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HAL Id: hal-01109899
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Submitted on 27 Jan 2015

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HARDY-LITTLEWOOD FIRST APPROXIMATION THEOREM FOR QUASI L-FUNCTIONS

BY
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(DEDICATED WITH DEEPEST REGARDS TO THE MEMORY OF LEONARD EULER)

(accepted on 12-06-2005)

§1. INTRODUCTION. We begin by stating the following theorem due to G.H. HARDY and J.E. LITTLEWOOD (see Theorem 4.11 of [ECT]).

**THEOREM 1.** We have

\[ \zeta(s) = \sum_{n \leq x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \]  

uniformly for \( \sigma \geq \sigma_0 > 0, |t| \leq \pi C^{-1} \) which \( C > 1 \) is any given constant.

**REMARK.** We call (1) as the first approximation theorem of HARDY and LITTLEWOOD. In [RB, KR] we proved (1) with the conditions \( t > 2 \) and \( x > (\frac{1}{2} + \delta)t \), where \( \delta > 0 \) is any constant.

The object of the present note (which is an addendum to [RB, KR]) is to make a few remarks on the results of [RB, KR] and prove the following theorem.

**THEOREM 2.** We define (quasi L-functions) by

\[ L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)(n+\alpha)^{-s}F(n+\alpha) \]  

where \( \chi(n) \) is any periodic sequence of complex numbers (not all zero) whose sum over any period is zero. \( \alpha > 0 \) is any constant and \( F(X) \) is any complex valued function of \( X \) which is infinitely often continuously differentiable in \( X \geq 1 \) with \( F^{(k)}(X) = O(X^{-k+\epsilon}) \) for every \( k \geq 0 \) and every \( \epsilon > 0. \) As usual \( s = \sigma + it, \ \sigma \geq \sigma_0 > 0. \) Then \( L(s, \chi) \) defined above is uniformly convergent in any compact sub-set. Moreover \( L(s, \chi) \) can be continued as an entire
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function and

\[ L(s, \chi) = \sum_{n \leq x} \chi(n)(n + \alpha)^{-s}F(n + \alpha) + E \]  \hspace{1cm} (3)

where

\[ |E| \leq C_{\epsilon}x^{-\sigma+\epsilon}, \sigma \geq -K, \]  \hspace{1cm} (4)

provided only that \( x \geq x_0 = D_\epsilon(\epsilon + 2)^{1+\epsilon}, D_\epsilon > 0 \) being a certain constant depending on \( K(> 0) \) and \( \epsilon \) (and of course on constants depending upon the quasi \( L \) function). In (4) \( C_\epsilon > 0 \) depends only on \( \epsilon(> 0) \) and \( K \) which are arbitrary.

REMARK 1. In (2) and (3) we have written \( F(n + \alpha) \) for some convenience. Certainly we can write \( F(n) \) instead.

REMARK 2. The proof that certain functions are entire (see [RB, KR]2) is a special case of theorem 2 obtained by treating \( \sum_{n=1}^{\infty}(-1)^n \int_0^1 f'(n + x)dx \).

REMARK 3. Examples of \( F(n) \) are \( \exp(\sqrt{\log n}) \) and so on.

§2. PROOF OF THEOREM 2. We begin by stating Theorem 2 of [RB, KR]1 as a lemma.

LEMMA. Let \( a \) and \( b \) integers with \( a < b, k \) a non-negative integer and \( f(x) \), a function of \( x \) which is \( k \)-times continuously differentiable in \( a \leq x \leq b - 1 + k \). Then

\[
\sum_{a \leq n < b} f(n) = \int_a^b f(x)dx + \left\{ -\frac{1}{2} \int_0^1 \int_0^1 (f(b + u_1 \nu_1^2) - f(a + u_1 \nu_1^2))du_1d\nu_1 \\
+ \frac{1}{2^2} \int_0^1 \int_0^1 \int_0^1 (f'(b + u_1 \nu_1^2 + u_2 \nu_2^2) - f'(a + u_1 \nu_1^2 + u_2 \nu_2^2))du_1d\nu_1du_2d\nu_2 \\
\ldots + \frac{(-1)^{k-1}}{2^{k-1}} \int_0^1 \ldots \int_0^1 (f^{(k-2)}(b + u_1 \nu_1^2 + \ldots + u_k \nu_k^2) - f^{(k-2)}(a + u_1 \nu_1^2 + \ldots + u_k \nu_k^2))du_1d\nu_1 \ldots du_{k-1}d\nu_{k-1} \\
- f^{(k-2)}(a + u_1 \nu_1^2 + \ldots + u_k \nu_k^2))du_1d\nu_1 \ldots du_{k-1}d\nu_{k-1} \right\} \\
+ \frac{(-1)^k}{2^k} \int_0^1 \ldots \int_0^1 \sum_{a \leq n < b} f^{(k)}(n + u_1 \nu_1^2 + \ldots + u_k \nu_k^2)du_1d\nu_1 \ldots du_kd\nu_k.
\]

REMARK 1. The terms in the curly brackets are absent if \( k = 1 \).

REMARK 2. We have corrected the following. In Theorem 2 of [RB, KR]1 \( a < n \leq b \) is not correct. It should read \( a \leq n < b \).
REMARK 3. Actually the proof of the lemma is simple and runs as follows.

\[ \sum_{a \leq n < b} f(n) = \sum_{a \leq n < b} (f(n) - \int_{n}^{n+1} f(u) du) + \int_{n}^{b} f(u) du \]

and here the first term on the RHS is

\[ \sum_{a \leq n < b} \int_{0}^{1} (f(n) - f(n + u)) du = - \sum_{a \leq n < b} \int_{0}^{1} \int_{0}^{1} u f'(n + u) dudv \]

\[ = - \frac{1}{2} \sum_{a \leq n < b} \int_{0}^{1} \int_{0}^{1} f'(n + uv^2) dudv \]

(by a change of variable). This gives the case \( k = 1 \) of the lemma. The lemma follows by a \( k \)-fold application of this case.

We now resume the proof of theorem 2. We define \( \chi(0) \) to be \( \chi(p) \) where \( p \) is the length of the period. Let \( X \) be any positive integer. We have

\[ \sum_{n \geq Xp} \chi(n)(n + \alpha)^{-s} F(n + \alpha) \]

\[ = \sum_{\nu=0}^{p-1} \chi(\nu) \sum_{n \equiv (mod \ p), n \geq Xp} (n + \alpha)^{-s} F(n + \alpha) \]

\[ = \sum_{\nu=0}^{p-1} \chi(\nu) \left\{ \sum_{n \equiv (mod \ p), n \geq Xp} (n + \alpha)^{-s} F(n + \alpha) - \sum_{n \equiv 0 (mod \ p), n \geq Xp} (n + \alpha)^{-s} F(n + \alpha) \right\} \]

\[ = \sum_{\nu=0}^{p-1} \chi(\nu) \left\{ \sum_{l \geq X} (lp + \nu + \alpha)^{-s} F(lp + \nu + \alpha) - \sum_{l \geq X} (lp + \alpha)^{-s} F(lp + \alpha) \right\} \]

\[ = \sum_{\nu=0}^{p-1} \chi(\nu) \left\{ \int_{X}^{\infty} (xp + \nu + \alpha)^{-s} F(xp + \nu + \alpha) dx - \int_{X}^{\infty} (xp + \alpha)^{-s} F(xp + \alpha) dx \right\} \]

\[ + \frac{1}{2} \frac{d}{dx} \left( (xp + \nu + \alpha)^{-s} F(xp + \nu + \alpha) - (xp + \alpha)^{-s} F(xp + \alpha) \right)_{x=X} \]

\[ - \frac{1}{2^2} \frac{d^2}{dx^2} \left( (xp + \nu + \alpha)^{-s} F(xp + \nu + \alpha) - (xp + \alpha)^{-s} F(xp + \alpha) \right)_{x=X} \]

\[ + \ldots \]
\[-\frac{(\lambda - 1)^{k-2}}{2^{k-2}} \frac{d^{k-2}}{dx^{k-2}} \left( (xp + \nu + \alpha)^{-s} F(xp + \nu + \alpha) - (xp + \alpha)^{-s} F(xp + \alpha) \right)_{x=X} \]
\[+ \frac{(\lambda - 1)^{k}}{2^k} \int_0^1 \ldots \int_0^1 \left\{ \frac{d^k}{dx^k} \left( \sum_{n \geq 1} ((X + x + n + \alpha + y + \nu)^{-s} F(X + x + n + \alpha + y + \nu)) \right) \right\} \text{d}u_1 \text{d}u_2 \ldots \text{d}u_k \text{d}u_k \]  
\[= O(X^{-\sigma + \epsilon}). \]

This follows by choosing \( f(x) \) (in the lemma) suitably and letting \( b \to \infty \) (provided \( \sigma \geq 2 \)). Thus we get analytic continuation in \( \sigma > 0 \). By choosing \( k = 1, 2, 3, \ldots, [K] + 10 \) (successively) we see that \( L(s, \chi) \) is entire (since \( K \) is arbitrary). We have
\[
\int_X^\infty (xp + \nu + \alpha)^{-s} F(xp + \nu + \alpha) \text{d}x = \int_{Xp+\nu+\alpha}^\infty W^{-s} F(W) \frac{dW}{p}
\]
(by putting \( xp + \nu + \alpha = W \)) and
\[
\int_X^\infty (xp + \alpha)^{-s} F(xp + \alpha) \text{d}x = \int_{Xp+\alpha}^\infty W^{-s} F(W) \frac{dW}{p}.
\]
Hence their difference is entire and is \( O(X^{-\sigma + \epsilon}) \).

In (5) all terms except the last are entire and are \( O(X^{-\sigma + \epsilon}) \). We give some details. For \( k \geq l \geq 0 \) and \( x \geq A \geq 10 \) and \( \beta(>0) \) bounded above by \( B \) we have
\[
\frac{d^l}{dx^l} \left( (x + \beta)^{-s} f(x) \right) \quad (= O(x^{-\sigma + \epsilon}) \text{ if } l = 0)
\]
and if \( l \geq 1 \) it is
\[
= O \left( \left| s \right| \left| s + 1 \right| \ldots \left| s + l - 1 \right| \frac{\left| f(x) \right|}{x^\sigma + l} + l \left| s \right| \left| s + 1 \right| \ldots \left| s + l - 2 \right| \frac{\left| f'(x) \right|}{x^\sigma + l - 1} \right)
\]
\[+ \frac{l(l - 1)}{2!} \left| s \right| \left| s + 1 \right| \ldots \left| s + l - 3 \right| \frac{\left| f''(x) \right|}{x^\sigma + l - 2} + \ldots \text{to } l + 1 \text{ terms}
\]
\[
= O \left( 2^l (l + 1) \max_{1 \leq j \leq l + 1} \left| s \right| \left( s + 1 \right) \ldots \left( s + l - j \right) \frac{\left| f^{(j-1)}(x) \right|}{x^\sigma + l - j + 1} \right)
\]
\[
= O \left( 2^l (l + 1) \max_{1 \leq j \leq l + 1} \left\{ \frac{\left| s \right| + l - j + 1}{x^l - j + 1} \right\} \right)
\]
\[= O \left( 2^l (l + 1) \max_{1 \leq j \leq l + 1} \left\{ \frac{\left| s \right| + \left| l \right| + k}{x^l} \right\} \right)
\]
\(= O(x^{-\sigma+\epsilon}) \text{ if } x \geq 4(|\sigma| + |t| + k).\)

On the other hand the last but one inequality shows us that the last multiple integral in (5) is

\[
O \left( \sum_{n \geq 1} 2^k (k + 1) \max_{1 \leq j \leq k+1} \left\{ \frac{|\sigma| + |t| + k}{X + n} \right\}^{k} (X + n)^{-\sigma+\epsilon} \right)
\]

\[
= O \left( 2^k (k + 1) \left( \frac{|\sigma| + |t| + k}{X} \right)^{k} X^{-\sigma+\epsilon+1} \right)
\]

\[
= O \left( X^{-\sigma+\epsilon} \right)
\]

provided \(k + \sigma \geq 3\) and \(X \geq (|\sigma| + |t| + k + 200)^{1+\epsilon}\) and also \(k = \left\lfloor \frac{100}{\epsilon} \right\rfloor\). This completes the proof of Theorem 2.
ACKNOWLEDGMENTS.

The authors are thankful to Professor ROGER HEATH-BROWN, FRS, for constant encouragement. They are thankful to Professor P.G.VAIDYA and an anamnious referee for their interest in this work. Finally they are thankful to Smt. J.N.SANDHYA for technical assistance.

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P.S. Sharper results are known for \( \zeta(s) \) and \( L(s, \chi) \) (\( \chi \) a character mod \( k \)). See the booklet 'RIEMANN ZETA-FUNCTION' published by Ramanujan Institute, Chennai (1979).