# ON SOME RAMANUJAN'S P-Q IDENTITIES

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**Abstract:** In this paper we obtain some P-Q eta-function identities of Ramanujan on employing some modular equations in Ramanujan's alternative theory of elliptic functions of signature 4.

**Key words:** Elliptic functions, Modular equations

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## 1. Introduction

Srinivasa Ramanujan has recorded several results in the theories of elliptic functions to alternative bases corresponding to the classical theory in his Second Notebook [6, pp. 257-262].

K. Venkatachaliengar [8, pp. 89-95] examined some of the Entries in Ramanujan's Notebooks [6] devoted to alternative theories. In order to establish several beautiful formulas for $1/\pi$ stated by Ramanujan [5], [7, pp.23-29], J. M. Borwein and P. B. Borwein [4, pp. 177-185] have developed the "corresponding theories".

In Entries 51-72 of Chapter 25 of the Second Notebook [6], Ramanujan states twenty-three elegant, so-called, P-Q identities. Eighteen of these P-Q identities have been proved by B. C. Berndt [1, pp. 204-237] and B. C. Berndt and L.-C. Zhang [3] on employing various modular equations belonging to the theory of signature 2 (classical theory). The main purpose of this paper is to establish some P-Q identities stated by Ramanujan on employing various modular equations belonging to the theory of signature 4.

## 2. Some Preliminary Results

First we shall provide some useful notations and definitions. As customary, for any complex number $a$ set

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \ |q| < 1.$$

Following Ramanujan, we define

$$f(-q = (q;q)_\infty.$$

Let

$$z(r) := z(r;x) :=_2 F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)$$

and

$$q_r := q_r(x) := \exp\left[-\pi csc(\frac{\pi}{r}) \frac{_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x)}{_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)}\right],$$

where $r = 2, 3, 4, 6$ and $0 < x < 1$,

$$_2F_1(a, b; c; x) = \sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k k!} x^k$$

with $(a)_k = (a)(a+1)\ldots(a+k-1)$.

Let $n$ denote a fixed natural number and assume that

$$n\frac{_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha)}{_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha)} = \frac{_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta)}{_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \beta)} \tag{2.1}$$

where $r = 2, 3, 4$ or $6$. Then a modular equation of degree $n$ in the theory of elliptic functions of signature $r$ is a relation between $\alpha$ and $\beta$ induced by (2.1). We often say that $\beta$ is of degree $n$ over $\alpha$ and $m(r) := \dfrac{z(r;\alpha)}{z(r;\beta)}$ is called the multiplier.

We also use the notations $z_1 := z_1(\alpha) := z(r;\alpha)$ and $z_n := z_n(\beta) := z(r;\beta)$ to indicate that $\beta$ has degree $n$ over $\alpha$. When the context is clear, we omit the argument $r$ in $q_r, z(r)$ and $m(r)$.

We now collect in the following theorem some of Ramanujan's modular equations belonging to the theory of signature 4.

**Theorem 2.1.** *The following modular equations hold in the theory of signature 4.*

*(i) if $\beta$ is of degree 3 over $\alpha$ then,*

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \frac{9}{m^2}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}. \tag{2.2}$$

*(ii) If $\beta$ is of degree 5 over $\alpha$, then*

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \frac{5}{m}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}. \tag{2.3}$$

(iii) *If $\beta$ is of degree 7 over $\alpha$, then*

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \frac{49}{m^2}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}$$

$$-8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}\left[\left(\frac{\beta}{\alpha}\right)^{1/6} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/6}\right]. \qquad (2.4)$$

(iv) *If $\beta$ is of degree 9 over $\alpha$, then*

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \frac{3}{\sqrt{m}}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}. \qquad (2.5)$$

(v) *If $\beta$ is of degree 13 over $\alpha$, then*

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \frac{13}{m}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}$$

$$-4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}\left[\left(\frac{\beta}{\alpha}\right)^{1/12} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/12}\right]. \qquad (2.6)$$

(vi) *If $\beta$ is of degree 25 over $\alpha$, then*

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \frac{5}{\sqrt{m}}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}$$

$$-2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/24}\left[\left(\frac{\beta}{\alpha}\right)^{1/24} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/24}\right]. \qquad (2.7)$$

**Proof.** For a proof of (i) see [2, p. 156. Theorem 10.5]. For a proof of (ii) see [2, p.157, Theorem 10.6]. For proofs of (iii) and (iv) see [2, p.158, Theorm 10.7 and Theorem 10.8]. For a proof of (v) see [2, p.159, Theorem 10.9]. For a proof of (vi) see [2, p.160, Theorem 10.10].

## 3. Main Theorem

In this Section we obtain six P-Q identities (3.1)-(3.6) due to Ramanujan which appears as Entries 51,53,55,56,57 and 58 of Chapter 25 of his Second Notebook on employing Ramanujan's modular equations in the theory of signature 4.

**Theorem 3.1**

*(i) If*

$$P = \frac{f^2(-q)}{q^{1/6} f^2(-q^3)} \ \ and \ Q = \frac{f^2(-q^2)}{q^{1/3} f^2(-q^6)},$$

*then*

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \tag{3.1}$$

*(ii) If*

$$P = \frac{f(-q)}{q^{1/6} f(-q^5)} \ \ and \ Q = \frac{f(-q^2)}{q^{1/3} f(-q^{10})},$$

*then*

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \tag{3.2}$$

*(iii) If*

$$P = \frac{f^2(-q)}{q^{1/2} f^2(-q^7)} \ \ and \ Q = \frac{f^2(-q^2)}{q \ f^2(-q^{14})},$$

*then*

$$PQ + \frac{49}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 - 8\left(\frac{Q}{P}\right) - 8\left(\frac{P}{Q}\right). \tag{3.3}$$

*(iv) If*

$$P = \frac{f(-q)}{q^{1/3} f(-q^9)} \ \ and \ Q = \frac{f(-q^2)}{q^{2/3} f(-q^{18})},$$

*then*

$$P^3 + Q^3 = P^2 Q^2 + 3PQ. \tag{3.4}$$

*(v) If*

$$P = \frac{f(-q)}{q^{1/2} f(-q^{13})} \ \ and \ \ Q = \frac{f(-q^2)}{q \ f(-q^{26})},$$

*then*

$$PQ + \frac{13}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 - 4\left(\frac{Q}{P}\right) - 4\left(\frac{P}{Q}\right). \tag{3.5}$$

*(vi) If*

$$P = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)} \ \ and \ Q = \frac{f(-q^{2/5})}{q^{2/5} f(-q^{10})},$$

*then*

$$P^2 Q^2 + 5PQ = P^3 - 2P^2 Q - 2PQ^2 + Q^3. \tag{3.6}$$

**Proof of (3.1).** When $y = \pi\sqrt{2}\ \dfrac{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; 1-x)}{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)}$ and $q = e^{-y}$, we find that [2,p.148]

$$q^{1/24} f(-q) = \sqrt{z} 2^{-1/4}\ x^{1/24}(1-x)^{1/12},$$

and

$$q^{1/12} f(-q^2) = \sqrt{z} 2^{-1/2}\ x^{1/12}(1-x)^{1/24}.$$

Using the above two identities, we deduce that

$$\sqrt{z} x^{1/8} = 2^{3/4} q^{1/8} \frac{f^2(-q^2)}{f(-q)}, \tag{3.7}$$

and

$$\sqrt{z}(1-x)^{1/8} = \frac{f^2(-q)}{f(-q^2)}, \tag{3.8}$$

If $\beta$ is of degree 3 over $\alpha$, then from (3.7) and (3.8), it follows that

$$\alpha^{1/8} = \frac{2^{3/4} q^{1/8}}{\sqrt{z_1}}\ \frac{f^2(-q^2)}{f(-q)},\quad \beta^{1/8} = \frac{2^{3/4} q^{3/8}}{\sqrt{z_3}}\ \frac{f^2(-q^6)}{f(-q^3)},$$

$$(1-\alpha)^{1/8} = \frac{1}{\sqrt{z_1}}\frac{f^2(-q)}{f(-q^2)},\quad (1-\beta)^{1/8} = \frac{1}{\sqrt{z_3}}\ \frac{f^2(-q^3)}{f(-q^6)}, \tag{3.9}$$

Employing (3.9) in (2.2), we obtain

$$1 = q\frac{f^8(-q^6)f^4(-q)}{f^4(-q^3)f^8(-q^2)}\ +\ \frac{f^8(-q)f^4(-q^3)}{f^4(-q^2)f^8(-q^6)}\ -\ 9q\frac{f^4(-q^6)f^4(-q^3)}{f^4(-q)f^4(-q^2)}. \tag{3.10}$$

Upon simplification, we find that (3.10) is same as (3.1).

**Proof of (3.2).** If $\beta$ is of degree 5 over $\alpha$, then from (3.7) and (3.8), it follows that

$$\alpha^{1/8} = \frac{2^{3/4} q^{1/8}}{\sqrt{z_1}}\ \frac{f^2(-q^2)}{f(-q)},\quad \beta^{1/8} = \frac{2^{3/4} q^{5/8}}{\sqrt{z_5}}\ \frac{f^2(-q^{10})}{f(-q^5)},$$

$$(1-\alpha)^{1/8} = \frac{1}{\sqrt{z_1}}\ \frac{f^2(-q)}{f(-q^2)},\quad (1-\beta)^{1/8} = \frac{1}{\sqrt{z_5}}\frac{f^2(-q^5)}{f(-q^{10})}, \tag{3.11}$$

Employing (3.11) in (2.3), we obtain

$$1 = q\frac{f^4(-q^{10})f^2(-q)}{f^2(-q^5)f^4(-q^2)}\ +\ \frac{f^4(-q^5)f^2(-q^2)}{f^2(-q^{10})f^4(-q)}\ -\ \frac{5}{q}\ \frac{f^2(-q^{10})f^2(-q^5)}{f^2(-q)f^2(-q^2)}. \tag{3.12}$$

On simplification we see that (3.12) is same as (3.2).

**Proof of (3.3).** If $\beta$ is of degree 7 over $\alpha$, then from (3.7) and (3.8), it follows that

$$\alpha^{1/8} = \frac{2^{3/4} q^{1/8}}{\sqrt{z_1}}\ \frac{f^2(-q^2)}{f(-q)},\quad \beta^{1/8} = \frac{2^{3/4} q^{7/8}}{\sqrt{z_7}}\ \frac{f^2(-q^{14})}{f(-q^7)},$$

$$(1-\alpha)^{1/8} = \frac{1}{\sqrt{z_1}} \frac{f^2(-q)}{f(-q^2)}, \quad (1-\beta)^{1/8} = \frac{1}{\sqrt{z_7}} \frac{f^2(-q^7)}{f(-q^{14})}, \tag{3.13}$$

Employing (3.13) in (2.4), we obtain

$$1 = q^3 \frac{f^8(-q^{14})f^4(-q)}{f^4(-q^7)f^8(-q^2)} + \frac{f^8(-q^7)f^4(-q^2)}{f^4(-q^{14})f^8(-q)} - 49q^3 \frac{f^4(-q^{14})f^4(-q^7)}{f^4(-q)f^4(-q^2)}.$$

$$-8q \frac{f^{4/3}(-q^{14})f^{4/3}(-q^7)}{f^{4/3}(-q)f^{4/3}(-q^2)} \left[ q \frac{f^{8/3}(-q^{14})f^{4/3}(-q)}{f^{4/3}(-q^7)f^{8/3}(-q^2)} + \frac{f^{8/3}(-q^7)f^{4/3}(-q^2)}{f^{4/3}(-q^{14})f^{8/3}(-q)} \right]. \tag{3.14}$$

On simplification we see that (3.14) is same as (3.3).

**Proof of (3.4).** If $\beta$ is of degree 9 over $\alpha$, then from (3.7) and (3.8), it follows that

$$\alpha^{1/8} = \frac{2^{3/4}q^{1/8}}{\sqrt{z_1}} \frac{f^2(-q^2)}{f(-q)}, \quad \beta^{1/8} = \frac{2^{3/4}q^{9/8}}{\sqrt{z_9}} \frac{f^2(-q^{18})}{f(-q^9)},$$

$$(1-\alpha)^{1/8} = \frac{1}{\sqrt{z_1}} \frac{f^2(-q)}{f(-q^2)}, \quad (1-\beta)^{1/8} = \frac{1}{\sqrt{z_9}} \frac{f^2(-q^9)}{f(-q^{18})}, \tag{3.15}$$

Employing (3.15) in (2.5), we obtain

$$1 = q \frac{f^2(-q^{18})f(-q)}{f(-q^9)f^2(-q^2)} + \frac{f^2(-q^9)f(-q^2)}{f(-q^{18})f^2(-q)} - 3q \frac{f(-q^9)f(-q^{18})}{f(-q)f(-q^2)}. \tag{3.16}$$

Further simplification of (3.16) yields (3.4).

**Proof of (3.5).** If $\beta$ is of degree 13 over $\alpha$, then from (3.7) and (3.8), it follows that

$$\alpha^{1/8} = \frac{2^{3/4}q^{1/8}}{\sqrt{z_1}} \frac{f^2(-q^2)}{f(-q)}, \quad \beta^{1/8} = \frac{2^{3/4}q^{13/8}}{\sqrt{z_{13}}} \frac{f^2(-q^{26})}{f(-q^{13})}$$

$$(1-\alpha)^{1/8} = \frac{1}{\sqrt{z_1}} \frac{f^2(-q)}{f(-q^2)}, \quad (1-\beta)^{1/8} = \frac{1}{\sqrt{z_{13}}} \frac{f^2(-q^{13})}{f(-q^{26})}, \tag{3.17}$$

Using (3.17) in (2.6), it follows that

$$1 = q^3 \frac{f^4(-q^{26})f^2(-q)}{f^2(-q^{13})f^4(-q^2)} + \frac{f^4(-q^{13})f^2(-q^2)}{f^2(-q^{26})f^4(-q)} - 13q^3 \frac{f^2(-q^{26})f^2(-q^{13})}{f^2(-q^2)f^2(-q)}$$

$$-4q \frac{f^{2/3}(-q^{26})f^{2/3}(-q^{13})}{f^{2/3}(-q)f^{2/3}(-q^2)} \left[ q \frac{f^{4/3}(-q^{26})f^{2/3}(-q)}{f^{2/3}(-q^{13})f^{4/3}(-q^2)} + \frac{f^{4/3}(-q^{13})f^{2/3}(-q^2)}{f^{2/3}(-q^{26})f^{4/3}(-q)} \right]. \tag{3.18}$$

Further simplification of (3.18) yields (3.5).

**Proof of (vi).** If $\beta$ is of degree 25 over $\alpha$, then from (3.7) and (3.8), it follows that

$$\alpha^{1/8} = \frac{2^{3/4}q^{1/8}}{\sqrt{z_1}} \frac{f^2(-q^2)}{f(-q)}, \quad \beta^{1/8} = \frac{2^{3/4}q^{25/8}}{\sqrt{z_{25}}} \frac{f^2(-q^{50})}{f(-q^{25})},$$

$$(1 - \alpha)^{1/8} = \frac{1}{\sqrt{z_1}} \frac{f^2(-q)}{f(-q^2)}, \ (1 - \beta)^{1/8} = \frac{1}{\sqrt{z_{25}}} \frac{f^2(-q^{25})}{f(-q^{50})}, \tag{3.19}$$

Using (3.19) in (2.7), it follows that

$$1 = q^3 \frac{f^2(-q^{50})f(-q)}{f(-q^{25})f^2(-q^2)} + \frac{f^2(-q^{25})f(-q^2)}{f(-q^{50})f^2(-q)} - 5q^3 \frac{f(-q^{50})f(-q^{25})}{f(-q)f(-q^2)}$$

$$-2q\frac{f^{1/3}(-q^{50})f^{1/3}(-q^{25})}{f^{1/3}(-q)f^{1/3}(-q^2)} \left[ q\frac{f^{2/3}(-q^{50})f^{1/3}(-q)}{f^{1/3}(-q^{25})f^{2/3}(-q^2)} + \frac{f^{2/3}(-q^{25})f^{1/3}(-q^2)}{f^{1/3}(-q^{50})f^{2/3}(-q)} \right]. \tag{3.20}$$

Upon simplification we see that (3.20) is same as (3.6).

## Acknowledgement

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