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ON A PROBLEM OF IVIĆ

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TO DR. A. SANKARANARAYANAN ON HIS FORTIETH BIRTHDAY

§1. INTRODUCTION AND STATEMENT OF RESULTS. In [A.I] A.IVIC has proved the following inequality. Let $\varepsilon > 0$ be any constant and let γ run over the ordinates of all the zeros (counted with multiplicity) of the Riemann zeta-function in the critical strip. Then

$$\sum_{T\leq\gamma\leq 2T} |\zeta(\frac{1}{2}+i\gamma)|^2 \ll_\varepsilon T(\log T)^2(\log\log T)^{\frac{3}{2}+\varepsilon} \tag{1.1}$$

where the implied constant depends only on ϵ. Here $T \geq T_0$, a large positive constant.

In the present paper it is our object to improve this inequality in two ways. First we replace the LHS by a bigger quantity and at the same time replace the RHS by a smaller quantity namely $T(\log T)^2 \log\log T$ (where now the implied constant is absolute). Also we consider further generalisations and sketch their proof. In place of $|\zeta(\frac{1}{2}+i\gamma)|^2$ our key function (associated with arbitrary (but fixed) constants $A > 0$ and $B > 0$) is

$$M(\gamma) = \max|\zeta(s)|^2, \tag{1.2}$$

where the maximum is taken over all $s(=\sigma+it)$ in the rectangle

$$\frac{1}{2} - A(\log T)^{-1} \leq \sigma \leq 2,\ |t-\gamma| \leq B(\log\ \log T)(\log T)^{-1}. \tag{1.3}$$

Plainly the investigations go through for $L(s,\chi)$ in place of $\zeta(s)$. Accordingly our Theorem is

THEOREM 1. *We have, for every fixed* $\varepsilon > 0$,

$$(i) \qquad \sum_{\gamma\epsilon I} M(\gamma) \ll_\varepsilon H(\log T)^2 \log\ \log T$$

where $I(\subset [T+1, 2T-1])$ *is any interval of length* $H(\geq T^{\frac{2}{3}+\epsilon})$ *and* $T \geq T_0(\epsilon)$.

$$(ii) \qquad \sum_{\gamma\epsilon I}(M(\gamma))^{\frac{1}{2}} \ll_{\varepsilon} H(\log T)^{\frac{5}{4}} \log\log T$$

where $I(\subset [T+1, 2T-1])$ *is any interval of length* $H(\geq T^{\frac{1}{2}+\varepsilon})$ *and* $T \geq T_0(\varepsilon)$.

REMARK. In (ii) we can replace $\frac{1}{2}$ by any constant $k > 0$ for which k^{-1} is an integer and correspondingly the number $\frac{5}{4}$ will have to be replaced by $1 + k^2$. (Of course we should have $H \geq T^{\frac{1}{2}+\varepsilon}$). In this connection we state the following Theorem 2.

THEOREM 2. *If* $k(>0)$ *is any constant, then*

$$\sum_{\gamma\epsilon I}(M(\gamma))^k \ll H(\log T)^{1+k^2} \log\log T$$

where $I(\subset [T+1, 2T-1])$ *is any interval of length* $H(>0)$ *satisfying the following two Hypotheses.*

HYPOTHESIS 1. *We should have*

$$\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)|^{2k} dt \ll (\log T)^{k^2}.$$

HYPOTHESIS 2. *For some constant* $k^* > k$, *we should have*

$$\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)|^{2k^*} \ll (\log T)^C$$

where $C(>0)$ *is some constant depending on* k^*.

REMARKS. If $k = 2$ we do not know whether Hypothesis 2 holds or not. However it holds if $k < 2$ (namely we can take $k^* = 2$ and $C = 4$ and $H \geq T^{\frac{2}{3}+\varepsilon}$). Naturally we have to restrict to $k < 2$. Even here we do not know the truth of Hypothesis 1 unless k^{-1} is an integer and $H \geq T^{\frac{1}{2}+\varepsilon}$. (Of course if $k = 1$ we can take things like $H \geq T^{\frac{1}{3}+\varepsilon}$ but the corresponding Hypothesis 2 forces us to take $H \geq T^{\frac{2}{3}+\varepsilon}$). It should be remarked that our method is completely different from that of A.IVIĆ.

Next we state Theorem 3 which gives lower bounds for

$$\sum_{\gamma\epsilon I}(M(\gamma))^k.$$

THEOREM 3. *We have, for any* $k > 0$

$$\sum_{\gamma\epsilon I}(M(\gamma))^k \gg (\int_I |\zeta(\frac{1}{2}+it)|^{2k} dt)(\log T)(\log\log T)^{-1}$$

where $I(\subset [T, 2T])$ *is any interval of length* H *satisfying*

$$H \geq 4B(\log T)^{-1} \log\log T$$

provided the intervals

$$|t - \gamma| \leq B(\log T)^{-1} \log\log T$$

associated with two successive γ*'s have at least one point in common.*

Since the proof of theorem 3 is very simple we give it here itself.

PROOF OF THEOREM 3. Contribution (to the integral on the right) from each interval

$$|t - \gamma| \leq B(\log T)^{-1} \log \log T$$

is clearly $\ll (M_0(\gamma))^k (\log T)^{-1} \log\log T$, where $M_0(\gamma)$ is the maximum of $|\zeta(\frac{1}{2} + it)|^2$ in the interval. Also these intervals cover I completely so that each γ is counted in $\sum_{\gamma \epsilon I}(M_0(\gamma))^k$. Clearly $M_0(\gamma) \leq M(\gamma)$ and this completes the proof of Theorem 3.

Next we state a Conjecture.

CONJECTURE: $\sum_{T \leq \gamma \leq 2T}(M(\gamma))^2 \ll T(\log T)^5 \log\log T$.

FURTHER REMARKS. Lower bounds (and sometimes upper bounds) for

$$\int_I |\zeta(\frac{1}{2} + it)|^{2k} dt$$

have been studied first by K.RAMACHANDRA and then by D.R.HEATH-BROWN and then by M.JUTILA (for these results see the book $[K.R]_1$ by K.RAMACHANDRA and also the famous classic [E.C.T, D.R.H-B] by E.C.TITCHMARSH and D.R.HEATH-BROWN). There have been some further work on this integral by K.RAMACHANDRA $[K.R]_2$. Another typical result of K.RAMACHANDRA $[K.R]_2$ is

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2\sqrt{2}} dt \gg (\log H)^2 (\log\log H)^{-1},$$

where H exceeds some positive constant times $\log\log T$. In particular

$$\frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2\sqrt{2}} dt \gg (\log T)^2 (\log\log T)^{-1}.$$

It seems very difficult to knock off the factor $(\log\ \log T)^{-1}$.

§2. **NOTATION.** We adopt standard notation. The Vinogradov symbols $\ll$ and $\gg$ mean "less than a positive constant times" and"greater than a positive constant times" respectively. Some times we use $\ll ...$ and $\gg ...$ to denote that these constants depend on .... (for example $\ll_\varepsilon$). We use the Landau symbod O(...) to denote "less than a positive constant times ...". In §3 the letters A, B, C, D, E, F and G will denote positive constants. A and B need not necessarily be the same as in §1. T will exceed a large positive constant. $\varepsilon(>0)$ will be an arbitrary constant. Some times we write $T_0(\varepsilon)$ to denote that the positive constant T_0 depends on ε.

§3. PROOF OF THEOREM 1. We divide the proof into seven steps for convenience. To illustrate our method we take $H = T-2$. The general case in (i) follows from the deep result

$$\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)|^4 dt \ll_\varepsilon (\log T)^4$$

where $T \geq H \geq T^{\frac{2}{3}+\varepsilon}$ due to N.ZAVOROTNYI [N.Z] (see also [D.R.H-B], [H.I], [M.J] and [Y.M] for the history to date) and (ii) from the crucial result

$$\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)| dt \ll_\varepsilon (\log T)^{\frac{1}{4}}$$

where $T \geq H \geq T^{\frac{1}{2}+\varepsilon}$ due to K.RAMACHANDRA (see (4.3.2) of $[K.R.]_1$).

STEP I. Consider the various rectangles

$$\left[\frac{1}{2} - \frac{D\log\log T}{\log T} \leq \sigma \leq 2\right] \times \left[T + (\log T)^E \leq t \leq T + 2(\log T)^E, T + 2(\log T)^E \leq t \leq T + 3(\log T)^E, .\right.$$

where the right extremity of the last t-interval does not exceed $2T - (\log T)^E$. Consider the maximum

$$\max |\zeta(s)|^4$$

over a typical rectangle. We will prove that

$$\sum \max |\zeta(s)|^4 \ll T(\log T)^{6+8D}. \tag{3.1}$$

(We can improve RHS to $T(\log T)^{5+8D}(\log\log T)^{-2}$; but we do not need this). Let $s_1, s_2 \ldots$ denote points where the maxima are attained. Then the required quantity is (see Theorem (1.7.1) of $[\text{K.R}]_1$)

$$\leq |\zeta(s_1)|^4 + |\zeta(s_2)|^4 + \ldots \leq \left(\frac{\log T}{D\log\log T}\right)^2 \pi^{-1}\left\{\int_{D_1}\int |\zeta(s)|^4 da + \int_{D_2}\int |\zeta(s)|^4 da + \ldots\right\}$$

(where D_j is the disc of radius $D(\log\log T)(\log T)^{-1}$ with centre s_j and da the element of area)

$$\leq 10\pi^{-1}(\log T)^2(D\log\log T)^{-2} \int\int |\zeta(s)|^4 d\sigma dt$$

where the last integration is over the rectangle

$$\left[\frac{1}{2} - \frac{2D\log\log T}{\log T} \le \sigma \le 3\right] \times [T \le t \le 2T].$$

The last integral is easily seen to be $\ll T(\log T)^{4+8D}$

STEP II. We now record a Corollary to step I. We have

$$\max|\zeta(s)| \le (\log T)^C$$

for all rectangles except N of them where

$$N(\log T)^{4C} \ll T(\log T)^{6+8D}$$

and so

$$N \ll T(\log T)^{6-4C+8D}. \tag{3.2}$$

STEP III. We now write

$$M(\gamma) = \max|\zeta(s)|^2$$

where the maximum is taken over the rectangle

$$\left[\frac{1}{2} - \frac{A}{\log T} \le \sigma \le 2\right] \times \left[|t-\gamma| \le \frac{B\log\log T}{\log T}\right].$$

Let the asterisk denote the sum over those γ for which $M(\gamma) \le (\log T)^{-F}$. Then

$$\sideset{}{^*}\sum_{T+1\le\gamma\le 2T-1} (M(\gamma)) \ll T(\log T)^{1-F} \tag{3.3}$$

since the total number of γ's is $\ll T\log T$ (we may fix F to be 2).

STEP IV. Denote the rectangle in step III by $R(\gamma)$ and by s_γ the point at which $|\zeta(s)|$ attains its maximum in $R(\gamma)$. Consider those γ's for which $M(\gamma) > (\log T)^{-F}$. Then the number of zeros of $\zeta(s)$ in the disc with centre s_γ and radius $G(\log\log T)(\log T)^{-1}$ is $O(\log\log T)$ uniformly provided that this disc is inside any of the rectangles of step II. (Note that the number of exceptional rectangles is $N \le T(\log T)^{6-4C+8D}$). This follows by Jensen's Theorem (see page 150 of $[\mathrm{K.R}]_1$).

STEP V. Now consider all the $M(\gamma)$ with $T+1 \le \gamma \le 2T-1$. We claim

$$\sum M(\gamma) \ll T(\log T)^2 \log\log T. \tag{3.4}$$

(this proves part (i) of Theorem 1). To see this we split up the sum on the LHS into $\sum_1 + \sum_2 + \sum_3$, where $\sum_3$ is the sum in (3.3) and $\sum_2$ is over all γ lying in all the exceptional N rectangles occuring in step II. We have

$$\sum_2 \leq (\sum (M(\gamma))^2)^{\frac{1}{2}} N^{\frac{1}{2}}.$$

Note that $\sum (M(\gamma))^2 \ll T(\log T)^{6+8D+E+40} \leq T(\log T)^{50+8D+E}$ since each of the $M(\gamma)$'s may be replaced by the maxima in step I. Also

$$N \ll T(\log T)^{6-4C+8D}.$$

Thus

$$\sum_2 \ll T(\log T)^{25+4D+E+3-2C+4D} \leq T(\log T)^{-1}$$

if $C = 4D + E + 20$.

We now look at the sum $\sum_1$. It consists of those γ's for which

$$(\log T)^{-F} \leq M(\gamma) \leq (\log T)^C \tag{3.5}$$

and so by step IV the number of zeros of $\zeta(s)$ in

$$|s_\gamma - s| \leq G(\log\log T)(\log T)^{-1} \; is \; O(\log\log T).$$

Here G is any constant such that this disc lies in

$$\left[\frac{1}{2} - \frac{D\log\log T}{\log T} \leq \sigma \leq 3\right] \times J$$

where J is any of the non exceptional intervals of step II. This is so if $D = 10G$.

STEP VI. We are now in a position to complete the estimation of $\sum_1$. Divide the $\sigma-$ range $\left[\frac{1}{2} - \frac{A}{\log T} \leq \sigma \leq 2\right]$ into intervals

$$I_0 = \left[\frac{1}{2} - \frac{A}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{A}{\log T}\right],$$

$$I_1 = \left[\frac{1}{2} + \frac{A}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{2A}{\log T}\right],$$

$$I_2 = \left[\frac{1}{2} + \frac{2A}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{3A}{\log T}\right], \ldots$$

$$I_n = \left[\frac{1}{2} + \frac{nA}{\log T} \le \sigma \le \frac{1}{2} + \frac{(n+1)A}{\log T}\right], \dots$$

the last interval projecting a little beyond 2. First consider I_0. Look at the s_γ of the sum $\sum_1$ for which $Re\ s_\gamma \epsilon I_0$. We have (see page 34 of $[K.R]_1$)

$$|\zeta(s_\gamma)|^2 \le (\pi R_0^2)^{-1} \int\int_{|s-s_\gamma|\le R_0} |\zeta(s)|^2 da$$

where da is the element of area and $R_0 > 0$. Choose $R_0 = \frac{20A}{\log T}$. We find

$$\sum_{\gamma, Re\ s_\gamma \epsilon I_0} |\zeta(s_\gamma)|^2 \ll (\log\log T)(\log T)^2 \int\int_{T\le t\le 2T, |\sigma-\frac{1}{2}|\le \frac{100A}{\log T}} |\zeta(s)|^2 d\sigma dt$$

$$\ll (\log\log T)\log T)^2 (T\log T)(\log T)^{-1}$$

$$= T(\log T)^2 \ \log\log\ T$$

Now for any fixed $n (n = 1, 2, 3, \dots)$ consider the interval I_n for $Re\ s_\gamma$. We have (see page 34 of $[\mathrm{K.R}]_1$)

$$|\zeta(s_\gamma)|^2 \le (\pi R_n^2)^{-1} \int\int_{|s-s_\gamma|\le R_n} |\zeta(s)|^2 da$$

where da is the element of area and $R_n > 0$. We choose $R_n = \frac{nA}{2\log T}$. We find

$$\sum_{\gamma, Re\ s_\gamma \epsilon I_n} |\zeta(s_\gamma)|^2$$

$$\ll (\log\log T)(n^{-1}\log T)^2 \int\int_{T\le t\le 2T, \frac{1}{2}+\frac{nA}{2\log T}\le\sigma\le\frac{1}{2}+\frac{2nA}{\log T}} |\zeta(s)|^2\ d\sigma dt$$

$$\ll (\log\log T)(\log T)^2 n^{-2} \int_{\sigma\epsilon I_n} (T\sum_{m=1}^{\infty} m^{-2\sigma}) d\sigma$$

$$\ll T(\log\log T)(\log T)^2 n^{-2}(n^{-1}\log T) n (\log T)^{-1}$$

$$\ll T(\log\log T)(\log T)^2 n^{-2}.$$

Summing up over $n = 1, 2, \dots, [\lambda\log T]$, where $\lambda > 0$ is a suitable constant (and square bracket denotes the integer part), we have

$$\sum_1 \ll T(\log T)^2(\log\log T).$$

This proves part (i) of Theorem 1 in the case $H = T - 2$, (the case $H \ge T^{\frac{2}{3}+\varepsilon}$ is similar).

STEP VII. To prove part (ii) we have to use

$$T^{-1}\int_T^{2T} |\zeta(\frac{1}{2} + it)| dt \ll (\log T)^{\frac{1}{4}}. \qquad (3.6)$$

From (3.6) the following two Corollaries can be drawn (by the use of convexity principles (see [K.R]$_2$)).

COROLLARY 1. For $|\sigma - \frac{1}{2}| \ll (\log T)^{-1}$, we have

$$\frac{1}{T}\int_T^{2T} |\zeta(\sigma + it)|dt \ll (\log T)^{\frac{1}{4}}. \tag{3.7}$$

COROLLARY 2. For $\sigma = \frac{1}{2} + \lambda$ with $0 < \lambda \leq 2, \lambda \gg (\log T)^{-1}$, We have

$$\frac{1}{T}\int_T^{2T} |\zeta(\sigma + it)|dt \ll \lambda^{-\frac{1}{4}}. \tag{3.8}$$

The rest of the proof of part (ii) of Theorem 1 is similar to that of part(i). (In this case also the proof with $H \geq T^{\frac{1}{2}+\epsilon}$ is similar).

§4. **PROOF OF THEOREM 2.** The proof is similar to that of Theorem 1 (part (i)). We have simply to note that Hypothesis 1 implies (by convexity principles refered to already) the following two Corollaries.

COROLLARY 1. For $|\sigma - \frac{1}{2}| \ll (\log T)^{-1}$, we have,

$$\frac{1}{H}\int_T^{T+H} |\zeta(\sigma + it)|^{2k}dt \ll (\log T)^{k^2}. \tag{4.1}$$

COROLLARY 2. For $\sigma = \frac{1}{2} + \lambda$, with $0 < \lambda \leq 2$ and $\lambda \gg (\log T)^{-1}$, we have,

$$\frac{1}{H}\int_T^{T+H} |\zeta(\sigma + it)|^{2k}dt \ll \lambda^{-k^2}. \tag{4.2}$$

This completes the proof of all our assertions.

§**5.REMARK**. We have not computed the constants in Theorems 1 and 2. We will take it up on a different occasion if there are some important applications.

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