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On the values of the Riemann zeta-function at rational arguments

S. Kanemitsu, Y. Tanigawa and M. Yoshimoto

Dedicated to Professor Takashi Yanagawa birthday with great respect

Abstract

In our previous papers [3], [4] we obtained a closed form evaluation of Ramanujan’s type of the values of the (multiple) Hurwitz zeta-function at rational arguments (with denominator even and numerator odd), which was in turn a vast generalization of D. Klusch’s and M. Katsurada’s generalization of Ramanujan’s formula. In this paper we shall continue our pursuit, specializing to the Riemann zeta-function, and obtain a closed form evaluation thereof at all rational arguments, with no restriction to the form of the rationals, in the critical strip. This is a complete generalization of the results of the aforementioned two authors. We shall obtain as a byproduct some curious identities among the Riemann zeta-values.

1 Introduction and notation

In this paper we shall give a closed form evaluation of Ramanujan’s type of the values of the Riemann zeta-function at positive rational arguments in the critical strip.

We have launched on this evaluation problem of zeta-values at rational arguments in [2], where we have succeeded in getting a Ramanujan type formula for $\zeta(2/3)$ after examining the results of D. Klusch [6] and M. Katsurada [5].

In [3] we have evaluated the values of the Hurwitz zeta-function at rational arguments smaller than 1 (of the form $1 - b/a$, with $b$ odd, $a$ even), which we have transformed into

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the evaluation of values of the multiple Hurwitz zeta-function through the Mellin-Vardi decomposition thereof into a linear combination of Hurwitz zeta-functions.

On the other hand, in [4] we have adopted a method closer to Katsurada’s to obtain the evaluation of the multiple Hurwitz zeta-function at positive rational arguments $b/a$.

In this paper we shall push forward our approach in [2] to the general situation and evaluate the values of the Riemann zeta-function at positive rational arguments, with no restriction to the form, thus supplying new information on them for $\zeta(b/a)$ with $b$ even, $a$ odd or both odd, missing in all preceding papers [2]-[4] (cf. Corollary 1).

We obtain as by-product some curious identities among modified Lambert series which look non-trivial and which follow by equating two expressions for them (cf. Corollary 2).

For readers’ convenience of comparison of the results in this paper and those in [2]-[4] we shall use the same notation and give the reference to the corresponding formulas in the latter.

**Notation.** Let $0 \leq h \leq N$ be integers. We put

\begin{align}
(1.1) \quad &a_{j,N} = \cos\left(\frac{\pi j}{2N}\right), \quad b_{j,N} = \sin\left(\frac{\pi j}{2N}\right), \\
(1.2) \quad &A_N(y) = \frac{1}{\pi} \arctan\left(\frac{4y}{1-y^2}\right), \\
(1.3) \quad &f_0(x; n, N) = \frac{e^{-A_N(\frac{n}{2})}}{2\sinh(A_N(\frac{n}{2}))} \\
\end{align}

and

\begin{align}
(1.4) \quad &f_j(x; n, N, h) = \frac{\cos(2A_N(\frac{n}{2})b_{j,N} + \frac{\pi(2h-1)j}{2N}) - e^{-2A_N(\frac{n}{2})a_{j,N}} \cos(\frac{\pi(2h-1)j}{2N})}{\cosh(2A_N(\frac{n}{2})a_{j,N}) - \cos(2A_N(\frac{n}{2})b_{j,N})}
\end{align}

for $j \geq 1$.

## 2 Values of the Riemann zeta-function

We shall prove the following Ramanujan type formula for $\zeta(\frac{N-2h+1}{N})$ on the lines as close as possible to those of proofs of the previous results. In the proof, we shall give the corresponding formula numbers in [2] and [4] (since those in [4] are suggestive of the corresponding ones in [3]).
Theorem 1. Let $N$ and $h$ be fixed natural numbers with $h \leq N/2$. Then we have for $x > 0$,

\begin{equation}
\sum_{n=1}^{\infty} \frac{n^{N-2h}}{e^{n^2x} - 1} = P(x) + S(x),
\end{equation}

where

\begin{equation}
P(x) = P(x; N, h) = -\frac{1}{2}\zeta(-N + 2h) + \zeta(2h)x^{-1}
+ \frac{1}{N} \Gamma \left( \frac{N - 2h + 1}{N} \right) \zeta \left( \frac{N - 2h + 1}{N} \right) x^{N-2h+1}
\end{equation}

and

\begin{equation}
S(x) = S(x; N, h) = \left( -\frac{1}{N} \right)^{h+1} \left( \frac{2\pi}{x} \right)^{N-2h+1} \times \sum_{n=0}^{\infty} \frac{1}{n} \sum_{j=1}^{\infty} f_{2j-1}(x; n, N, h)
\end{equation}

for $N$ odd, and

\begin{equation}
S(x) = S(x; N, h)
= \left( -\frac{1}{N} \right)^{h+1} \left( \frac{2\pi}{x} \right)^{N-2h+1} \sum_{n=0}^{\infty} \frac{1}{n} \sum_{j=1}^{\infty} f_{2j-1}(x; n, N, h),
\end{equation}

for $N$ even, with $f_j$ defined by (1.3) and (1.4).

Proof. We consider the integral with $c_0 > 1$

\begin{equation}
I(x) = \frac{1}{2\pi i} \int_{(c_0)} \Gamma(s)\zeta(s)\zeta(Ns - (N - 2h))x^{-s}ds,
\end{equation}

where $(c_0)$ denotes the vertical line $\sigma = c_0$, $-\infty < t < \infty$ (cf. [2],[4,(2.6)*]).

On the one hand, we have

\begin{equation}
I(x) = \sum_{n=1}^{\infty} \frac{n^{N-2h}}{e^{n^2x} - 1},
\end{equation}

i.e. the left-hand side of (2.1) (cf.[2], [4,(2.6)*]).
We now shift the line of integration to \((-c)\) with \(c > 0\), whereby we encounter the poles of the integrand at \(s = 0, 1, \frac{N-2h+1}{N}\), whence we deduce that

\[
I(x) = P(x) + J(x)
\]

where

\[
J(x) = J_{-c}(x) = \frac{1}{2\pi i} \int_{(-c)} \Gamma(s)\zeta(s)\zeta(Ns - (N - 2h))x^{-s}ds
\]

and \(P(x)\) is given by (2.2) (cf.\([2], [4, (2.8)*]\)).

Hence it is enough to prove that \(J(x)\) coincides with \(S(x)\).

We make the change of variables \(s \leftrightarrow 1 - s\) and use the functional equation for the Riemann zeta-function in the form

\[
\Gamma(1-s)\zeta(1-s)\zeta(2h - Ns) = (2\pi)^{2h-(N+1)s}(-1)^{h+1}C_N \left(\frac{\pi s}{2}\right)
\]

\[
\times \Gamma(Ns - 2h + 1)\zeta(Ns - 2h + 1)\zeta(s)
\]

(cf. \([2], [4, (2.9)*]\)), where \(C_N(z)\) denotes the Chebyshev polynomial

\[
C_N(z) = \frac{\sin Nz}{\sin z} = \sum_{j=-(N-1)}^{N-1} e^{ijz},
\]

and where \(\sum^\prime\) means the summation over \(j\) which increases by 2. Hence the integrand of

\[
J(x) = \frac{1}{2\pi i} \int_{(1+c)} \Gamma(1-s)\zeta(1-s)\zeta(2h - Ns)x^{-(1-s)}ds
\]

becomes

\[
G(s)x^{-(1-s)} = (-1)^{h+1}(2\pi)^{2h-(N+1)s}C_N \left(\frac{\pi s}{2}\right)
\]

\[
\times \Gamma(Ns - 2h + 1)\zeta(Ns - 2h + 1)\zeta(s)x^{-(1-s)}.
\]

Now make the change of variables

\[Ns - 2h + 1 = s_1, \quad s = \frac{s_1 + 2h - 1}{N}.\]

Then \(G(s)x^{s-1}\) becomes

\[
G \left(\frac{s_1 + 2h - 1}{N}\right)x^{1+\frac{s_1+2h-1}{N}} = (-1)^{h+1} \left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{2}} \left(\frac{x}{(2\pi)^{N+1}}\right)^{\frac{2h-1}{N}}
\]

\[
\times \Gamma(s_1)\zeta(s_1) \left(\frac{s_1 + 2h - 1}{N}\right) C_N \left(\frac{\pi (s_1 + 2h - 1)}{2N}\right).
\]
Incorporating these, we successively get

\begin{equation}
J(x) = \frac{1}{2\pi i} \int_{(1+c)} G(s)x^{s-1}ds
\end{equation}

\begin{equation}
= (-1)^{h+1} \left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \frac{1}{2\pi i} \int_{(c_1)} \left(\frac{(2\pi)^{N+1}}{x}\right)^{-\frac{s}{N}} \Gamma(s)
\end{equation}

\begin{equation}
\times \zeta(s_1)\zeta\left(\frac{s_1 + 2h - 1}{N}\right) C_N \left(\frac{\pi(s_1 + 2h - 1)}{2N}\right) ds_1, \frac{1}{N},
\end{equation}

where \(c_1 = N(1+c) - 2h + 1\) (cf. [2,(3.7)], [4, (2.18)*], (2.18)*]).

We substitute the Dirichlet series

\[\zeta(s)\zeta\left(\frac{s + 2h - 1}{N}\right) = \sum_{m,n=1}^{\infty} n^{-\frac{2h-1}{N}} (mn)^s\]

in (2.12) to obtain

\begin{equation}
J(x) = \frac{(-1)^{h+1}}{N} \left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \sum_{m,n=1}^{\infty} n^{-\frac{2h-1}{N}} E(X_{m,n})
\end{equation}

(cf. [2],[4,(2.18)*3,(2.19)]), where \((\kappa = c_1)\)

\begin{equation}
E(X_{m,n}) = \frac{1}{2\pi i} \int_{(\kappa)} X_{m,n}^{-s} C_N \left(\frac{\pi(s + 2h - 1)}{2N}\right) \Gamma(s) ds
\end{equation}

(cf. [2,(3.9)] [4,(2.21)*]) with \(X_{m,n}\) denoting

\begin{equation}
X_{m,n} = 2\pi m \left(\frac{2\pi n}{x}\right)^{\frac{1}{N}} = 2m A_N \left(\frac{n}{x}\right),
\end{equation}

where \(A_N(n/x)\) is defined by (1.3) (cf. [4,(2.22)*]).

To evaluate \(E(X_{m,n})\) we use (2.10) to write

\begin{equation}
X_{m,n}^{-s} C_N \left(\frac{\pi(s + 2h - 1)}{2N}\right) = \sum_{j=-(N-1)}^{N-1} e^{\pi i (2h-1)j} \left(X_{m,n} e^{-\frac{\pi i j}{2N}}\right)^{-s}
\end{equation}

with

\[|\arg X_{m,n} e^{-\frac{\pi i j}{2N}}| < \frac{\pi}{2}.
\]

Hence the well-known Mellin inversion formula

\[\frac{1}{2\pi i} \int_{(\kappa)} \Gamma(s) Y^{-s} ds = e^{-Y}\]
applies, and we have

\[(2.17) \quad E(X_{m,n}) = \sum_{j=-(N-1)}^{N-1} e^{\frac{\pi i (2h-1)j}{2N}} \left( e^{-2A_N(\frac{n}{2})} e^{-\frac{\pi ij}{2N}} \right)^m \]

(cf. [4,(2.24)*]).

Substituting (2.17) in (2.13) and summing the geometric progression in \(m\), we deduce that

\[(2.18) \quad J(x) = \frac{(-1)^{h+1}}{N} \left( \frac{2\pi}{x} \right)^{\frac{N-2h+1}{N}} \sum_{j=-(N-1)}^{N-1} e^{\frac{\pi i (2h-1)j}{2N}} \times \sum_{n=1}^{\infty} \left( \frac{1}{n^{2\frac{h-1}{N}}} e^{2A_N(\frac{n}{2})} e^{-\frac{\pi ij}{2N}} - 1 \right) \]

\[= \frac{(-1)^{h+1}}{N} \left( \frac{2\pi}{x} \right)^{\frac{N-2h+1}{N}} \sum_{n=1}^{\infty} n^{\frac{2h-1}{N}} \sum_{j=-(N-1)}^{N-1} e^{\frac{\pi i (2h-1)j}{2N}} Z_{j,n}(x), \]

where

\[Z_{j,n}(x) = e^{\frac{\pi i (2h-1)j}{2N}} e^{-A_N(\frac{n}{2})} e^{-\frac{\pi ij}{2N}} \]

\[= \frac{2 \sinh(A_N(\frac{n}{2}) e^{-\frac{\pi ij}{2N}})}{2 \sin(A_N(\frac{n}{2}) e^{-\frac{\pi ij}{2N}})} \]

(cf. [4,(2.30)*]).

We sum \(Z_j\)'s in pairs: \(Z_{j,n}(x) + Z_{-j,n}(x)\). Distinguishing two cases according to the parity of \(N\), we deduce that \(J(x)\) coincides with \(S(x)\) in (2.3) and (2.4) (cf. [4,(2.31)-(2.34)]

**Corollary 1.** Under the same notation as in Theorem 1, let \(L(x; N, h)\) be the function defined by

\[(2.19) \quad L(x; N, h) = \sum_{n=1}^{\infty} \frac{n^{N-2h} e^{nN x}}{\sin(nN x - 1)} - S(x; N, h) + \frac{1}{2} \zeta(-N + 2h) - \zeta(2h)x^{-1}. \]

Then we have

\[(2.20) \quad \zeta \left( \frac{N-2h+1}{N} \right) = \frac{N}{\Gamma(\frac{N-2h+1}{N})} x^{\frac{N-2h+1}{N}} L(x; N, h) \]

and

\[(2.21) \quad \zeta \left( \frac{2h-1}{N} \right) = \frac{N}{\pi} (2\pi)^{\frac{2h-1}{N}} \sin \pi(2h-1) x^{\frac{N-2h+1}{N}} L(x; N, h) \]

**Proof.** The first assertion is just the restatement of Theorem 1 and the second assertion follows from the functional equation of the Riemann zeta-function.
3 Identities among Lambert series

We shall prove the following consequence of our Corollary 1 which may be of interest for its own sake.

**Corollary 2.** In the notation of Corollary 1, we have

\[(3.1) \quad lL\left(x; lN, hl - \frac{l-1}{2}\right) = L(x; N, l)\]

holds for all \(l \geq 1, \text{ odd.}\)

Further suppose \(N\) even and put

\[(3.2) \quad \tilde{L}(x; N, h) = \left(\frac{2\sqrt{\pi}}{x}\right)^{\frac{2h}{N}} \Gamma\left(\frac{2h - 1 + N}{2N}\right) L(x; N, h).\]

Then

\[(3.3) \quad \tilde{L}(x; N, h) = \tilde{L}(x; N, N/2 - h + 1)\]

holds for all \(h, 1 \leq h \leq N/2.\)

**Proof.** Formula (3.1) follows from two distinct representations of \(\zeta(s)\) at

\[s = 1 - \frac{2h - 1}{N} = 1 - \frac{2(hl - \frac{l-1}{2}) - 1}{Nl}\]

for any odd integer \(l.\)

We now turn to (3.3) and suppose \(N\) even. Equate the expressions for \(\zeta(\frac{2h-1}{N})\) given by (2.20) and by (2.21) with \(h\) replaced by \(h_1 = \frac{N}{2} - h + 1\) to obtain

\[
\frac{N}{\pi} (2\pi)^{\frac{2h-1}{N}} \sin \frac{\pi(2h - 1)}{2N} x^{\frac{N-2h+1}{N}} L(x; N, h) = \frac{N}{\Gamma\left(\frac{N-2h+1}{N}\right)} x^{\frac{N-2h+1}{N}} L(x; N, h_1) \]

or

\[(3.4) \quad \frac{1}{\pi} (2\pi)^{\frac{2h-1}{N}} \sin \frac{\pi(2h - 1)}{2N} \Gamma\left(\frac{2h - 1}{N}\right) x^{\frac{N-4h+2}{N}} L(x; N, h) = L(x; N, h_1).\]

By the duplication formula and the reciprocity relation of the gamma function, we see that

\[
\sin \frac{\pi(2h - 1)}{2N} \Gamma\left(\frac{2h - 1}{N}\right) = 2^{\frac{2h-1}{N}-1} \sqrt{\pi} \Gamma\left(\frac{2h-1+N}{2N}\right) \Gamma\left(\frac{2(N/2-h+1)-1+N}{2N}\right),
\]
and therefore (3.4) becomes

\[
\frac{1}{\pi} (2\pi)^{\frac{2h+1}{N}} 2^{\frac{2h+1}{N}} \sqrt{\pi x} \left( \frac{2h-1+N}{2N} \right)^2 \Gamma \left( \frac{2h-1+N}{2N} \right) L(x; N, h) = \Gamma \left( \frac{2(N/2-h+1)-1+N}{2N} \right) L(x; N/2-h+1).
\]

Combining (3.2) and (3.5) implies the assertion (3.3). This completes the proof.

**Remark 1.** It may be worth recording that the main deformation in (3.1) occurs in the first summation and the coefficients \(f_j\)'s in the definition of \(L\), i.e. in place of \(\sum_{n=1}^{\infty} \frac{n^{N-2h}}{e^{nx} x-1}\) and \(f_j\)'s, we have

\[
\sum_{n=1}^{\infty} \frac{n^{(N-h+1)l+1}}{e^{nx} x-1},
\]

and

\[
f_j(x; n, lN, hl - \frac{l-1}{2}) = \frac{\cos(2A_{lN}(\frac{n}{x}) \sin \frac{\pi j}{2N}) + \pi(2l-1)j}{\cosh(2A_{lN}(\frac{n}{x}) \cos \frac{\pi j}{2N}) - \cos(2A_{lN}(\frac{n}{x}) \sin \frac{\pi j}{2N})}
\]

respectively.

**Examples**

(i) In (3.1), we take \(N = 2, h = 1, l = 3\). Then we get \(L(x; 2, 1) = 3L(x; 6, 2)\). More explicitly, it reads that

\[
\sum_{n=1}^{\infty} \frac{1}{e^{nx} x-1} - \frac{1}{2} \left( \frac{2\pi}{x} \right)^{\frac{3}{2}} \sum_{n=1}^{\infty} f_1(x; n, 2, 1) \sqrt{n} - \frac{1}{4} - \frac{\pi^2}{6x} = 3 \sum_{n=1}^{\infty} \frac{n^2}{e^{nx} x-1} + \frac{1}{2} \left( \frac{2\pi}{x} \right)^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left\{ f_1(x; n, 6, 2) + f_3(x; n, 6, 2) \right\} - \frac{\pi^4}{30x}.
\]

(ii) Next we take \(N = 4, h = 1\) in (3.2). Then \(\tilde{L}(x; 4, 1) = \tilde{L}(x; 4, 2)\). This means that

\[
\Gamma \left( \frac{5}{8} \right) \left\{ \sum_{n=1}^{\infty} \frac{n^2}{e^{nx} x-1} - \frac{1}{4} \left( \frac{2\pi}{x} \right)^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{1/4}} \left( f_1(x; n, 4, 1) + f_3(x; n, 4, 1) \right) - \frac{\pi^2}{6x} \right\} = \left( \frac{2\sqrt{\pi}}{x} \right)^{\frac{3}{2}} \Gamma \left( \frac{7}{8} \right) \left\{ \sum_{n=1}^{\infty} \frac{1}{e^{nx} x-1} - \frac{1}{4} - \frac{\pi^4}{90x} + \frac{1}{4} \left( \frac{2\pi}{x} \right)^{\frac{3}{2}} \times \sum_{n=1}^{\infty} \frac{1}{n^{3/4}} \left( f_1(x; n, 4, 2) + f_3(x; n, 4, 2) \right) \right\}.
\]
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