Hardy-Ramanujan Journal
Vol.29 (2006) 18-36

On the periodic Hurwitz zeta-function

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Abstract. In the paper an universality theorem in the Voronin sense for the periodic Hurwitz zeta-function is proved.

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, non-negative integers, real and complex numbers, respectively, and let $\mathfrak{a} = \{a_m, m \in \mathbb{Z}\}$ be a periodic sequence of complex numbers with period $k \geq 1$. Denote by $s = \sigma + it$ a complex variable. The periodic zeta-function $\zeta(s; \mathfrak{a})$, for $\sigma > 1$, is defined by

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and by analytic continuation elsewhere. Define

$$a = \frac{1}{k} \sum_{m=0}^{k-1} a_m.$$

If $a = 0$, then $\zeta(s; \mathfrak{a})$ is an entire function, while if $a \neq 0$, then the point $s = 1$ is a simple pole of $\zeta(s; \mathfrak{a})$ with residue a. The function $\zeta(s; \mathfrak{a})$ satisfies the functional equation [22]

$$\zeta(1-s; \mathfrak{a}_\pm) = \left(\frac{k}{2\pi}\right)^s \frac{\Gamma(s)}{\sqrt{k}} \left(\exp\left\{\frac{\pi i s}{2}\right\} \zeta(s; \mathfrak{a}_\mp) + \exp\left\{-\frac{\pi i s}{2}\right\} \zeta(s; \mathfrak{a}_\pm)\right),$$

where $\Gamma(s)$ is the Euler gamma-function, and

$$\mathfrak{a}_\pm = \left\{\frac{1}{\sqrt{k}} \sum_{l=1}^{k} a_l \exp\left\{\frac{\pm 2\pi i l m}{k}\right\} : m \in \mathbb{Z}\right\}.$$

The function $\zeta(s; \mathfrak{a})$ was studied by many authors. In [3] the function $\zeta(s; \mathfrak{a})$ appears as a special case of the periodic Lerch zeta-function with its functional equation. The papers [7] and [22] are devoted to Hamburger-type theorems for $\zeta(s; \mathfrak{a})$. In [6] the Kronecker limit formula for $\zeta(s; \mathfrak{a})$ is obtained. The mean square of $\zeta(s; \mathfrak{a})$ is studied in [11], [18] and [19]. Probabilistic limit theorems in various spaces are proven in [10] and [12]. The paper [9] contains some expansions for $\zeta(s; \mathfrak{a})$ and its derivatives as well as a Voronoi type formula for

$$d_l(m) = \sum_{d_1 \ldots d_l = m} a_{d_1} \ldots a_{d_l},$$

which is obtained by using the properties of $\zeta(s; \mathfrak{a})$. The zero-distribution of $\zeta(s; \mathfrak{a})$ is investigated in [24]. The zero-free regions and formulas for the number of non-trivial zeroes of $\zeta(s; \mathfrak{a})$ are established. Moreover, in [25] an important result on the universality of $\zeta(s; \mathfrak{a})$ is obtained. Let $\operatorname{meas}\{A\}$ denote the Lebesgue measure of the set $A \subset \mathbb{R}$, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T} \operatorname{meas}\{\tau \in [0, T] \,:\, \ldots\},$$

where in place of dots a condition satisfied by τ is to be written. Suppose that k is an odd prime, a_m is not a multiple of a character modulo k, and $a_k = 0$.

Let K be a compact subset of the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then in [25] it is proved that, for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon \right) > 0.$$

Note that the universality of the Riemann zeta-function was discovered by S.M. Voronin [26]. Later, the Voronin theorem was improved and generalized by many authors, see, the survey papers [8], [16] and [21].

In this paper we consider a generalization of the function $\zeta(s;\mathfrak{a})$. Let $0 < \alpha \le 1$, and, for $\sigma > 1$,

$$\zeta(s,\alpha;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

For $\sigma > 1$, we have

$$\begin{aligned}\zeta(s,\alpha;\mathfrak{a}) &= \sum_{l=0}^{k-1} a_l \sum_{\substack{r=0\\ m=l+rk}}^{\infty} \frac{1}{(m+\alpha)^s} = \sum_{l=0}^{k-1} a_l \sum_{r=0}^{\infty} \frac{1}{(l+rk+\alpha)^s} \\ &= \frac{1}{k^s}\sum_{l=0}^{k-1} a_l \sum_{r=0}^{\infty} \frac{1}{\left(r+\frac{l+\alpha}{k}\right)^s} = \frac{1}{k^s}\sum_{l=0}^{k-1} a_l \zeta\left(s, \frac{l+\alpha}{k}\right),\end{aligned} \tag{1}$$

where $\zeta(s,\alpha)$ is the Hurwitz zeta-function. Therefore, the function $\zeta(s,\alpha;\mathfrak{a})$ is a linear combination of the Hurwitz zeta-functions, and equality (1) gives analytic continuation of the function $\zeta(s,\alpha;\mathfrak{a})$ to the whole complex plane, where it is regular, except, maybe, for a simple pole at $s = 1$ with residue a (if $a \neq 0$).

If $\{a_m\} = \{1\}$ and $k = 1$, the function $\zeta(a,\alpha;\mathfrak{a})$ reduces to the Hurwitz zeta-function $\zeta(s,\alpha)$. If additionally $\alpha = 1$, then $\zeta(s,\alpha;\mathfrak{a})$ becomes the Riemann zeta-function. The sequence $\mathfrak{a}_l = \left\{e^{2\pi i \frac{lm}{k}},\ m \in \mathbb{N}_0\right\}$, $(l,k) = 1$, clearly, is periodic with period k. Therefore, in the case $\mathfrak{a} = \mathfrak{a}_l$ the function $\zeta(s, \alpha; \mathfrak{a})$ reduces to the Lerch zeta-function

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}, \ \sigma > 1,$$

with rational parameter λ. Thus, the function $\zeta(s, \alpha; \mathfrak{a})$ is a generalization of classical zeta-functions, and it is reasonable to call $\zeta(s, \alpha; \mathfrak{a})$ either the periodic Hurwitz zeta-function, or the Hurwitz zeta-function with periodic coefficients.

The function $\zeta(s, \alpha; \mathfrak{a})$ has been investigated in [2], true with a small difference in the definition of $\zeta(s, \alpha, \mathfrak{a})$. Indeed, in our notation, in [2] the function

$$F(s) = \zeta(s, \alpha; \mathfrak{a}) - \frac{a_0}{\alpha^s}$$

has been considered. Applying their original approximate functional equation for the function $\zeta(s, \alpha) - \alpha^{-s}$, the authors obtained in [2] the following interesting estimate. Suppose that $a = 0$, $\max_j |a_j| \leq A$ and $H = T^{1/3}$. Then, for $T \geq 2$,

$$\frac{1}{H} \int_T^{T+H} |F(1/2 + it)|^2 \ll A^2 k \log^3 T + A^2 k \log k$$

uniformly in α. Also, in [2] a mean-square estimate for twists of $F(s)$ with some sequence has been obtained.

The aim of this note is to prove the universality of the function $\zeta(s, \alpha; \mathfrak{a})$ for some values of α and certain sequence $\mathfrak{a}$.

Theorem 1. *Suppose that α is a transcendental number and* $\min_{0 \leq m \leq k-1} |a_m| > 0$. *Let K be a compact subset of the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\left(\sup_{s\in K}|\zeta(s+i\tau,\alpha;\mathfrak{a})-f(s)|<\varepsilon\right)>0.$$

Note that the universality theorem for the Hurwitz zeta-function with transcendental parameter was first proved in the Ph.D. thesis of B. Bagchi [1]. He proposed a new method for the proof of universality for Dirichlet series based on limit theorems in the sense of weak convergence of probability measures in the space of analytic functions. We will apply this method of limit theorems as well as other tools used in [1] for the proof of Theorem 1.

2. The mean square of $\zeta(s,\alpha;\mathfrak{a})$

As it was mentioned above, for the proof of Theorem 1 we need a limit theorem in the sense of week convergence of probability measures on the space of analytic functions for the function $\zeta(s,\alpha;\mathfrak{a})$. The proof of theorem of such a kind is based on the mean square estimate for $\zeta(s,\alpha;\mathfrak{a})$ in the region $\sigma>1/2$.

Theorem 2. *Let $\sigma>1/2$. Then*

$$\frac{1}{T}\int_0^T|\zeta(\sigma+it,\alpha;\mathfrak{a})|^2dt\ll_{\sigma,\alpha,\mathfrak{a}}1.$$

Proof. Suppose that $\max_{0\le j\le k-1}|a_j|\le C$. Then in view of (1) we find that

$$\begin{aligned}|\zeta(s,\alpha;\mathfrak{a})|^2&\le k^{-2\sigma}2^{k-1}\sum_{l=0}^{k-1}|a_l|^2\left|\zeta\left(s,\frac{l+\alpha}{k}\right)\right|^2\\&\le k^{-2\sigma}2^{k-1}C^2\sum_{l=0}^{k-1}\left|\zeta\left(s,\frac{l+\alpha}{k}\right)\right|^2.\end{aligned}\tag{2}$$

By Theorems 3.3.1 and 3.3.2 from [16], for $\sigma>1/2$,

$$\int_1^T |\zeta(\sigma+it,\alpha)|^2 dt \ll_{\sigma,\alpha} T.$$

Therefore, taking into account (2), we obtain the theorem.

3. A limit theorem

Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S.

Define

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where γ_m is the unit circle $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ for every $m \in \mathbb{N}_0$. With the product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m.

For $\sigma > 1/2$, define

$$\zeta(s,\alpha,\omega;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^s}.$$

Since $\{\omega(m),\, m \in \mathbb{N}_0\}$ is a sequence of pairwise orthogonal random variables and

$$\sum_{m=1}^{\infty} \frac{|a_m|^2 \log^2 m}{(m+\alpha)^{2\sigma}} < \infty,$$

we obtain by Rademacher's theorem [19] on series of pairwise orthogonal random variables that the series

$$\sum_{m=1}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^s}$$

for almost all $\omega \in \Omega$ with respect to the measure m_H converges uniformly on compact subsets of the half-plane $\{s \in \mathbb{C} : \sigma > 1/2\}$. This is obtained similarly to the proof of Lemma 5.2.1 [17]. Therefore, $\zeta(s, \alpha, \omega; \mathfrak{a})$ is an $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let V be an arbitrary positive number, and $D_V = \{s \in \mathbb{C} : 1/2 < \sigma < 1, |t| < V\}$. Then, clearly, $\zeta(s, \alpha, \omega; \mathfrak{a})$ is also an $H(D_V)$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by P_ζ the distribution of $\zeta(s, \alpha, \omega; \mathfrak{a})$, i.e.

$$P_\zeta(A) = m_H\left(\omega \in \Omega : \zeta(s, \alpha, \omega; \mathfrak{a}) \in A\right), \; A \in \mathcal{B}(H(D_V)).$$

Theorem 3. *The probability measure*

$$P_T(A) = \nu_T\left(\zeta(s + i\tau, \alpha; \mathfrak{a}) \in A\right), \; A \in \mathcal{B}(H(D_V)),$$

converges weakly to P_ζ as $T \to \infty$.

We begin the proof of Theorem 3 with the following statement.

Lemma 4. *The probability measure*

$$Q_T(A) = \nu_T\left(\left((m+\alpha)^{-i\tau}, \, m \in \mathbb{N}_0\right) \in A\right), \; A \in \mathcal{B}(\Omega),$$

converges weakly to the Haar measure m_H as $T \to \infty$.

Proof. The dual group of Ω is

$$\bigoplus_{m=0}^{\infty} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}_0$.

$$\mathbf{k} = (k_1, k_2, \ldots) \in \bigoplus_{m=0}^{\infty} \mathbb{Z}_m,$$

where only the finite number of integers k_j are distinct from zero, acts on Ω by

$$\mathbf{x} \to \mathbf{x}^{\mathbf{k}} = \prod_{m=0} x_m^{k_m}, \ \mathbf{x} = (x_1, x_2, \ldots) \in \Omega.$$

Therefore, the Fourier transform $g_T(\mathbf{k})$ of the measure Q_T is

$$\begin{aligned} g_T(\mathbf{k}) &= \int_{\Omega} \prod_{m=0}^{\infty} x_m^{k_m} dQ_T = \frac{1}{T} \int_0^T \prod_{m=0}^{\infty} (m+\alpha)^{-i\tau k_m} d\tau \\ &= \frac{1}{T} \int_0^T \exp\left\{ -i\tau \sum_{m=0}^{\infty} k_m \log(m+\alpha) \right\} d\tau. \end{aligned} \tag{3}$$

Since α is transcendental, the system $\{\log(m+\alpha),\ m \in \mathbb{N}_0\}$ is linearly independent over the field of rational numbers, and in view of (3)

$$g_T(\mathbf{k}) = \begin{cases} 1, & \text{if } \ \mathbf{k} = \mathbf{0}, \\ \frac{1 - \exp\left\{-iT \sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\}}{iT \exp\left\{-iT \sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\}}, & \text{if } \ \mathbf{k} \neq \mathbf{0}, \end{cases}$$

where in the second case only a finite number of k_j are non zero. Thus,

$$\lim_{T\to\infty} g_T(\mathbf{k}) = \begin{cases} 1, & \text{if } \mathbf{k}=\mathbf{0}, \\ 0, & \text{if } \mathbf{k}\neq\mathbf{0}, \end{cases}$$

and therefore the measure Q_T converges weakly to m_H as $T\to\infty$.

Proof of Theorem 3. We will give only a sketch of the proof, since it only in some places differs from the case of the Lerch zeta-function [17].

Let $\sigma_1 > 0$, and

$$v(m,n) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_1}\right\}.$$

Define the Dirichlet polynomials

$$\zeta_{n,N}(s,\alpha;\mathfrak{a}) = \sum_{m=1}^{N} \frac{a_m v(m,n)}{(m+\alpha)^s}$$

and

$$\zeta_{n,N}(s,\alpha,\omega;\mathfrak{a}) = \sum_{m=1}^{N} \frac{a_m \omega(m) v(m,n)}{(m+\alpha)^s},\ \omega\in\Omega.$$

Since the function $h:\Omega\to H(D_V)$ given by the formula $h(\omega)=\zeta_{n,N}(s,\alpha,\omega;\mathfrak{a})$ is continuous and $\zeta_{n,N}(s+i\tau,\alpha,\omega;\mathfrak{a}) = h(\omega_\tau)$ where $\omega_\tau = (\alpha^{-i\tau}, (1+\alpha)^{-i\tau}, (2+\alpha)^{-i\tau}, \ldots)$, Lemma 4 and Theorem 5.1 of [4] show that the probability measure

$$P_{T,n,N}(A) = \nu_T\left(\zeta_{n,N}(s+i\tau,\alpha,\omega;\mathfrak{a})\in A\right),\ A\in\mathcal{B}(H(D_V)),$$

converges weakly to the measure $P_{n,N} = m_H h^{-1}$ as $T\to\infty$. In view of the invariance property of the Haar measure m_H, this is also true for the probability measure

$$\hat{P}_{T,n,N}(A) = \nu_T\left(\zeta_{n,N}(s+i\tau,\alpha,\omega;\mathfrak{a}) \in A\right),\ A \in \mathcal{B}(H(D_V)).$$

Thus, both the measures $P_{T,n,N}$ and $\hat{P}_{T,n,N}$ converge weakly to $P_{n,N}$ as $T \to \infty$.

Now let

$$\zeta_n(s,\alpha;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v(m,n)}{(m+\alpha)^s}$$

and

$$\zeta_n(s,\alpha,\omega;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m) v(m,n)}{(m+\alpha)^s},\ \omega \in \Omega.$$

It is not difficult to see that the series for $\zeta_n(s,\alpha;\mathfrak{a})$ as well as for $\zeta_n(s,\alpha,\omega;\mathfrak{a})$ converge absolutely and, therefore, uniformly in t for $\sigma > 1/2$. Define on $(H(D_V),\mathcal{B}(H(D_V)))$ two probability measures

$$P_{T,n}(A) = \nu_T\left(\zeta_n(s+i\tau,\alpha;\mathfrak{a})\right)$$

and

$$\hat{P}_{T,n}(A) = \nu_T\left(\zeta_n(s+i\tau,\alpha,\omega;\mathfrak{a})\right).$$

Then, using the weak convergence of the probability measures $P_{T,n,N}$ and $\hat{P}_{T,n,N}$ to $P_{n,N}$ as $T \to \infty$, we can prove that on $(H(D_V),\mathcal{B}(H(D_V)))$ there exists a probability measure P_n such that both the measures $P_{T,n}$ and $\hat{P}_{T,n}$ converge weakly to P_n as $T \to \infty$.

In the next step we approximate $\zeta(s,\alpha;\mathfrak{a})$ and $\zeta(s,\alpha,\omega;\mathfrak{a})$ in the mean by $\zeta_n(s,\alpha;\mathfrak{a})$ and $\zeta_n(s,\alpha,\omega;\mathfrak{a})$ respectively. Let K be a compact subset of D_V. Then, applying Theorem 2, we find that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}|\zeta(s+i\tau,\alpha;\mathfrak{a})-\zeta_n(s+i\tau,\alpha;\mathfrak{a})|dt=0. \tag{4}$$

To obtain a similar relation for $\zeta(s+i\tau,\alpha,\omega;\mathfrak{a})$, we use the Birkhoff-Khinchine theorem, see, for example, [4]. Let $a_\tau=((m+\alpha)^{-i\tau},\ m\in\mathbb{N}_0)$, $\tau\in\mathbb{R}$. Then $\{a_\tau:\tau\in\mathbb{R}\}$ is a one-parameter group. Define a one-parameter group $\{h_\tau:\tau\in\mathbb{R}\}$ of measurable transformations of Ω by $h_\tau(\omega)=a_\tau\omega$, $\omega\in\Omega$. We recall that a set $A\in B(\Omega)$ is called invariant with respect to $\{h_\tau:\tau\in\mathbb{R}\}$ if for each τ the sets A and $A_\tau=h_\tau(A)$ differ at most by a set of zero m_H-measure. All invariant sets form a σ-field. A one-parameter group $\{h_\tau:\tau\in\mathbb{R}\}$ is called ergodic if its σ-field of invariant sets consists only of sets having m_H-measure equal to 0 or 1. In [10] it was proved that the one-parameter group $\{h_\tau:\tau\in\mathbb{R}\}$ is ergodic. Hence we deduce that the process $|\zeta(\sigma+it,\alpha,\omega;\mathfrak{a})|^2$ is ergodic. Therefore, the Birkhoff-Khinchine theorem shows that, for $\sigma>1/2$,

$$\int_0^T|\zeta(\sigma+it,\alpha,\omega;\mathfrak{a})|^2dt\ll_{\sigma,\alpha,\mathfrak{a}}T$$

for almost all $\omega\in\Omega$. Now, from this we find the analogue of relation (4):

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}|\zeta(s+i\tau,\alpha,\omega;\mathfrak{a})-\zeta_n(s+i\tau,\alpha,\omega;\mathfrak{a})|dt=0 \tag{5}$$

for almost all ω. Now we are ready to prove limit theorems for $\zeta(s,\alpha;\mathfrak{a})$ and $\zeta(s,\alpha,\omega;\mathfrak{a})$. Define

$$\hat{P}_T(A)=\nu_T\left(\zeta(s+i\tau,\alpha,\omega;\mathfrak{a})\in A\right),\ A\in\mathcal{B}(H(D_V)).$$

Since the probability measures $P_{T,n}$ and $\hat{P}_{T,n}$ both converge weakly to the measure P_n as $T\to\infty$, we deduce from (4) and (5) that on $(H(D_V),\mathcal{B}(H(D_V)))$

there exists a probability measure P such that the measures P_T and $\hat{P}_T$ both converge weakly to P as $T \to \infty$. For this, as well as for the measures $P_{T,n}$ and $\hat{P}_{T,n}$, the Prokhorov theorems, see [3], are applying. Thus it remains to identify the limit measure P. For this, we fix a continuity set $A \in \mathcal{B}(H(D))$ of the measure P, and define a random variable θ on $(\Omega, \mathcal{B}(\Omega))$ by

$$\theta(\omega) = \begin{cases} 1, & \text{if } \zeta(s, \alpha, \omega; \mathfrak{a}) \in A, \\ 0, & \text{if } \zeta(s, \alpha, \omega; \mathfrak{a}) \notin A. \end{cases}$$

Then the expectation $E(\theta)$ is

$$E(\theta) = \int_{\Omega} \theta dm_H = m_H\left(\omega \, : \, \zeta(s, \alpha, \omega; \mathfrak{a}) \in A\right) = P_\zeta(A). \tag{6}$$

Since the one-parameter group $\{h_\tau \, : \, \tau \in \mathbb{R}\}$ is ergodic, the process $\theta\left(h_\tau(\omega)\right)$ is also ergodic. Therefore, the Birkhoff-Khinchine theorem yields

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \theta\left(h_\tau(\omega)\right) d\tau = E(\theta) \tag{7}$$

for almost all $\omega \in \Omega$. On the other hand,

$$\frac{1}{T} \int_0^T \theta\left(h_\tau(\omega)\right) d\tau = \nu_T\left(\zeta(s + i\tau, \alpha, \omega; \mathfrak{a}) \in A\right).$$

This, (6) and (7) show that

$$\lim_{T\to\infty} \nu_T\left(\zeta(s + i\tau, \alpha, \omega; \mathfrak{a}) \in A\right) = P_\zeta(A).$$

However,

$$\lim_{T\to\infty} \nu_T\left(\zeta(s + i\tau, \alpha, \omega; \mathfrak{a}) \in A\right) = P(A).$$

Therefore, $P(A) = P_\zeta(A)$ for all continuity sets A of the measure P. Since the continuity sets constitute a determining class, hence $P(A) = P_\zeta(A)$ for all $A \in \mathcal{B}(H(D))$. The theorem is proved.

4. Proof of Theorem 1

We begin with the support of the probability measure P_ζ. We recall that the support of P_ζ is a minimal closed set $A \subset H(D_V)$ such that $P_\zeta(A) = 1$. Since P_ζ is the distribution of the random element $\zeta(s, \alpha, \omega; \mathfrak{a})$, its support coincides with the support of $\zeta(s, \alpha, \omega; \mathfrak{a})$.

By the definition, $\{\omega(m) : m \in \mathbb{N}_0\}$ is a sequence of independent complex-valued random variable, defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, and the support of each $\omega(m)$ is the unit circle γ. Hence the support of $a_m\omega(m)/(m+\alpha)^s$, $m = 0, 1, 2, \dots$ is the set

$$\left\{ g \in H(D_V) \,:\, g(s) = \frac{a_m a}{(m+\alpha)^s},\ |a| = 1 \right\}.$$

$\{a_m\omega(m)/(m+\alpha)^s,\ m \in \mathbb{N}_0\}$ is a sequence of independent $H(D_V)$-valued random elements, therefore by Theorem 1.7.10 of [13], which was first used in [1] as Lemma 5.2.11, the support of the random element $\zeta(s, \alpha, \omega; \mathfrak{a})$ is the closure of the set of all convergent series

$$\sum_{m=0}^{\infty} \frac{a_m a(m)}{(m+\alpha)^s},\ a(m) \in \gamma. \tag{8}$$

Lemma 5. *The support of the measure P_ζ is the whole of $H(D_V)$.*

Proof. We will apply Lemma 5.2.9 of [1], see also Theorem 6.3.10 of [13]. Let K be a compact subset of D_V. Then, clearly, for $|b(m)| = 1$,

$$\sum_{m=0}^{\infty} \sup_{s \in K} \left| \frac{a_m b(m)}{(m+\alpha)^s} \right|^2 < \infty.$$

Moreover, in Section 3 we have seen that the series

$$\sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^s}$$

converges uniformly on compact subsets of D_V for almost all $\omega \in \Omega$. Therefore, there exists $b(m)$, $b(m) \in \gamma$, such that

$$\sum_{m=0}^{\infty} \frac{a_m b(m)}{(m+\alpha)^s}$$

converges in $H(D_V)$. Thus we have that conditions 2° and 3° of Theorem 6.3.10 from [13] for the sequence $\{a_m b(m)/(m+\alpha)^s,\ m \in \mathbb{N}_0\}$ are satisfied. It remains to verify its condition 1°.

Let μ be a complex Borel measure with compact support contained in D_V such that

$$\sum_{m=0}^{\infty} \left| \int_{\mathbb{C}} \frac{a_m b(m)}{(m+\alpha)^s} d\mu(s) \right| < \infty. \tag{9}$$

Since the sequence $\{a_m\}$ is periodic and $\min_{0 \le m \le k-1} |a_m| > 0$, (9) shows that

$$\sum_{m=0}^{\infty} \left| \int_{\mathbb{C}} \frac{d\mu(s)}{(m+\alpha)^s} \right| < \infty.$$

Hence we have that

$$\sum_{m=0}^{\infty} |\varrho(\log(m+\alpha)| < \infty, \tag{10}$$

where

$$\varrho(z) = \int_{\mathbb{C}} e^{-sz} d\mu(s) = \int_{\mathbb{C}} e^{sz} d\hat{\mu}(s),$$

and $\hat{\mu}(A) = \mu h^{-1}(a) = \mu(h^{-1}A)$, $A \in B(\mathbb{C})$, $h(s) = -s$. It is clear that the support of the measure $\hat{\mu}$ is contained in $\{s \in \mathbb{C} : -1 < \sigma < -1/2, |t| < V\}$. Since

$$\int_{\mathbb{C}} e^{sz} d\hat{\mu}(s) \ll e^{V},$$

the function $\varrho(z)$ is of exponential type. Therefore, in view of Lemma 5.2.2 from [1], see also Lemma 6.4.10 of [13], either $\varrho(z) \equiv 0$, or

$$\limsup_{r\to\infty} \frac{\log|\varrho(r)|}{r} > -1.$$

This and Lemma 5.2.5 of [1], see also Theorem 6.4.14 of [13], imply

$$\sum_{p} |\varrho(\log p)| = \infty, \tag{11}$$

where the summing runs over all primes p. Clearly, for $m \geq 2$,

$$\log(m+\alpha) - \log m \ll m^{-1},$$

and consequently,

$$\varrho(\log(m+\alpha)) - \varrho(\log m) \ll m^{-3/2}.$$

Thus, taking into account (10), we have that

$$\sum_{m=2}^{\infty} |\varrho(\log m)| < \infty.$$

However, this contradicts (11). Therefore, the case $\varrho(z) \equiv 0$ takes place, and the differentiation of $\varrho(z) \equiv 0$ at the point $z = 0$ yields

$$\int_{\mathbb{C}} s^r d\mu(s) = 0$$

for $r \in \mathbb{N}_0$. This is condition 1° of Theorem 6.3.10 from [12], and we have that the set of all convergent series

$$\sum_{m=0}^{\infty} \frac{a_m b(m) a(m)}{(m+\alpha)^s}, \ a(m) \in \gamma,$$

is dense in $H(D_V)$. Obviously, this shows that the set of all convergent series (8), has the same property. Since the support of the random element is the closure of the latter set, the lemma is proved.

Proof of Theorem 1. Clearly, there exists $V > 0$ such that $K \subset D_V$. Suppose that the function $f(s)$ is analytically continuable to the region D_V. Define an open set G by

$$G = \left\{ g \in H(D_V) : \sup_{s \in K} |g(s) - f(s)| < \frac{\varepsilon}{4} \right\}.$$

Since by Lemma 5 the function $f(s)$ belongs to the support of the measure P_ζ, we have by Theorem 2.1 of [3] that

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{a}) - f(s)| < \varepsilon \right) \geq P_\zeta(G) > 0.$$

Now let $f(s)$ satisfy the hypotheses of the theorem. Then by the Mergelyan theorem, see, for example, [23], there exists a polynomial $p_n(s)$ such that

$$\sup_{s\in K} |f(s) - p_n(s)| < \frac{\varepsilon}{2}.$$

Moreover, since $p_n(s)$ is an entire function, by the beginning of the proof we have that

$$\limsup_{T\to\infty} \nu_T \left(\sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{a}) - p_n(s)| < \frac{\varepsilon}{2} \right) > 0.$$

This proves the theorem.

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Received on 18-10-2005, Accepted on 10-04-2006