and \( Q(s) \) be any two Dirichlet polynomials of which \( Q(s) \) has non-zero constant term. Let \( F(s) \) defined by \( F(s) = P(s)(Q(s))^{-1} \) have a non-terminating Dirichlet series expansion in some far off right half plane. Then \( F(s) \) has infinitely many poles and the gaps between successive poles are bounded.

**REMARK.** Successive in the sense of poles with ordinates being arranged in the non-decreasing order.
THEOREM 5. Let $P(s)$ and $Q(s)$ be two Dirichlet polynomials of which $Q(s)$ has a non zero constant term. Let $F(s)$ defined by

$$F(s) = P(s)(Q'(s))^2 \frac{1}{Q(s)}$$

have a non-terminating Dirichlet series expansion in some far off right half plane. Then $F(s)$ has infinitely many poles and the gaps between successive poles are bounded.

REMARK Note that the poles of $F(s)$ are precisely the simple zeros of $Q(s)(P(s))^{-1}$.

\section{SOME REMARKS ABOUT THE METHOD OF PROOFS.}

In [KR, AS] we dealt with 'Explicit formula' for $\sum_{n \leq x} a_n \frac{1}{n-s}$ in some far off right half plane. In our new method we consider "Explicit formula" for $E$ defined by $E = \sum_{n \leq x} a_n (x - n)^q$, where $q$ is a large positive integer constant. This enables us to cope up with "large growth of $F(s)$" and to knock off the condition $\sum^* < 1$ completely. Note that

$$E = E(x) = \frac{1}{2\pi i} \int_{L_1} \frac{F(s) x^q}{s(s+1)\ldots(s+q)} ds$$

over a suitable vertical line $L_1$ (far off to the right). It is not hard to prove that $E(x)$ is $q-1$ times continuously differentiable and that

$$\frac{d^{q-1}}{dx^{q-1}} E(x) = \sum_{n \leq x} a_n (x - n).$$

This helps us to observe that $E(x)$ does not have a continuous derivative of order $q$, provided that $a_n \neq 0$ for infinitely many $n$. On the other hand the explicit formula for $E(x)$ (which is easy to derive), shows that if there are no poles of $F(s)$ in $|t| \geq t_0 (> 0)$, a large constant then $E(x)$ is $q$ times continuously differentiable. To verify this all that we have to verify is that each of the functions $E_1(x)$ defined by

$$E_1(x) = \frac{1}{2\pi i} \int_L \frac{F(s)x^q}{s(s+1)\ldots(s+q)} ds$$

(where $L$ is the (anti-clockwise) boundary of the rectangles

$$\{|t|=2t_0, -N \leq \sigma \leq -q_1\}$$

for suitable sequence $N = N_\nu (\nu = 1, 2, 3,\ldots)$ even in the limit as $N$ tends to $\infty$) are $q$ times continuously differentiable provided $x$ exceeds a large positive constant. Here $q_1$ is a large
positive constant. This contradiction proves that $F(s)$ has infinitely many poles in $|t| \geq t_0$, since the poles in $\sigma > -q_1$ contribute a $C^\infty$ function to the sum $E(x)$. Once we have the infinitude of poles in $|t| \geq t_0$ we can apply the results of [RB. KR] (see remark 1 on page 194). We take this opportunity to correct a misprint on page 193, last line: $(F(s)s)$ should read $(F(s))$. According to this result it follows that $F(s)$ is a good Dirichlet series to be defined presently.

**DEFINITION.** $F(s)$ is said to be good if there exist positive constants $C_1, C_2, C_3$ such that for all $X \geq C_1$ there holds

$$\max_{X \leq n \leq X^{C_2}} |a_n| > X^{-C_3}.$$ 

Next the paper [RB, KR, AS, KS]-III comes to our rescue to prove all that we want. For the convenience of readers we will give a complete statement of the main result of [RB,KR, AS,KS]-III in §3. Apart from goodness the other conditions needed therein are easy to verify.

**§3. MAIN RESULT OF [RB, KR, AS, KS] III.** In what follows $T$ will exceed a large positive constant. The letters $\psi, \varphi$ and $H$ will denote positive functions of $T$ bounded below by large positive constants. They are assumed to satisfy

$$H = o(T), \log \log \varphi = O(H)$$ 

and

$$\log \psi = O(H)$$

where the two $O$-constants are assumed to be sufficiently small.

(A) Let $F_1(s)$ and $F_2(s) (s = \sigma + it)$ be two Dirichlet series (which may depend on a parameter $T$ and we consider only the interval $T - H \leq t \leq T + H$) convergent absolutely in $\sigma \geq C_0 (\geq 100)$ and bounded there. The letter $g > 0$ will denote a large absolute constant and we assume that $F_1(s)$ and $F_2(s)$ can be continued analytically in

$$\sigma \geq -g(\log \psi)(\log \log \psi)^{-1}.$$ 

Also we assume that $F_2(\sigma) \to 1$ as $\sigma \to \infty$ and that in $\sigma \geq C_0$ the function $\log F_2(s)$ is bounded

(B) Let

$$|F_2(s)| < \varphi^g \text{ in } \sigma \geq -20g.$$
(C) Let \( F_2(s) \neq 0 \) in \(-g \leq \sigma \leq -g(\log H)(\log \log H)^{-1}\) and also in the same region we have

\[
|F_1(s)(F_2(s))^{-1}| \leq C_4 \psi(C_4 > 0),
\]

where \( C_4 \) is a constant. For convenience we assume that the constant \( C_4 \) is bounded below and also above.

(D) Let \(|F_1(s)| \leq \exp((g^2 \log \varphi)^2)\) in \( \sigma \geq -g \). Under these conditions we have the following main theorem.

**THEOREM.** We have a pole \( p_1 + ip_2 \) with

\[
T - H \leq p_2 \leq T + H,
\]

provided \( F(s) \) defined as \( F_1(s)(F_2(s))^{-1} \) is good.

**REMARK 1.** For theorem 3, it is not difficult to show that \( F(s) \) is a non-terminating Dirichlet series whenever \( P(s) \neq 0 \). If we let \( P(s) = \sum_{\nu \leq M} a_{\nu} \nu^{-s} \) and \( \nu_0 \) is the least \( \nu \) for which \( a_{\nu} \neq 0 \), then we find that for all large \( p \), the coefficient of \( p^{-2\nu_0} \) in \( F(s) \) is

\[
\frac{a_{\nu}(\log p)^2}{(1-\nu)(1-b)}
\]

which in fact gives the "goodness" of \( F(s) \). These remarks go through for number fields case etc with the additional condition \( \sum n(K) \leq \sum n(K') \). For theorem 3, we find that \( \varphi \ll T^A \) and \( \psi \ll (\log T)^A \) and hence the proof of theorem 3 follows from the above theorem of [RB, KR, AS, KS]III. Of course we have to use the functional equation of \( \zeta(s) \) appropriately here.

**REMARK 2.** For theorems 4 and 5, we find that \( \varphi \) and \( \psi \) of the above theorem satisfy \( \varphi \ll B \) and \( \psi \ll B \) for a large positive constant \( B \). Hence the proofs follow from the above theorem of the paper [RB, KR, AS, KS]III.
REFERENCES


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