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HARDY-LITTLEWOOD FIRST APPROXIMATION THEOREM FOR QUASI L-FUNCTIONS

BY

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(DEDICATED WITH DEEPEST REGARDS TO
THE MEMORY OF LEONARD EULER)

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§1. **INTRODUCTION**. We begin by stating the following theorem due to G.H.HARDY and J.E.LITTLEWOOD (see Theorem 4.11 of [ECT]).

THEOREM 1. *We have*

$$\zeta(s) = \sum_{n\leq x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \tag{1}$$

uniformly for $\sigma \geq \sigma_0 > 0, |t| \leq \pi x C^{-1}$ *which* $C > 1$ *is any given constant.*

REMARK. We call (1) as the first approximation theorem of HARDY and LITTLEWOOD. In $[\text{RB, KR}]_1$ we proved (1) with the conditions $t > 2$ and $x > (\frac{1}{2} + \delta)t$, where $\delta > 0$ is any constant.

The object of the present note (which is an addendum to $[\text{RB, KR}]_1$) is to make a few remarks on the results of $[\text{RB, KR}]_1$ and prove the following theorem.

THEOREM 2. *We define (quasi L-functions) by*

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)(n+\alpha)^{-s}F(n+\alpha) \tag{2}$$

where $\chi(n)$ *is any periodic sequence of complex numbers (not all zero) whose sum over any period is zero.* $\alpha > 0$ *is any constant and* $F(X)$ *is any complex valued function of* X *which is infinitely often continuously differentiable in* $X \geq 1$ *with* $F^{(k)}(X) = O(X^{-k+\epsilon})$ *for every* $k \geq 0$ *and every* $\epsilon > 0$*. As usual* $s = \sigma + it$, $\sigma \geq \sigma_0 > 0$*. Then* $L(s,\chi)$ *defined above is uniformly convergent in any compact sub-set. Moreover* $L(s,\chi)$ *can be continued as an entire* *function and*

$$L(s,\chi)=\sum_{n\le x}\chi(n)(n+\alpha)^{-s}F(n+\alpha)+E \tag{3}$$

where

$$|E|\le C_\epsilon x^{-\sigma+\epsilon}, \sigma\ge -K, \tag{4}$$

provided only that $x\ge x_0=D_\epsilon(|t|+2)^{1+\epsilon}, D_\epsilon>0$ *being a certain constant depending on* $K(>0)$ *and* ϵ *(and of course on constants depending upon the quasi L function). In* $(4) C_\epsilon>0$ *depends only on* $\epsilon(>0)$ *and* K *which are arbitrary.*

REMARK 1. In (2) and (3) we have written $F(n+\alpha)$ for some convenience. Certainly we can write $F(n)$ instead.

REMARK 2. The proof that certain functions are entire (see [RB, KR]$_2$) is a special case of theorem 2 obtained by treating $\sum_{n=1}^{\infty}(-1)^n \int_0^1 f'(n+x)dx$.

REMARK 3. Examples of $F(n)$ are $exp(\sqrt{\log n})$ and so on.

§2. **PROOF OF THEOREM 2.** We begin by stating Theorem 2 of [RB, KR]$_1$ as a lemma.

LEMMA. Let a and b integers with $a<b, k$ a non-negative integer and $f(x)$, a function of x which is $k-$times continuously differentiable in $a\le x\le b-1+k$. Then

$$\sum_{a\le n<b} f(n)=\int_a^b f(x)dx+\left\{-\frac{1}{2}\int_0^1\int_0^1 (f(b+u_1\ \nu_1^{\frac{1}{2}})-f(a+u_1\nu_1^{\frac{1}{2}}))du_1d\nu_1\right.$$

$$+\frac{1}{2^2}\int_0^1\int_0^1\int_0^1\int_0^1(f'(b+u_1\nu_1^{\frac{1}{2}}+u_2\ \nu_2^{\frac{1}{2}})-f'(a+u_1\nu_1^{\frac{1}{2}}+u_2\nu_2^{\frac{1}{2}}))du_1d\nu_1du_2d\nu_2$$

$$\ldots+\frac{(-1)^{k-1}}{2^{k-1}}\int_0^1\ldots\int_0^1 (f^{(k-2)}(b+u_1\nu_1^{\frac{1}{2}}+\ldots+u_{k-1}\ \nu_{k-1}^{\frac{1}{2}})$$

$$\left.-f^{(k-2)}(a+u_1\nu_1^{\frac{1}{2}}+\ldots+u_{k-1}\nu_{k-1}^{\frac{1}{2}}))du_1d\nu_1\ldots du_{k-1}d\nu_{k-1}\right\}$$

$$+\frac{(-1)^k}{2^k}\int_0^1\ldots\int_0^1\sum_{a\le n<b} f^{(k)}(n+u_1\ \nu_1^{\frac{1}{2}}+\ldots+u_k\ \nu_k^{\frac{1}{2}})du_1d\nu_1\ldots du_kd\nu_k.$$

REMARK 1. The terms in the curly brackets are absent if $k=1$.

REMARK 2. We have corrected the following. In Theorem 2 of [RB, KR]$_1$ $a<n\le b$ is not correct. It should read $a\le n<b$.

REMARK 3. Actually the proof of the lemma is simple and runs as follows.

$$\sum_{a\le n<b} f(n) = \sum_{a\le n<b} (f(n) - \int_n^{n+1} f(u)du) + \int_a^b f(u)du$$

and here the first term on the RHS is

$$\sum_{a\le n<b} \int_0^1 (f(n) - f(n+u))du = - \sum_{a\le n<b} \int_0^1 \int_0^u f'(n+v)dvdu$$
$$= - \sum_{a\le n<b} \int_0^1 \int_0^1 uf'(n+uv)dudv$$
$$= -\frac{1}{2} \sum_{a\le n<b} \int_0^1 \int_0^1 f'(n+uv^{\frac{1}{2}})dudv$$

(by a change of variable). This gives the case $k = 1$ of the lemma. The lemma follows by a k-fold application of this case.

We now resume the proof of theorem 2. We define $\chi(0)$ to be $\chi(p)$ where p is the length of the period. Let X be any positive integer. We have

$$\sum_{n\ge Xp} \chi(n)(n+\alpha)^{-s} F(n+\alpha)$$
$$= \sum_{\nu=0}^{p-1} \chi(\nu) \sum_{n\equiv\nu(mod\ p), n\ge Xp} (n+\alpha)^{-s} F(n+\alpha)$$
$$= \sum_{\nu=0}^{p-1} \chi(\nu) \left\{ \sum_{n\equiv\nu(mod\ p), n\ge Xp} (n+\alpha)^{-s} F(n+\alpha) - \sum_{n\equiv 0(mod\ p), n\ge Xp} (n+\alpha)^{-s} F(n+\alpha) \right\}$$
$$= \sum_{\nu=0}^{p-1} \chi(\nu) \left\{ \sum_{lp+\nu\ge Xp} (lp+\nu+\alpha)^{-s} F(lp+\nu+\alpha) - \sum_{l\ge X} (lp+\alpha)^{-s} F(lp+\alpha) \right\}$$
$$= \sum_{\nu=0}^{p-1} \chi(\nu) \left\{ \sum_{l\ge X} (lp+\nu+\alpha)^{-s} F(lp+\nu+\alpha) - \sum_{l\ge X} (lp+\alpha)^{-s} F(lp+\alpha) \right\}$$
$$= \sum_{\nu=0}^{p-1} \chi(\nu) \left[\left\{ \int_X^\infty (xp+\nu+\alpha)^{-s} F(xp+\nu+\alpha)dx - \int_X^\infty (xp+\alpha)^{-s} F(xp+\alpha)dx \right\} \right.$$
$$+\frac{1}{2} \frac{d}{dx} \Big((xp+\nu+\alpha)^{-s} F(xp+\nu+\alpha) - (xp+\alpha)^{-s} F(xp+\alpha) \Big)_{x=X}$$
$$-\frac{1}{2^2} \frac{d^2}{dx^2} \Big((xp+\nu+\alpha)^{-s} F(xp+\nu+\alpha) - (xp+\alpha)^{-s} F(xp+\alpha) \Big)_{x=X}$$
$$+ \dots$$

$$-\frac{(-1)^{k-2}}{2^{k-2}}\ \frac{d^{(k-2)}}{dx^{k-2}}\ \Big((xp+\nu+\alpha)^{-s}\ F(xp+\nu+\alpha)-(xp+\alpha)^{-s}F(xp+\alpha)\Big)_{x=X}$$

$$+\frac{(-)^k}{2^k}\ \int_0^1\ldots\int_0^1\left\{\frac{d^k}{dx^k}\left(\sum_{n\geq 1}((X+x+n+\alpha+y+\nu)^{-s}F(X+x+n+\alpha+y+\nu))\right.\right.$$

$$\left.\left.-\sum_{n\geq 1}\Big(\big(X+x+n+\alpha+y)^{-s}F(X+x+n+\alpha+y)\big)\Big)\right\}_{x=0}du_1d\nu_1\ldots du_kd\nu_k\Bigg] \qquad (5)$$

where $y=u_1\nu_1^{\frac{1}{2}}+\ldots+u_k\ \nu_k^{\frac{1}{2}}$.

This follows by choosing $f(x)$ (in the lemma) suitably and letting $b\to\infty$ (provided $\sigma\geq 2$). Thus we get analytic continuation in $\sigma>0$. By choosing $k=1,2,3,\ldots,[K]+10$ (successively) we see that $L(s,\chi)$ is entire (since K is arbitrary). We have

$$\int_X^\infty(xp+\nu+\alpha)^{-s}F(xp+\nu+\alpha)dx=\int_{Xp+\nu+\alpha}^\infty\ W^{-s}F(W)\ \frac{dW}{p}$$

(by putting $xp+\nu+\alpha=W$) and

$$\int_X^\infty(xp+\alpha)^{-s}\ F(xp+\alpha)dx=\int_{Xp+\alpha}^\infty\ W^{-s}F(W)\frac{dW}{p}.$$

Hence their difference is entire and is $O(X^{-\sigma+\epsilon})$.

In (5) all terms except the last are entire and are $O(X^{-\sigma+\epsilon})$. We give some details. For $k\geq l\geq 0$ and $x\geq A\geq 10$ and $\beta(>0)$ bounded above by B we have

$$\frac{d^l}{dx^l}\left((x+\beta)^{-s}f(x)\right)\ \left(=O(x^{-\sigma+\epsilon})\text{ if }l=0\right)$$

and if $l\geq 1$ it is

$$\begin{aligned}
&= O\left(\frac{|s|(|s|+1)\ldots(|s|+l-1)}{x^{\sigma+l}}|f(x)|+l\frac{|s|(|s|+1)\ldots(|s|+l-2)}{x^{\sigma+l-1}}|f'(x)|\right.\\
&+\ \frac{l(l-1)}{2!}\ \frac{|s|(|s|+1)\ldots(|s|+l-3)}{x^{\sigma+l-2}}\ |f''(x)|+\ldots to\ l+1\text{ terms})\\
&= O\left(2^l(l+1)\ max_{1\leq j\leq l+1}\ \frac{|s|(s)+1)\ldots(|s|+l-j)}{x^{\sigma+l-j+1}}\ |f^{(j-1)}(x)|\right)\\
&= O\left(2^l(l+1)\ max_{1\leq j\leq l+1}\ \left\{\frac{(|s|+l-j+1)^{l-j+1}}{x^{l-j+1}}\ x^{-\sigma}\ x^{-(j-1)+\epsilon}\right\}\right)\\
&= O\left(2^l(l+1)\ max_{1\leq j\leq l+1}\left\{\frac{(|\sigma|+|t|+k)^l}{x^l}\ x^{-\sigma+\epsilon}\right\}\right)
\end{aligned}$$

$$= \quad O(x^{-\sigma+\epsilon}) \text{ if } \quad x \geq 4(|\sigma| + |\mathrm{t}| + \mathrm{k}).$$

On the other hand the last but one inequality shows us that the last multiple integral in (5) is

$$\begin{aligned} & O \quad \left(\sum_{n\geq 1} 2^k (k+1) \max_{1\leq j\leq k+1} \left\{ \frac{(|\sigma| + |t| + k)^k}{(X+n)^k} \; (X+n)^{-\sigma+\epsilon} \right\} \right) \\ = & \quad O \left(2^k (k+1) \left(\frac{|\sigma| + |t| + k}{X} \right)^k \; X^{-\sigma+\epsilon+1} \right) \\ = & \quad O \left(X^{-\sigma+\epsilon} \right) \end{aligned}$$

provided $k + \sigma \geq 3$ and $X \geq (|\sigma| + |t| + k + 200)^{1+\epsilon}$ and also $k = \left[\frac{100}{\epsilon} \right]$. This completes the proof of Theorem 2.

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P.S. Sharper results are known for $\zeta(s)$ and $L(s,\chi)$ ($\chi - a$ character *mod* k). See the booklet 'RIEMANN ZETA-FUNCTION' published by Ramanujan Institute, Chennai (1979).