Hardy-Ramanujan Journal
Vol.26 (2003) 2-17

Mean-Square Upper Bound of Hecke L- Functions On The Critical Line

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ABSTRACT. We prove the upper bound for the mean-square of the absolute value of the Hecke L-functions (attached to a holomorphic cusp form) defined for the congruence subgroup $\Gamma_0(N)$ on the critical line uniformly with respect to its conductor N.

1. Introduction

Let χ denote a character mod q and $N_\chi(\alpha, T)$, the number of zeros of $L(s,\chi)$ in the rectangle $\alpha < \sigma < 1, |t| < T$. Gallagher (see [7]) proved that

$$\sum_{\chi} \left| L\left(\frac{1}{2}+it,\chi\right)\right|^2 \ll (q+|t|)\log(q|t|+2), \tag{1.1}$$

$$\int_{-T}^{T} \left| L\left(\frac{1}{2}+it,\chi\right)\right|^2 \ll (q+T)\log qT, \tag{1.2}$$

and

$$N_\chi(\alpha,T) \ll T^{3(1-\alpha)}(\log T)^C \tag{1.3}$$

uniformly for all $q \leq T$. He also established that

$$\sum_{\chi} \left(N_\chi(\alpha,T+1) - N_\chi(\alpha,T)\right) \ll q^{3(1-\alpha)}(\log q)^C \tag{1.4}$$

for $T \leq q$.

Meurman (see [24]) proved that for any $q \geq 1$,

$$\sum_{\chi(mod\ \ q)} \int_{T}^{T+H} \left| L\left(\frac{1}{2}+it,\chi\right)\right|^2 \ll_\epsilon \left(qH + (qT)^{\frac{1}{3}}\right)(q(T+H))^\epsilon \tag{1.5}$$

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*2000 *Mathematics subject classification :* primary 11M , secondary 11M06 .

†*Key Words :* Hecke L-functions, Holomorphic cusp forms, Mean-value theorems, Montgomery-Vaughan Theorem.

for any $\epsilon > 0$ and $3 \leq H \leq T$.

Balasubramanian and Ramachandra have given a simple proof of the above result (1.5) (see [2]) and they could replace $(qT)^\epsilon$ by $\log qT$ times a power of $d(q)$ for primitive characters mod q. Jutila has proved ([20] and [21]) the fourth power mean for $\zeta(s)$ and mean-square for L-functions attached to cusp forms (holomorphic as well Maass wave forms) via Laplace transformation method. In fact he studies the error term mean values in a greater generality with an emphasis as a function of T.

It should be mentioned here that mean values of derivatives of modular L-series had been studied by Ram Murty and Kumar Murty in [34]. For several interesting mean value results which were obtained by many mathematicians, see for example [1], [11], [16], [19], [22], [23], [28], [37], [38].

Let $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}, \Im z > 0$ be a cusp form of even integral weight $k > 2$ for the full modular group $SL(2, Z)$. Hecke L-function attached to f is defined as

$$L_f(s) = \sum_{n=1}^{\infty} a_f(n) n^{-s} \tag{1.6}$$

which is absolutely convergent in $\Re s > \frac{k+1}{2}$, and it satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L_f(s) = (-1)^{\frac{k}{2}}(2\pi)^{s-k}\Gamma(k-s)L_f(k-s). \tag{1.7}$$

In [10], A. Good proved the following

THEOREM A. *If C_{-1} denotes the residue and $C_{-1}C_0$ the constant term in the Laurent expansion of*

$$D(s) = \sum_{n=1}^{\infty} |a_n|^2 n^{-s} \quad \text{at} \quad \text{s} = \text{k},$$

then

$$\int_0^T \left| L_f\left(\frac{k}{2} + it\right)\right|^2 dt = 2C_{-1}T\left(\log\left(\frac{T}{2\pi e}\right) + C_0\right) + O\left(T(\log T)^{\frac{2}{3}}\right)$$

as $T \to \infty$.

As a corollary, he deduced that

$$L_f\left(\frac{k}{2} + it\right) \ll |t|^{\frac{1}{3}} (\log|t|)^{\frac{5}{6}}.$$

Our main goal in this paper is the following. Let $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}, \Im z > 0$ be a holomorphic cusp form of even integral weight $k > 2$ with level N (i.e in $\Gamma_0(N)$) and

$a_f(1) = 1$. We also assume that f is a Hecke eigenform so that $a_f(n)$ are eigenvalues of all the Hecke operators. We are interested to estimate an upper bound for the average integral namely

$$I = \sum_f{}' \frac{1}{< f, f >} \int_T^{2T} \left| L_f\left(\frac{k}{2} + it\right)\right|^2 dt$$

where $'$ indicates that f runs over an orthonormal basis set, uniformly with respect to the conductor N. We prove

MAIN THEOREM. *We have*

$$\begin{aligned} I &= \sum_f{}' \frac{1}{< f, f >} \int_T^{2T} \left| L_f\left(\frac{k}{2} + it\right)\right|^2 dt \\ &\ll_{k,\epsilon} NT(\log NT)^3(\log\log T)^2 + N^{5+10\epsilon}e^{-C(\log T)^2} \end{aligned}$$

uniformly for all levels N of f.

Remark 1. This main theorem may be compared with the result of A. Good. As a corollary to this main theorem, we obtain uniformly for all $N \ll T^\epsilon$, the inequality

$$I = \sum_f{}' \frac{1}{< f, f >} \int_T^{2T} \left| L_f\left(\frac{k}{2} + it\right)\right|^2 dt \ll_{k,\epsilon} T^{1+\epsilon}$$

Remark 2. The main feature of the proof here is that we avoid the approximate functional equation (whereas Good's proof depends upon the approximate functional equation). Whenever a Dirichlet series has a functional equation, in [4] K. Chandrasekharan and Raghavan Narasimhan established that a form of the approximate functional equation can be proved and it has a nice form whenever the coefficients of the Dirichlet series under consideration are positive which can be utilised to study mean value theorems. Of course, in some special circumstances even if we do not have the coefficients of the Dirichlet series to be positive it is possible to use that form to study mean-value theorems. However K. Ramachandra observed that just with the functional equation, one can prove reasonable upper bounds for mean-value questions on certain lines and he has used this idea in many of his research work (see for example [28] and [29]). This crucial idea what we are going to use in proving our main theorem.

Remark 3. It should be mentioned here that a general mean value theorem (see theorem 1 of [22]) is available. It is not difficult to see that (choosing the parameters appropriately and combining lemma 3.1 of section 3)

$$I = \sum_f{}' \frac{1}{< f, f >} \int_T^{2T} \left| L_f\left(\frac{k}{2} + it\right) \right|^2 dt \ll_{k,\epsilon} T^{1+\epsilon}.$$

However, this estimate holds only when the conductor N of f satisfies the condition $N \ll 1$.

Acknowledgement : The author has great pleasure in thanking Professor Ram Murty for posing this problem some times back and for having fruitful and stimulating discussions related to the Poincare series topic. The author is highly indebted to Instituto de Matemáticas, UNAM, Morelia, Mexico for it's warm hospitality and grand financial support during his sabbatical stay there. He also wishes to express his gratitude to the referee for valuable comments.

2. Notation and preliminaries

The letters C and A (with or without suffixes) denote effective positive constants unless it is specified. It need not be the same at every occurrence. Throughout the paper we assume $T \geq T_0$ where T_0 is a large positive constant. We write $f(x) \ll g(x)$ to mean $|f(x)| < C_1 g(x)$ (sometimes we denote this by the O notation also). Let $s = \sigma + it$, and $w = u + iv$. The implied constants are all effective.

Let us recall some basic facts concerning the Poincaré series (see for example [33]).

Let Γ_∞ be the stabilizer of $i\infty$ in $\Gamma_0(N)$. The space $S_k(N)$ of cusp forms of weight k and level N is a finite dimensional vector space over C, spanned by the Poincaré series : for $m \geq 1$,

$$P_m(z, k, N) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} j(\gamma, z)^{-k} e(m\gamma z)$$

where

$$e(z) = e^{2\pi iz}, \quad j(\gamma, z) = (\det \gamma)^{-\frac{1}{2}} (cz + d)$$

and

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$$

In the case of $k = 2$, we do not have absolute convergence of the Poincaré series. However, the discussion below holds in this case as well. If $f \in S_k(N)$, we write the Fourier expansion of f as :

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz)$$

at $i\infty$. The space $S_k(N)$ has an inner product (Petersson inner product) :

$$< f, g >= \int_{\Gamma_0(N)\backslash h} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}$$

where h denotes the upper half plane. Petersson proved that

$$a_f(n) = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} < f, P_n(., k, N) >$$

so that if $f_1, \cdots, f_r$ is an orthonormal basis for $S_k(N)$, and

$$P_n(., k, N) = \sum_i c_i f_i$$

we obtain

$$c_i =< f_i, P_n(., k, N) >= \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a_{f_i}(n).$$

Therefore, we have

$$\frac{(4\pi n)^{k-1}}{\Gamma(k-1)} P_n(., k, N) = \sum_i a_{f_i}(n) f_i.$$

If we compare the m-th coefficients on both sides, then we obtain

$$\frac{(4\pi n)^{k-1}}{\Gamma(k-1)} a_{P_n(m,k,N)} = \sum_i a_{f_i}(n) a_{f_i}(m).$$

The m-th Fourier coefficient of the n-th Poincaré series $P_n(z, k, N)$ can be computed as :

$$a_{P_n(m,k,N)} = \left(\frac{m}{n}\right)^{\frac{k-1}{2}} \left\{ \delta_{mn} + 2\pi i^{-k} \sum_{c \equiv 0 (mod N)} c^{-1} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) S_{\infty\infty}(m, n, c) \right\}$$

where $\delta_{mn} = 0$ unless $m = n$ in which case it is 1.

It is well known that (see [12], [13]), $L_f(s)$ (f with level N) satisfies the functional equation

(2.1) $$L_f(s) = \chi_f(s) L_f(k-s),$$

where

(2.2) $$\chi_f(s) = C\left(\frac{2\pi}{N}\right)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)},$$

with $|C| = 1$. We also know that, in any fixed strip $a \leq \sigma \leq b$, as $t \to \infty$, we have

(2.3) $$\Gamma(\sigma + it) = t^{\sigma+it-1/2} e^{-\pi/2-it+(i\pi/2)(\sigma-1/2)} \sqrt{2\pi}\left(1 + O\left(\frac{1}{t}\right)\right).$$

We observe that

$$|\chi_f(s)| = \left(\frac{2\pi}{N}\right)^{2\sigma-k} |t|^{k-2\sigma}\left(1+O\left(\frac{1}{|t|}\right)\right) \tag{2.4}$$

as $t \to \infty$. Since, $L_f(s)$ is absolutely convergent in $\sigma > \frac{k+1}{2}$, we have

$$L_f\left(\frac{k+1}{2}+\epsilon+it\right) \ll_\epsilon 1. \tag{2.5}$$

From the functional equation, we have

$$\begin{aligned} \left|L_f\left(\frac{k}{2}-1+it\right)\right| &= \left|\chi_f\left(\frac{k}{2}-1+it\right)L_f\left(\frac{k}{2}+1+it\right)\right| \\ &\ll \left(\frac{2\pi}{N}\right)^{2\left(\frac{k}{2}-1\right)-k} |t|^{k-2\left(\frac{k}{2}-1\right)} \\ &\ll \left(\frac{2\pi}{N}\right)^{-2} |t|^2. \end{aligned} \tag{2.6}$$

Let H be the set of all holomorphic cusp forms (Hecke eigenforms) of even integral weight $k(\geq 2)$ of level N. We assume that $a_f(1) = 1 \; \forall \; f \in H$. We also observe that the set

$$\left\{\frac{f}{< f, f >} \quad : \quad f \in H\right\}$$

forms an orthonormal basis for H. From the Poincare series discussion above (see for example [33] also), we have (for $k \geq 2$)

$$\begin{aligned} E &=: \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_f{}' \frac{a_f(n)\bar{a_f(m)}}{< f, f > (mn)^{\frac{k-1}{2}}} \\ &= \delta_{mn} + 2\pi i^{-k} \sum_{c\equiv 0 (mod N)} c^{-1} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) S_{\infty\infty}(m,n,c), \end{aligned} \tag{2.7}$$

where $\delta_{mn} = 0$ unless $m = n$ in which case it is 1. $J_{k-1}(x)$ is the Bessel function of order $k-1$, defined as

$$J_k(z) = \frac{\left(\frac{z}{2}\right)^k}{\sqrt{\pi}\Gamma\left(k+\frac{1}{2}\right)} \int_0^\pi (\sin\theta)^{2k}\cos(z\cos\theta)d\theta \tag{2.8}$$

and $S_{\infty\infty}(m,n,c)$ is the Kloosterman sum

$$(2.9) \qquad S_{\infty\infty}(m,n,c) = S(m,n,Nl) = \sum_{\substack{d(modNl) \\ d\bar{d}\equiv 1(modNl)}} e\left(\frac{md+n\bar{d}}{Nl}\right).$$

Here $e(z) = e^{2\pi i z}$. We have (from Deligne's estimate)

$$(2.10) \qquad a_f(n) \ll d(n) n^{\frac{k-1}{2}},$$

and (from Weil's estimate)

$$(2.11) \qquad S(m,n,c) \leq (m,n,c)^{\frac{1}{2}} d(c) c^{\frac{1}{2}}.$$

It is known that

$$(2.12) \qquad J_k(x) \ll \frac{x^k}{2^k \Gamma\left(k+\frac{1}{2}\right)}$$

3. Some lemmas

LEMMA 3.1 (i) For $k=2$

$$\sum_f{}' \frac{1}{< f, f >} = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} + O\left(N^{-\frac{3}{2}+\epsilon}\right)$$

and
(ii) for $k > 2$,

$$\sum_f{}' \frac{1}{< f, f >} = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} + O\left(N^{-(k-1)+\epsilon}\right).$$

Proof. From (2.7), (2.11) and (2.12), we get (for $k=2$)

$$\begin{aligned} \sum_{c\equiv 0(modN)} c^{-1} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) S(m,n,c) &\ll (mn)^{\frac{1}{2}} (m,n)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} \frac{d(Nj)}{(Nj)^{\frac{3}{2}}}\right) \\ (3.1) \qquad &\ll N^{-\frac{3}{2}+\epsilon} (mn)^{\frac{1}{2}} (m,n)^{\frac{1}{2}}, \end{aligned}$$

and for $k > 2$,

$$\begin{aligned} \sum_{c\equiv 0 (modN)} c^{-1} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) S(m,n,c) &\ll_k \sum_{c\equiv 0(modN)} \frac{(mn)^{\frac{k-1}{2}} d(c)}{c^{k-1}} \\ &\ll_k N^{-(k-1)+\epsilon}(mn)^{\frac{k-1}{2}}. \end{aligned} \tag{3.2}$$

From (3.1.1) and (3.1.2), we get

$$\begin{aligned} E &=: \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_f{}' \frac{a_f(n)a_f\bar{(m)}}{< f,f > (mn)^{\frac{k-1}{2}}} \\ &= \delta_{mn} + O\left(\max\left(N^{-\frac{3}{2}+\epsilon}(mn)^{\frac{1}{2}}(m,n)^{\frac{1}{2}}, N^{-(k-1)+\epsilon}(mn)^{\frac{k-1}{2}}\right)\right). \end{aligned} \tag{3.3}$$

Taking $m = n = 1$ in (3.1.3) and noticing that $a_f(1) = 1$, we obtain

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_f{}' \frac{1}{< f,f >} = 1 + O\left(\max\left(N^{-\frac{3}{2}+\epsilon}, N^{-(k-1)+\epsilon}\right)\right). \tag{3.4}$$

This proves the lemma.

LEMMA 3.2 For σ in the range $\frac{k}{2} - 1 \leq \sigma \leq \frac{k+1}{2} + \epsilon$, uniformly, we have the estimate

$$L_f(\sigma + it) \ll \left(\frac{|t|N}{2\pi}\right)^{\frac{4}{3}\left(\frac{k+1}{2}-\sigma\right)+\epsilon}.$$

Proof. Follows by applying maximum-modulus principle to a suitable function, namely

$$F(w) = L_f(w)e^{(w-s)^2}X^{w-s}$$

over a suitable rectangle and we can choose $X = \left(\frac{|t|N}{2\pi}\right)^{\frac{4}{3}+\epsilon}$.

LEMMA 3.3. *(Montgomery-Vaughan) If b_n is an infinite sequence of complex numbers such that $\sum_{n=1}^{\infty} n|b_n|^2$ is convergent, then*

$$\int_T^{T+H} \left|\sum_{n=1}^{\infty} b_n n^{-it}\right|^2 dt = \sum_{n=1}^{\infty} |b_n|^2 \left(H + O(n)\right).$$

Proof. See for example lemma 3.3 of [26] or [31].

4. Proof of the theorem

Let $s_1 = \frac{k}{2} + it$ and let Y and Y_1 be two parameters satisfying $T^{\frac{1}{2}} \leq Y, Y_1 \leq (NT)^A$ to be chosen appropriately later. Let $\epsilon_1 = (\log T)^{-1}$. By Mellin's transformation,

$$\begin{aligned}
S &=: \sum_{n=1}^{\infty} \frac{a_f(n)}{n^{s_1}} e^{-\frac{n}{Y}} \\
&= \frac{1}{2\pi i} \int_{\substack{\Re w = 1/2+\epsilon, \\ |v| \le (\log T)^2}} L_f(s_1+w) Y^w \Gamma(w) dw + O\left(Y^{\frac{1}{2}+\epsilon} e^{-C(\log T)^2}\right) \\
&= L_f(s_1) + \frac{1}{2\pi i} \int_{\substack{\Re w = -1+2\epsilon_1, \\ |v| \le (\log T)^2}} L_f(s_1+w) Y^w \Gamma(w) dw + O\left(Y^{\frac{1}{2}+\epsilon} e^{-C(\log T)^2}\right) \\
&+ O\left(Y^{\frac{1}{2}+\epsilon} \left(\frac{TN}{2\pi}\right)^{2+\epsilon} e^{-C(\log T)^2}\right) \\
&= L_f(s_1) + I + O\left(Y^{\frac{1}{2}+\epsilon} N^{2+\epsilon} e^{-C(\log T)^2}\right) \text{ (say)} \qquad (4.1)
\end{aligned}$$

by moving the line of integration to $\Re w = -1 + 2\epsilon_1$. Now

$$\begin{aligned}
I &= \frac{1}{2\pi i} \int_{\substack{\Re w = -1+2\epsilon_1, \\ |v| \le (\log T)^2}} L_f(s_1+w) Y^w \Gamma(w) dw \\
&= \frac{1}{2\pi i} \int_{\substack{\Re w = -1+2\epsilon_1, \\ |v| \le (\log T)^2}} \chi(s_1+w) \left\{Q_1 + Q_2\right\} Y^w \Gamma(w) dw \\
&= I_1 + I_2, \qquad (4.2)
\end{aligned}$$

where

$$Q_1 = \sum_{n \le Y_1} a_f(n) n^{s_1+w-k} \ ; \ Q_2 = \sum_{n > Y_1} a_f(n) n^{s_1+w-k}.$$

We note that in I_2, $\Re(s_1+w) = \frac{k}{2} - 1 + 2\epsilon_1$. We have

$$\begin{aligned}
\sum_{n > Y_1} a_f(n) n^{s_1+w-k} &= \sum_{Y_1 < n \le Y_1^{10}} a_f(n) n^{s_1+w-k} \\
&+ O\left(\sum_{n > Y_1^{10}} d(n) n^{\frac{k-1}{2} + \frac{k}{2} - 1 + 2\epsilon_1 - k}\right) \\
&= \sum_{Y_1 < n \le Y_1^{10}} a_f(n) n^{s_1+w-k} + O\left(Y_1^{-5+20\epsilon_1}\right). \qquad (4.3)
\end{aligned}$$

Note that we have used the inequality (2.10). Using Hölder's inequality and a theorem of Montgomery and Vaughan (see [2]), we get

$$\int_T^{2T} |I_2|^2 dt \ll \int_T^{2T} \left| \int_{\substack{\Re w = -1+2\epsilon_1, \\ |v| \le (\log T)^2}} \chi(s_1+w) \{ \sum_{Y_1^{10} \ge n > Y_1} a_f(n) n^{s_1+w-k} \} Y^w \Gamma(w) dw \right|^2 dt$$

$$
\begin{aligned}
&+ \left(\frac{2\pi}{N}\right)^{-4+4\epsilon_1} T^{5+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1} \\
&\ll (\log T)^2 \left(\frac{2\pi}{N}\right)^{-4+4\epsilon_1} \frac{T^{4+20\epsilon_1}}{Y^2} \int_{T-(\log T)^2}^{2T+(\log T)^2} \sum_U \left| \sum_{(U,2U)} a_f(n) n^{s_1+\Re w-k} \right|^2 dt \\
&+ \left(\frac{2\pi}{N}\right)^{-4+4\epsilon_1} T^{5+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1} \\
&\ll (\log T)^2 \left(\frac{2\pi}{N}\right)^{-4+4\epsilon_1} \frac{T^{4+20\epsilon_1}}{Y^2} \sum_U \sum_{(U\le n\le 2U)} \frac{|a_f(n)|^2}{n^{2(2k-1-(k-\frac{3}{2}+2\epsilon_1))}} n \\
&+ \left(\frac{2\pi}{N}\right)^{-4+4\epsilon_1} T^{5+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1} \\
&\ll (\log T)^2 \left(\frac{2\pi}{N}\right)^{-4+4\epsilon_1} \frac{T^{4+20\epsilon_1}}{Y^2} \sum_U \sum_{(U\le n\le 2U)} \frac{(d(n))^2 n^{k-1} n}{n^{2(k-(\frac{k}{2}-1+2\epsilon_1))}} \\
&+ \left(\frac{2\pi}{N}\right)^{-4+4\epsilon_1} T^{5+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1} \\
&\ll (\log T)^2 \left(\frac{2\pi}{N}\right)^{-4+4\epsilon_1} \frac{T^{4+20\epsilon_1}}{Y^2} \sum_{j=0}^{C\log Y_1} (\log Y_1)^3 (2^j Y_1)^{-1+4\epsilon_1} \\
&+ \left(\frac{2\pi}{N}\right)^{-4+4\epsilon_1} T^{5+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1} \\
&\ll (\log T)^2 N^{4-4\epsilon_1} T^{4+20\epsilon_1} Y^{-2} Y_1^{-1+4\epsilon_1} (\log Y_1)^3 \\
&+ N^{4-4\epsilon_1} T^{5+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1}. \qquad (4.4)
\end{aligned}
$$

Also we have

$$
\begin{aligned}
I_1 &= \frac{1}{2\pi i} \int_{\substack{\Re w=-1+2\epsilon_1,\\ |v|\le(\log T)^2}} \chi(s_1+w) \left\{ \sum_{n\le Y_1} a_f(n) n^{s_1+w-k} \right\} Y^w \Gamma(w) dw \\
&= \frac{1}{2\pi i} \int_{\substack{\Re w=-\frac{1}{2},\\ |v|\le(\log T)^2}} \chi(s_1+w) \left\{ \sum_{n\le Y_1} a_f(n) n^{s_1+w-k} \right\} Y^w \Gamma(w) dw \\
&+ O\left((NT)^2 (\log Y_1)^2 Y^{-\frac{1}{2}} e^{-C\log T)^2} \right) \qquad (4.5)
\end{aligned}
$$

by moving the line of integration to $\Re w = -\frac{1}{2}$. Notice that, if $-1+2\epsilon_1 \le u \le -\frac{1}{2}$, then $\frac{k}{2}-1+2\epsilon_1 \le \Re(s_1+w) \le \frac{k}{2}-\frac{1}{2}$ and $-1-\frac{k}{2}+2\epsilon_1 \le \Re(s_1+w-k) \le -\frac{k}{2}-\frac{1}{2}$. Therefore, for $T\le t\le 2T$, we have

$$
|\chi(s_1+w)| \ll \left(\frac{NT}{2\pi}\right)^{k-2\left(2\epsilon_1+\frac{k}{2}-1\right)} \ll (NT)^2 .
$$

Also, we have

$$
\sum_{n\le Y_1} a_f(n) n^{s_1+w-k} \ll \sum_{n\le Y_1} d(n) n^{\frac{k-1}{2}+\frac{k-1}{2}-k} \ll_k (\log Y_1)^2 ,
$$

$$Y^w \ll Y^{-\frac{1}{2}} \text{ and } \Gamma(\mathrm{w}) \ll \mathrm{e}^{-\mathrm{C}(\log \mathrm{T})^2}$$

on horizontal lines. Again using Montgomery- Vaughan Theorem and the inequality (2.10), we obtain

$$\begin{aligned}
\int_T^{2T} |I_1|^2 dt &\ll (\log T)^2 \left(\frac{2\pi}{N}\right)^{-2} \frac{T^2}{Y} \int_{T-(\log T)^2}^{2T+(\log T)^2} \left| \sum_{n \le Y_1} a_f(n) n^{s_1 + \Re w - k} \right|^2 dt \\
&+ (NT)^4 (\log Y_1)^4 Y^{-1} e^{-C(\log T)^2} \\
&\ll (\log T)^2 N^2 \frac{T^2}{Y} \sum_{n \le Y_1} \frac{|a_f(n)|^2 n}{n^{2(\frac{k}{2}+\frac{1}{2})}} \\
&+ (NT)^4 (\log Y_1)^4 Y^{-1} e^{-C(\log T)^2} \\
&\ll (\log T)^2 \frac{T^3}{Y} \sum_{n \le Y_1} \frac{(d(n))^2 n^k}{n^{k+1}} \\
&+ (NT)^4 (\log Y_1)^4 Y^{-1} e^{-C(\log T)^2} \\
&\ll (\log T)^2 N^2 \frac{T^2}{Y} (\log Y_1)^3 \\
&+ (NT)^4 (\log Y_1)^4 Y^{-1} e^{-C(\log T)^2}. \qquad (4.6)
\end{aligned}$$

Now, we have

$$S = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^{s_1}} e^{-n/Y}.$$

$$\begin{aligned}
\int_T^{2T} |S|^2 dt &= \sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^k} e^{-\frac{2n}{Y}} (T + O(n)) \\
&= T \sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^k} e^{-\frac{2n}{Y}} \\
&+ O\left(\sum_{n \le Y/2} \frac{|a_f(n)|^2}{n^k} n + \sum_{n \ge Y/2} \frac{|a_f(n)|^2}{n^k} n \left(\frac{Y}{n}\right)^{2\alpha+2} \right) \\
&= T \sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^k} e^{-\frac{2n}{Y}} + O\left(Y (\log Y)^3\right) \qquad (4.7)
\end{aligned}$$

with a suitable α.

From (4.1) to (4.7), we obtain,

$$\int_T^{2T} |L_f(s_1)|^2 dt = T \sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^k} e^{-\frac{2n}{Y}}$$

$$
\begin{aligned}
&+ \; O\left((\log T)^2 N^{4-4\epsilon_1} Y^{-2} T^4 Y_1^{-1+4\epsilon_1} (\log Y_1)^3\right) \\
&+ \; O\left(N^{4-4\epsilon_1} T^5 Y^{-2} Y_1^{-10+40\epsilon_1}\right) + O\left(N^2 Y^{-1} T^2 (\log Y_1)^3 (\log T)^2\right) \\
&+ \; O\left(Y (\log Y)^3\right) + O\left(Y^{1+2\epsilon} N^{4+2\epsilon} e^{-C(\log T)^2}\right) \\
&+ \; O\left(N^4 (\log Y_1)^4 Y^{-1} e^{-C(\log T)^2}\right) \\
&= \; T \sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^k} e^{-\frac{2n}{Y}} + O\left(E_1 + E_2 + E_3 + E_4 + E_5 + E_6\right) \quad \text{say,}
\end{aligned} \tag{4.8}
$$

where E_i are with obvious notion. It is clear from lemma 3.1 that

$$
\sum_f{}' \frac{1}{< f, f >} E_i \ll E_i \quad (i = 1, 2, 3, 4, 5, 6). \tag{4.9}
$$

Therefore, we get

$$
\begin{aligned}
\sum_f{}' \frac{1}{< f, f >} \int_T^{2T} |L_f(s_1)|^2 \, dt \qquad &= T \sum_f{}' \tfrac{1}{<f,f>} \textstyle\sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^k} e^{-\frac{2n}{Y}} \\
&+ O\left(E_1 + E_2 + E_3 + E_4 + E_5 + E_6\right).
\end{aligned} \tag{4.10}
$$

We observe that

$$
\begin{aligned}
S_1 \quad &=: \; \sum_f{}' \frac{1}{< f, f >} \sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^k} e^{-\frac{2n}{Y}} \\
&= \; \sum_{n=1}^{\infty} \frac{e^{-\frac{2n}{Y}}}{n} \sum_f{}' \frac{|a_f(n)|^2}{< f, f > n^{k-1}} \\
&\ll \; \sum_{n=1}^{\infty} \frac{e^{-\frac{2n}{Y}} (d(n))^2}{n} \sum_f{}' \frac{1}{< f, f >} \quad (\text{by}(2.10)) \\
&\ll_k \; \sum_{n=1}^{\infty} \frac{e^{-\frac{2n}{Y}} (d(n))^2}{n} \quad (\text{from lemma 3.1}) \\
&\ll_k \; \sum_{n \le \frac{Y}{2}} \frac{(d(n))^2}{n} \left(1 + O\left(\frac{n}{Y}\right)\right) + \sum_{n \le \frac{Y}{2}} \frac{(d(n))^2}{n} \frac{Y}{n} \\
&\ll_k \; (\log Y)^3.
\end{aligned} \tag{4.11}
$$

Hence, we obtain

$$
\begin{aligned}
\sum_f{}' \frac{1}{< f, f >} \int_T^{2T} |L_f(s_1)|^2 \, dt \qquad &\ll_k T (\log Y)^3 \\
&+ E_1 + E_2 + E_3 + E_4 + E_5 + E_6.
\end{aligned} \tag{4.12}
$$

First, we choose Y such that E_3 and E_4 to be of the same order. This suggests us to choose $Y = NT$. This means that

$$
\max(E_3, E_4) \ll NT \left\{ (\log Y_1)^3 (\log T)^2 + (\log Y)^3 \right\}.
$$

Now, we are forced to choose $Y_1 = NT$ and we notice that

$$E_1 \ll NT\left(\log Y_1\right)^3\left(\log T\right)^2.$$

Hence, for this choice $Y = Y_1 = NT$, we get

$$\begin{aligned} \sum_f{}' \frac{1}{< f, f >} \int_T^{2T} \left|L_f\left(s_1\right)\right|^2 dt \quad &\ll_k T(\log(NT))^3 + NT(\log(NT))^3(\log T)^2 \\ &+ N^{5+10\epsilon} e^{-C(\log T)^2} + N^3(\log(NT))^4 e^{-C(\log T)^2}, \end{aligned} \tag{4.13}$$

which proves the theorem, since, on the r.h.s of (4.13), the first and the last terms are dominated by the other two terms.

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(Accepted on 03-09-2003)