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On the values of the Riemann zeta-function at rational arguments

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Dedicated to Professor Takashi Yanagawa birthday with great respect

Abstract

In our previous papers [3], [4] we obtained a closed form evaluation of Ramanujan's type of the values of the (multiple) Hurwitz zeta-function at rational arguments (with denominator even and numerator odd), which was in turn a vast generalization of D. Klusch's and M. Katsurada's generalization of Ramanujan's formula. In this paper we shall continue our pursuit, specializing to the Riemann zeta-function, and obtain a closed form evaluation thereof at all rational arguments, with no restriction to the form of the rationals, in the critical strip. This is a complete generalization of the results of the aforementioned two authors. We shall obtain as a byproduct some curious identities among the Riemann zeta-values.

1 Introduction and notation

In this paper we shall give a closed form evaluation of Ramanujan's type of the values of the Riemann zeta-function at positive rational arguments in the critical strip.

We have launched on this evaluation problem of zeta-values at rational arguments in [2], where we have succeeded in getting a Ramanujan type formula for $\zeta(2/3)$ after examining the results of D. Klusch [6] and M. Katsurada [5].

In [3] we have evaluated the values of the Hurwitz zeta-function at rational arguments smaller than 1 (of the form $1-b/a$, with b odd, a even), which we have transformed into

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the evaluation of values of the multiple Hurwitz zeta-function through the Mellin-Vardi decomposition thereof into a linear combination of Hurwitz zeta-functions.

On the other hand, in [4] we have adopted a method closer to Katsurada's to obtain the evaluation of the multiple Hurwitz zeta-f unction at positive rational arguments b/a.

In this paper we shall push forward our approach in [2] to the general situation and evaluate the values of the Riemann zeta-function at positive rational arguments, with no restriction to the form, thus supplying new information on them for $\zeta(b/a)$ with b even, a odd or both odd, missing in all preceding papers [2]-[4] (cf. Corollary 1).

We obtain as by-product some curious identities among modified Lambert series which look non-trivial and which follow by equating two expressions for them (cf. Corollary 2).

For readers' convenience of comparison of the results in this paper and those in [2]-[4] we shall use the same notation and give the reference to the corresponding formulas in the latter.

Notation. Let $0 \leq h \leq N$ be integers. We put

$$a_{j,N} = \cos\left(\frac{\pi j}{2N}\right), \quad b_{j,N} = \sin\left(\frac{\pi j}{2N}\right) \tag{1.1}$$

$$A_N(y) = \pi(2\pi y)^{\frac{1}{N}}, \tag{1.2}$$

$$f_0(x; n, N) = \frac{e^{-A_N(\frac{n}{x})}}{2\sinh(A_N(\frac{n}{x}))} \tag{1.3}$$

and

$$f_j(x; n, N, h) = \frac{\cos(2A_N(\frac{n}{x})b_{j,N} + \frac{\pi(2h-1)j}{2N}) - e^{-2A_N(\frac{n}{x})a_{j,N}}\cos(\frac{\pi(2h-1)j}{2N})}{\cosh(2A_N(\frac{n}{x})a_{j,N}) - \cos(2A_N(\frac{n}{x})b_{j,N})} \tag{1.4}$$

for $j \geq 1$.

2 Values of the Riemann zeta-function

We shall prove the following Ramanujan type formula for $\zeta(\frac{N-2h+1}{N})$ on the lines as close as possible to those of proofs of the previous results. In the proof, we shall give the corresponding formula numbers in [2] and [4] (since those in [4] are suggestive of the corresponding ones in [3]).

Theorem 1. *Let N and h be fixed natural numbers with $h \le N/2$. Then we have for $x > 0$,*

$$\sum_{n=1}^{\infty} \frac{n^{N-2h}}{e^{n^N x} - 1} = P(x) + S(x), \tag{2.1}$$

where

$$P(x) = P(x; N, h) = -\frac{1}{2}\zeta(-N+2h) + \zeta(2h)x^{-1} + \frac{1}{N}\Gamma\left(\frac{N-2h+1}{N}\right)\zeta\left(\frac{N-2h+1}{N}\right)x^{-\frac{N-2h+1}{N}} \tag{2.2}$$

and

$$S(x) = S(x; N, h) = \frac{(-1)^{h+1}}{N}\left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \times \sum_{n=0}^{\infty} \frac{1}{n^{\frac{2h-1}{N}}} \left\{ f_0(x; n, N) + \sum_{j=1}^{\frac{N-1}{2}} f_{2j}(x; n, N, h) \right\} \tag{2.3}$$

for N odd, and

$$S(x) = S(x; N, h) = \frac{(-1)^{h+1}}{N}\left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \sum_{n=0}^{\infty} \frac{1}{n^{\frac{2h-1}{N}}} \sum_{j=1}^{\frac{N}{2}} f_{2j-1}(x; n, N, h), \tag{2.4}$$

for N even, with f_j defined by (1.3) and (1.4).

Proof. We consider the integral with $c_0 > 1$

$$I(x) = \frac{1}{2\pi i} \int_{(c_0)} \Gamma(s)\zeta(s)\zeta(Ns - (N-2h))x^{-s} ds, \tag{2.5}$$

where (c_0) denotes the vertical line $\sigma = c_0$, $-\infty < t < \infty$ (cf. [2],[4,(2.6)*]).

On the one hand, we have

$$I(x) = \sum_{n=1}^{\infty} \frac{n^{N-2h}}{e^{n^N x} - 1}, \tag{2.6}$$

i.e. the left-hand side of (2.1) (cf.[2], [4,(2.6)*]).

We now shift the line of integration to $(-c)$ with $c > 0$, whereby we encounter the poles of the integrand at $s = 0, 1, \frac{N-2h+1}{N}$, whence we deduce that

$$I(x) = P(x) + J(x) \tag{2.7}$$

where

$$J(x) = J_{-c}(x) = \frac{1}{2\pi i}\int_{(-c)} \Gamma(s)\zeta(s)\zeta(Ns-(N-2h))x^{-s}ds \tag{2.8}$$

and $P(x)$ is given by (2.2) (cf.[2], [4, $(2.8)_1^*$]).

Hence it is enough to prove that $J(x)$ coincides with $S(x)$.

We make the change of variables $s \leftrightarrow 1-s$ and use the functional equation for the Riemann zeta-function in the form

$$\Gamma(1-s)\zeta(1-s)\zeta(2h-Ns) = (2\pi)^{2h-(N+1)s}(-1)^{h+1}C_N\left(\frac{\pi s}{2}\right) \tag{2.9}$$
$$\times\Gamma(Ns-2h+1)\zeta(Ns-2h+1)\zeta(s)$$

(cf. [2], [4,(2.9)*]), where $C_N(z)$ denotes the Chebyshev polynomial

$$C_N(z) = \frac{\sin Nz}{\sin z} = \sum_{j=-(N-1)}^{N-1}{}'' e^{ijz}, \tag{2.10}$$

and where $\sum''$ means the summation over j which increases by 2. Hence the integrand of

$$J(x) = \frac{1}{2\pi i}\int_{(1+c)} \Gamma(1-s)\zeta(1-s)\zeta(2h-Ns)x^{-(1-s)}ds$$

becomes

$$G(s)x^{-(1-s)} = (-1)^{h+1}(2\pi)^{2h-(N+1)s}C_N(\frac{\pi s}{2}) \tag{2.11}$$
$$\times\Gamma(Ns-2h+1)\zeta(Ns-2h+1)\zeta(s)x^{-(1-s)}.$$

Now make the change of variables

$$Ns-2h+1 = s_1, \quad s = \frac{s_1+2h-1}{N}.$$

Then $G(s)x^{s-1}$ becomes

$$G\left(\frac{s_1+2h-1}{N}\right)x^{-1+\frac{s_1+2h-1}{N}} = (-1)^{h+1}\left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}}\left(\frac{x}{(2\pi)^{N+1}}\right)^{\frac{s_1}{N}}$$
$$\times\Gamma(s_1)\zeta(s_1)\zeta\left(\frac{s_1+2h-1}{N}\right)C_N\left(\frac{\pi(s_1+2h-1)}{2N}\right).$$

Incorporating these, we successively get

$$J(x) = \frac{1}{2\pi i}\int_{(1+c)} G(s)x^{s-1}ds \tag{2.12}$$

$$= (-1)^{h+1}\left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \frac{1}{2\pi i}\int_{(c_1)}\left(\frac{(2\pi)^{N+1}}{x}\right)^{-\frac{s_1}{N}}\Gamma(s_1)$$

$$\times\zeta(s_1)\zeta\left(\frac{s_1+2h-1}{N}\right)C_N\left(\frac{\pi(s_1+2h-1)}{2N}\right)\frac{ds_1}{N},$$

where $c_1 = N(1+c) - 2h + 1$ (cf. [2,(3.7)],[4, $(2.18)_1^*, (2.18)_2^*$]).

We substitute the Dirichlet series

$$\zeta(s)\zeta\left(\frac{s+2h-1}{N}\right) = \sum_{m,n=1}^{\infty} n^{-\frac{2h-1}{N}}(mn^{\frac{1}{N}})^{-s}$$

in (2.12) to obtain

$$J(x) = \frac{(-1)^{h+1}}{N}\left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}}\sum_{m,n=1}^{\infty} n^{-\frac{2h-1}{N}}E(X_{m,n}) \tag{2.13}$$

(cf. [2],[4,$(2.18)_3^*, (2.19)^*$]), where ($\kappa = c_1$)

$$E(X_{m,n}) = \frac{1}{2\pi i}\int_{(\kappa)} X_{m,n}^{-s}C_N\left(\frac{\pi(s+2h-1)}{2N}\right)\Gamma(s)ds \tag{2.14}$$

(cf. [2,(3.9)] [4,$(2.21)^*$]) with $X_{m,n}$ denoting

$$X_{m,n} = 2\pi m\left(\frac{2\pi n}{x}\right)^{\frac{1}{N}} = 2mA_N\left(\frac{n}{x}\right), \tag{2.15}$$

where $A_N(n/x)$ is defined by (1.3) (cf. [4,$(2.22)^*$]).

To evaluate $E(X_{m,n})$ we use (2.10) to write

$$X_{m,n}^{-s}C_N\left(\frac{\pi(s+2h-1)}{2N}\right) = \sum_{j=-(N-1)}^{N-1}{}'' e^{\frac{\pi i(2h-1)j}{2N}}\left(X_{m,n}e^{-\frac{\pi ij}{2N}}\right)^{-s} \tag{2.16}$$

with

$$|\arg X_{m,n}e^{-\frac{\pi ij}{2N}}| < \frac{\pi}{2}.$$

Hence the well-known Mellin inversion formula

$$\frac{1}{2\pi i}\int_{(\kappa)}\Gamma(s)Y^{-s}ds = e^{-Y}$$

applies, and we have

$$E(X_{m,n}) = \sum_{j=-(N-1)}^{N-1}{}'' e^{\frac{\pi i(2h-1)j}{2N}} \left(e^{-2A_N(\frac{n}{x})e^{-\frac{\pi ij}{2N}}}\right)^m \tag{2.17}$$

(cf. [4,(2.24)*]).

Substituting (2.17) in (2.13) and summing the geometric progression in m, we deduce that

$$\begin{aligned} J(x) &= \frac{(-1)^{h+1}}{N}\left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \sum_{j=-(N-1)}^{N-1}{}'' e^{\frac{\pi i(2h-1)j}{2N}} \\ &\quad \times \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2h-1}{N}}} \frac{1}{e^{2A_N(\frac{n}{x})e^{-\frac{\pi ij}{2N}}}-1} \\ &= \frac{(-1)^{h+1}}{N}\left(\frac{2\pi}{x}\right)^{\frac{N-2h+1}{N}} \sum_{n=1}^{\infty} n^{-\frac{2h-1}{N}} \sum_{j=-(N-1)}^{N-1}{}'' Z_{j,n}(x), \end{aligned} \tag{2.18}$$

where

$$Z_{j,n}(x) = \frac{e^{\frac{\pi i(2h-1)j}{2N}} e^{-A_N(\frac{n}{x})e^{-\frac{\pi ij}{2N}}}}{2\sinh(A_N(\frac{n}{x})e^{-\frac{\pi ij}{2N}})}$$

(cf. [4,(2.30)*]).

We sum Z_j's in pairs : $Z_{j,n}(x) + Z_{-j,n}(x)$. Distinguishing two cases according to the parity of N, we deduce that $J(x)$ coincides with $S(x)$ in (2.3) and (2.4) (cf. [4,(2.31)-(2.34)]

Corollary 1. *Under the same notation as in Theorem 1, let $L(x;N,h)$ be the function defined by*

$$L(x;N,h) = \sum_{n=1}^{\infty} \frac{n^{N-2h}}{e^{n^N x}-1} - S(x;N,h) + \frac{1}{2}\zeta(-N+2h) - \zeta(2h)x^{-1}. \tag{2.19}$$

Then we have

$$\zeta\left(\frac{N-2h+1}{N}\right) = \frac{N}{\Gamma(\frac{N-2h+1}{N})} x^{\frac{N-2h+1}{N}} L(x;N,h) \tag{2.20}$$

and

$$\zeta\left(\frac{2h-1}{N}\right) = \frac{N}{\pi}(2\pi)^{\frac{2h-1}{N}} \sin\frac{\pi(2h-1)}{2N} x^{\frac{N-2h+1}{N}} L(x;N,h) \tag{2.21}$$

Proof. The first assertion is just the restatement of Theorem 1 and the second assertion follows from the functional equation of the Riemann zeta-function.

3 Identities among Lambert series

We shall prove the following consequence of our Corollary 1 which may be of interest for its own sake.

Corollary 2. *In the notation of Corollary 1, we have*

$$lL\left(x; lN, hl - \frac{l-1}{2}\right) = L(x;N,l) \tag{3.1}$$

holds for all $l \geq 1$, odd.

Further suppose N even and put

$$\tilde{L}(x;N,h) = \left(\frac{2\sqrt{\pi}}{x}\right)^{\frac{2h}{N}} \Gamma\left(\frac{2h-1+N}{2N}\right) L(x;N,h). \tag{3.2}$$

Then

$$\tilde{L}(x;N,h) = \tilde{L}(x;N,N/2-h+1) \tag{3.3}$$

holds for all h, $1 \leq h \leq N/2$.

Proof. Formula (3.1) follows from two distinct representations of $\zeta(s)$ at

$$s = 1 - \frac{2h-1}{N} = 1 - \frac{2(hl - \frac{l-1}{2}) - 1}{Nl}.$$

for any odd integer l.

We now turn to (3.3) and suppose N even. Equate the expressions for $\zeta(\frac{2h-1}{N})$ given by (2.20) and by (2.21) with h replaced by $h_1 = \frac{N}{2} - h + 1$ to obtain

$$\frac{N}{\pi}(2\pi)^{\frac{2h-1}{N}} \sin\frac{\pi(2h-1)}{2N} x^{\frac{N-2h+1}{N}} L(x;N,h) = \frac{N}{\Gamma\left(\frac{N-2h_1+1}{N}\right)} x^{\frac{N-2h_1+1}{N}} L(x;N,h_1)$$

or

$$\frac{1}{\pi}(2\pi)^{\frac{2h-1}{N}} \sin\frac{\pi(2h-1)}{2N} \Gamma(\frac{2h-1}{N}) x^{\frac{N-4h+2}{N}} L(x;N,h) = L(x;N,h_1). \tag{3.4}$$

By the duplication formula and the reciprocity relation of the gamma function, we see that

$$\sin\frac{\pi(2h-1)}{2N} \Gamma\left(\frac{2h-1}{N}\right) = 2^{\frac{2h-1}{N}-1}\sqrt{\pi}\frac{\Gamma(\frac{2h-1+N}{2N})}{\Gamma\left(\frac{2(N/2-h+1)-1+N}{2N}\right)},$$

and therefore (3.4) becomes

$$\frac{1}{\pi}(2\pi)^{\frac{2h-1}{N}}2^{\frac{2h-1}{N}-1}\sqrt{\pi}x^{\frac{N-4h+2}{N}}\Gamma\left(\frac{2h-1+N}{2N}\right)L(x;N,h)$$
$$=\Gamma\left(\frac{2(N/2-h+1)-1+N}{2N}\right)L(x;N,N/2-h+1). \tag{3.5}$$

Combing (3.2) and (3.5) implies the assertion (3.3). This completes the proof.

*Remark 1.*It may be worth recording that the main deformation in (3.1) occurs in the first summation and the coefficients f_j's in the definition of L, i.e. in place of $\sum_{n=1}^{\infty}\frac{n^{N-2h}}{e^{n^N x}-1}$ and f_j's, we have

$$\sum_{n=1}^{\infty}\frac{n^{(N-h+1)l-1}}{e^{n^{Nl}x}-1},$$

and

$$f_j(x;n,lN,hl-\frac{l-1}{2})$$
$$=\frac{\cos(2A_{lN}(\frac{n}{x})\sin\frac{\pi j}{2lN}+\frac{\pi(2h-1)j}{2N})-e^{-2A_{lN}(\frac{n}{x})\cos\frac{\pi j}{2lN}}\cos(\frac{\pi(2h-1)j}{2N})}{\cosh(2A_{lN}(\frac{n}{x})\cos\frac{\pi j}{2lN})-\cos(2A_{lN}(\frac{n}{x})\sin\frac{\pi j}{2lN})}$$

respectively.

Examples (i) In (3.1), we take $N=2$, $h=1$, $l=3$. Then we get $L(x;2,1)=3L(x;6,2)$. More explicitly, it reads that

$$\sum_{n=1}^{\infty}\frac{1}{e^{n^2x}-1}-\frac{1}{2}\left(\frac{2\pi}{x}\right)^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{f_1(x;n,2,1)}{\sqrt{n}}-\frac{1}{4}-\frac{\pi^2}{6x}$$
$$=3\sum_{n=1}^{\infty}\frac{n^2}{e^{n^6x}-1}+\frac{1}{2}\left(\frac{2\pi}{x}\right)^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\Big\{f_1(x;n,6,2)+f_3(x;n,6,2)$$
$$+f_5(x;n,6,2)\Big\}-\frac{\pi^4}{30x}.$$

(ii) Next we take $N=4$, $h=1$ in (3.2). Then $\tilde{L}(x;4,1)=\tilde{L}(x;4,2)$. This means that

$$\Gamma(\frac{5}{8})\left\{\sum_{n=1}^{\infty}\frac{n^2}{e^{n^4x}-1}-\frac{1}{4}\left(\frac{2\pi}{x}\right)^{\frac{3}{4}}\sum_{n=1}^{\infty}\frac{1}{n^{1/4}}\Big(f_1(x;n,4,1)+f_3(x;n,4,1)\Big)-\frac{\pi^2}{6x}\right\}$$
$$=\left(\frac{2\sqrt{\pi}}{x}\right)^{\frac{1}{2}}\Gamma(\frac{7}{8})\left\{\sum_{n=1}^{\infty}\frac{1}{e^{n^4x}-1}-\frac{1}{4}-\frac{\pi^4}{90x}+\frac{1}{4}\left(\frac{2\pi}{x}\right)^{\frac{1}{4}}\right.$$
$$\left.\times\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}\Big(f_1(x;n,4,2)+f_3(x;n,4,2)\Big)\right\}.$$

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