Hardy-Ramanujan Journal 47 (2024), 65-91 submitted 28/05/2024, accepted 24/03/2025, revised 03/04/2025

Explicit formulas for the Fourier coefficients of a class of eta quotients

Florian Münkel and Kenneth S. Williams

Abstract. We determine explicit formulas for the Fourier coefficients of a class of eta quotients by making use of some results from the theory of ternary quadratic forms.

Keywords. q-series, eta quotients.

2010 Mathematics Subject Classification. 11E20, 11F20, 11F27, 11F30, 11F37.

1. Introduction

As usual $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ denote the sets of all integers, nonnegative integers, and positive integers, respectively and $\mathbb{Q}$, $\mathbb{C}$ denote the fields of rational numbers, complex numbers, respectively. We let $\mathcal{H} := \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$. For $z \in \mathcal{H}$ the Dedekind eta function is defined by

$$\eta(z) := e^{\frac{2\pi i z}{24}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right).$$

Dedekind's eta function is holomorphic on $\mathcal{H}$ and satisfies $\eta(z) \neq 0$ for all $z \in \mathcal{H}$.

If S is a finite subset of $\mathbb{N}$ and $\{a_s | s \in S\} \subset \mathbb{Z}$, then the product

$$\prod_{s \in S} \eta^{a_s}(sz)$$

is called an eta product or eta quotient in the literature. We use the term eta quotient for such products. Köhler[Köh11, p. 31] uses eta product. Some authors use eta product when $a_s \geq 0 \quad (s \in S)$ and eta quotient otherwise, see for example Ono [Ono04, p. 18].

The determination of the coefficients of the Fourier expansion of an eta quotient is an important and ongoing area of research in number theory. Köhler's comprehensive book on eta quotients [Köh11] provides a great many such determinations. We just mention one of his examples [Köh11, p. 282], namely,

$$\frac{\eta^2(2z)\eta^2(14z)}{\eta(z)\eta(7z)} = \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n \\ 2\nmid d}} \left(\frac{-7}{d}\right) \right) e^{2\pi i n z},$$

where $\left(\frac{m}{n}\right)$ is the Legendre-Jacobi-Kronecker symbol. The notation $d|n$ under a summation sign indicates that d runs through the positive divisors of the nonzero integer n.

We now define the class of eta quotients for which we determine the Fourier coefficients explicitly.

Definition 1.1. *We denote by $\mathcal{C}$ the class of eta quotients*

$$f(\underline{a}; z) := \prod_{d|24} \eta^{a_d}(dz), \quad \underline{a} = (a_1, a_2, a_3, a_4, a_6, a_8, a_{12}, a_{24}) \in \mathbb{Z}^8, \tag{1.1}$$

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having exponents a_d $(d|24)$ *which satisfy the following conditions*

$$\sum_{d|24} a_d = 3, \tag{1.2}$$

$$\sum_{d|24} da_d = 0, \tag{1.3}$$

$$\sum_{d|24} \frac{24}{d} a_d = 0, \tag{1.4}$$

$$\sum_{d|24} \frac{24(c,d)^2}{d} a_d \geq 0, \tag{1.5}$$

for all positive divisors $c(\neq 1, 24)$ *of* 24. *(Here* (c,d) *denotes the gcd of* c *and* d.*)*

Clearly $f(\underline{a}; z)$ is holomorphic on $\mathcal{H}$. From (1.1) we see that the level of $f(\underline{a}; z)$ is 24. By (1.2) the weight of $f(\underline{a}; z)$ is

$$\frac{1}{2}\sum_{d|24} a_d = \frac{3}{2}.$$

By (1.5) $f(\underline{a}; z)$ is holomorphic at its cusps [Köh11, p. 37]. Using Petersson's formula for the multiplier system of the Dedekind eta function [Kno93, p. 51], [Köh11, p. 14], we deduce using (1.2)–(1.4) that $f(\underline{a}; z)$ transforms for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(24)$ like a half-integral weight modular form of weight $\frac{3}{2}$ and Nebentypus χ given by

$$\chi(d) = \left(\frac{2}{|d|}\right)^{a_2+a_6+a_8+a_{24}+1} \left(\frac{3}{|d|}\right)^{a_3+a_6+a_{12}+a_{24}}.$$

$\Gamma_0(N)$ $(N \in \mathbb{N})$ denotes the congruence subgroup $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) | ad - bc = 1, c \equiv 0 \pmod{N}\right\}$. We have established that if $f(\underline{a}; z) \in \mathcal{C}$, then

$$f(\underline{a}; z) \in M_{\frac{3}{2}}\left(\Gamma_0(24), \left(\frac{2}{*}\right)^{a_2+a_6+a_8+a_{24}+1} \left(\frac{3}{*}\right)^{a_3+a_6+a_{12}+a_{24}}\right). \tag{1.6}$$

We partition the class $\mathcal{C}$ into 4 nonoverlapping subclasses $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_6$ as follows:

$$\mathcal{C}_1 := \{f(\underline{a}; z) \in \mathcal{C} | a_2 + a_6 + a_8 + a_{24} \equiv 1 \pmod 2, a_3 + a_6 + a_{12} + a_{24} \equiv 0 \pmod 2\}, \tag{1.7}$$

$$\mathcal{C}_2 := \{f(\underline{a}; z) \in \mathcal{C} | a_2 + a_6 + a_8 + a_{24} \equiv 0 \pmod 2, a_3 + a_6 + a_{12} + a_{24} \equiv 0 \pmod 2\}, \tag{1.8}$$

$$\mathcal{C}_3 := \{f(\underline{a}; z) \in \mathcal{C} | a_2 + a_6 + a_8 + a_{24} \equiv 1 \pmod 2, a_3 + a_6 + a_{12} + a_{24} \equiv 1 \pmod 2\}, \tag{1.9}$$

$$\mathcal{C}_6 := \{f(\underline{a}; z) \in \mathcal{C} | a_2 + a_6 + a_8 + a_{24} \equiv 0 \pmod 2, a_3 + a_6 + a_{12} + a_{24} \equiv 1 \pmod 2\}, \tag{1.10}$$

so that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_6$ and, by (1.6)–(1.10),

$$\begin{cases} \mathcal{C}_1 \subset M_{\frac{3}{2}}\left(\Gamma_0(24), 1\right), & \mathcal{C}_2 \subset M_{\frac{3}{2}}\left(\Gamma_0(24), \left(\frac{2}{*}\right)\right), \\ \mathcal{C}_3 \subset M_{\frac{3}{2}}\left(\Gamma_0(24), \left(\frac{3}{*}\right)\right), & \mathcal{C}_6 \subset M_{\frac{3}{2}}\left(\Gamma_0(24), \left(\frac{6}{*}\right)\right). \end{cases}$$

From this point on, we make use of some simplifying notation. We set $q := e^{2\pi i z}$ $(z \in \mathcal{H})$ so that $|q| < 1$ and, for $k \in \mathbb{N}$, we set $E_k := E_k(q) := \prod_{n=1}^{\infty}\left(1 - q^{kn}\right)$ so that $\eta(kz) = q^{\frac{k}{24}} E_k$. Hence, we have

$$f(\underline{a}; z) = \prod_{d|24}\left(q^{\frac{d}{24}} E_d\right)^{a_d} = q^{\frac{1}{24}\sum_{d|24} da_d} \prod_{d|24} E_d^{a_d}$$

so, by (1.3), we have

$$f(\underline{a};z) = \prod_{d|24} E_d^{a_d}. \tag{1.11}$$

It is clear from (1.11) that, for $|q|<1$, $f(\underline{a};z)$ has a Fourier expansion

$$f(\underline{a};z) = 1 + \sum_{n=1}^{\infty} c(\underline{a};n) q^n, \tag{1.12}$$

where the Fourier coefficients $c(\underline{a};n)$ are integers. In our main result (Theorem 1.1) we give explicit formulas for the $c(\underline{a};n)$ according as $f(\underline{a};z) \in \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_6$, respectively. These formulas involve the exact powers of 2 and 3 dividing n as well as the largest square dividing n which is not divisible by 2 or 3. We therefore express $n \in \mathbb{N}$ in the form

$$n = 2^{\alpha} 3^{\beta} g h^2, \tag{1.13}$$

where $\alpha, \beta \in \mathbb{N}_0$, $g, h \in \mathbb{N}$, and

$$(g,6) = (h,6) = 1, \quad g \text{ squarefree}. \tag{1.14}$$

In Section 3 we require the number of representations of n by certain ternary quadratic forms of type $ax^2+by^2+cz^2$ $(a,b,c \in \mathbb{N})$ of discriminant Δ, where

$$\Delta := \frac{1}{2} \begin{vmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{vmatrix} = 4abc \in \mathbb{N}.$$

In order to give a formula for the number of such representations, it is convenient to define

$$s(n,\Delta) := \operatorname{sqf}(n\Delta), \tag{1.15}$$

where $\operatorname{sqf}(m)$ denotes the squarefree part of $m \in \mathbb{N}$. We also define

$$l(n,\Delta) := \prod_{p|h} \left(\sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-s(n,\Delta)}{p}\right) \sigma\left(p^{\nu_p(h)-1}\right)\right), \tag{1.16}$$

where σ is the sum of divisors function and $\nu_p(h)$ denotes the exponent of the largest power of the prime p dividing h.

We require the following two simple properties of $s(n,\Delta)$ and $l(n,\Delta)$. Let $k,n \in \mathbb{N}$ be such that $k|n$. Let Δ and Δ' be discriminants of ternary quadratic forms $ax^2+by^2+cz^2$ $(a,b,c\in\mathbb{N})$ such that $\Delta = k\Delta'$. Then

$$s(n,\Delta) = \operatorname{sqf}(n\Delta) = \operatorname{sqf}(nk\Delta') = \operatorname{sqf}\left(\frac{n}{k}\Delta' k^2\right) = \operatorname{sqf}\left(\frac{n}{k}\Delta'\right) = s\left(\frac{n}{k},\Delta'\right). \tag{1.17}$$

Now, restrict k to be one of $2,3,6$. Hence, by (1.14), we have $(k,g)=(k,h)=1$, and by (1.13)

$$\frac{n}{k} = 2^{\alpha-1}3^{\beta}gh^2, \quad 2^{\alpha}3^{\beta-1}gh^2, \quad 2^{\alpha-1}3^{\beta-1}gh^2$$

according as $k=2,3,6$. Thus, appealing to (1.17), we obtain

$$l\left(\frac{n}{k},\Delta'\right) = \prod_{p|h}\left(\sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-s(\frac{n}{k},\Delta')}{p}\right)\sigma\left(p^{\nu_p(h)-1}\right)\right)$$

$$= \prod_{p \mid h} \left(\sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-s(n,\Delta)}{p}\right) \sigma\left(p^{\nu_p(h)-1}\right) \right)$$

that is

$$l\left(\frac{n}{k}, \Delta'\right) = l(n, \Delta). \tag{1.18}$$

In preparation for the proof of Theorem 1.1 in Section 4, we determine in Section 2 the theta quotients in each of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, and $\mathcal{C}_6$, and in Section 3 the number of representations of n by certain ternary quadratic forms $ax^2 + by^2 + cz^2$.

Our main result (Theorem 1.1) gives an explicit formula for the Fourier coefficients $c(\underline{a}; n)$ $(n \in \mathbb{N})$ of every eta quotient $f(\underline{a}; z)$ in the class $\mathcal{C}$. The proof is given in Section 4 and follows from the results in Sections 1–3.

Theorem 1.1. *Let $\underline{a} = (a_1, a_2, a_3, a_4, a_6, a_8, a_{12}, a_{24})$, where the a_i are integers satisfying (1.2)–(1.5). Let $n \in \mathbb{N}$. We express n in the form (1.13), where α, β, g, h satisfy (1.14). If*

$$(a_2 + a_6 + a_8 + a_{24}, a_3 + a_6 + a_{12} + a_{24}) \equiv (1, 0), (0, 0), (1, 1), (0, 1) \pmod 2$$

then the eta quotient

$$\prod_{d \mid 24} \eta^{a_d}(dz) = 1 + \sum_{n=1}^{\infty} c(\underline{a}; n) q^n \qquad (|q| < 1)$$

belongs to the class $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_6$, respectively. We define $\Delta = \operatorname{disc}(x^2 + y^2 + mz^2) = 4m$ if the eta quotient is in $\mathcal{C}_m$ $(m \in \{1, 2, 3, 6\})$. We define $s(n, \Delta)$ and $l(n, \Delta)$ as in (1.15) and (1.16), respectively. Then the Fourier coefficient $c(\underline{a}; n)$ $(n \in \mathbb{N})$ is given by

$$c(\underline{a}; n) = \left(R 2^{[\alpha/2]} + S 3^{[\beta/2]} + T\right) l(n, \Delta) h\left(\mathbb{Q}\left(\sqrt{-s(n, \Delta)}\right)\right),$$

where $[\theta]$ denotes the greatest integer less than or equal to the real number θ, $h\left(\mathbb{Q}\left(\sqrt{-s(n,\Delta)}\right)\right)$ is the class number of the imaginary quadratic field $\mathbb{Q}\left(\sqrt{-s(n,\Delta)}\right)$, and the values of R, S, and T are given in Tables 4.1, 4.2, 4.3, 4.4 according as the eta quotient is in $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_6$, respectively. The values of v, w, x, y, z occurring in Tables 4.1, 4.2, 4.3, 4.4 are given in terms of the values of $c(\underline{a}; t)$ $(t \in \{1, 2, 3, 4\})$ in (2.19), (2.21), (2.23), (2.25) according as the eta quotient is in $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_6$, respectively.

2. Theta quotients in $\mathcal{C}$

Ramanujan's theta function $\varphi(q)$ [Ber06, p. 6] is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad |q| < 1. \tag{2.1}$$

Jacobi's triple product identity gives $\varphi(q)$ as an eta quotient, namely

$$\varphi(q) = E_1^{-2} E_2^5 E_4^{-2} \qquad \text{[Ber06, p. 11]}. \tag{2.2}$$

Let $r, s, t, u \in \mathbb{Z}$. Appealing to (2.2), we obtain

$$\varphi^r(q)\varphi^s(q^2)\varphi^t(q^3)\varphi^u(q^6) = E_1^{-2r} E_2^{5r-2s} E_3^{-2t} E_4^{-2r+5s} E_6^{5t-2u} E_8^{-2s} E_{12}^{-2t+5u} E_{24}^{-2u}.$$

Then (1.11) gives

$$\varphi^r(q)\varphi^s(q^2)\varphi^t(q^3)\varphi^u(q^6) = f(\underline{a}; z),$$

where $\underline{a} = (a_1, a_2, a_3, a_4, a_6, a_8, a_{12}, a_{24})$ is given by

$$\begin{cases} a_1 = -2r, a_2 = 5r - 2s, a_3 = -2t, a_4 = -2r + 5s, \\ a_6 = 5t - 2u, a_8 = -2s, a_{12} = -2t + 5u, a_{24} = -2u. \end{cases}$$

We now choose r, s, t, u so that $f(\underline{a}; z) \in \mathcal{C}$. We have

$$\sum_{d|24} a_d = r + s + t + u,$$

$$\sum_{d|24} \frac{24(c,d)^2}{d} a_d = \begin{cases} 0 & \text{if } c = 1, \\ 24(6r + 2t) & \text{if } c = 2, \\ 0 & \text{if } c = 3, \\ 24(12s + 4u) & \text{if } c = 4, \\ 24(6r + 18t) & \text{if } c = 6, \\ 0 & \text{if } c = 8, \\ 24(12s + 36u) & \text{if } c = 12, \\ 0 & \text{if } c = 24. \end{cases}$$

Hence, if we choose $r, s, t, u \in \mathbb{N}_0$ such that $r + s + t + u = 3$ then $f(\underline{a}; z) \in \mathcal{C}$. There are 20 4-tuples $(r, s, t, u) \in \mathbb{N}_0^4$ satisfying $r + s + t + u = 3$. These are listed in Table 2.1 from which we observe that

$$\varphi^3(q), \quad \varphi(q)\varphi^2(q^2), \quad \varphi(q)\varphi^2(q^3), \quad \varphi(q)\varphi^2(q^6), \quad \varphi(q^2)\varphi(q^3)\varphi(q^6) \in \mathcal{C}_1, \tag{2.3}$$

$$\varphi^3(q^2), \quad \varphi(q^2)\varphi^2(q^3), \quad \varphi(q^2)\varphi^2(q^6), \quad \varphi^2(q)\varphi(q^2), \quad \varphi(q)\varphi(q^3)\varphi(q^6) \in \mathcal{C}_2, \tag{2.4}$$

$$\varphi^3(q^3), \quad \varphi(q^3)\varphi^2(q^6), \quad \varphi^2(q)\varphi(q^3), \quad \varphi^2(q^2)\varphi(q^3), \quad \varphi(q)\varphi(q^2)\varphi(q^6) \in \mathcal{C}_3, \tag{2.5}$$

$$\varphi^3(q^6), \quad \varphi^2(q)\varphi(q^6), \quad \varphi^2(q^2)\varphi(q^6), \quad \varphi^2(q^3)\varphi(q^6), \quad \varphi(q)\varphi(q^2)\varphi(q^3) \in \mathcal{C}_6. \tag{2.6}$$

We show next that the theta products in (2.3) are linearly independent. Suppose that

$$c_1\varphi^3(q) + c_2\varphi(q)\varphi^2(q^2) + c_3\varphi(q)\varphi^2(q^3) + c_4\varphi(q)\varphi^2(q^6) + c_5\varphi(q^2)\varphi(q^3)\varphi(q^6) = 0 \tag{2.7}$$

for some $c_1, c_2, c_3, c_4, c_5 \in \mathbb{C}$. Then, using

$$\varphi(q) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + \cdots$$

in (2.7), we obtain

$$\begin{aligned} (c_1 + c_2 + c_3 + c_4 + c_5) + (6c_1 + 2c_2 + 2c_3 + 2c_4)q + (12c_1 + 4c_2 + 2c_5)q^2 \\ +(8c_1 + 8c_2 + 4c_3 + 2c_5)q^3 + (6c_1 + 6c_2 + 10c_3 + 2c_4)q^4 \\ +(24c_1 + 8c_2 + 4c_5)q^5 + (24c_1 + 8c_2 + 4c_3 + 4c_4 + 2c_5)q^6 + \cdots = 0, \end{aligned}$$

so that

$$\begin{cases} c_1 + c_2 + c_3 + c_4 + c_5 = 0, \\ 6c_1 + 2c_2 + 2c_3 + 2c_4 = 0, \\ 12c_1 + 4c_2 + 2c_5 = 0, \\ 8c_1 + 8c_2 + 4c_3 + 2c_5 = 0, \\ 6c_1 + 6c_2 + 10c_3 + 2c_4 = 0, \\ 24c_1 + 8c_2 + 4c_5 = 0, \\ 24c_1 + 8c_2 + 4c_3 + 4c_4 + 2c_5 = 0. \end{cases}$$

$r\ s\ t\ u$	$\varphi^r(q)\varphi^s(q^2)\varphi^t(q^3)\varphi^u(q^6)$	a_1	a_2	a_3	a_4	a_6	a_8	a_{12}	a_{24}	ternary	class
3 0 0 0	$\varphi^3(q)$	-6	15	0	-6	0	0	0	0	$x^2+y^2+z^2$	$\mathcal{C}_1$
0 3 0 0	$\varphi^3(q^2)$	0	-6	0	15	0	-6	0	0	$2x^2+2y^2+2z^2$	$\mathcal{C}_2$
0 0 3 0	$\varphi^3(q^3)$	0	0	-6	0	15	0	-6	0	$3x^2+3y^2+3z^2$	$\mathcal{C}_3$
0 0 0 3	$\varphi^3(q^6)$	0	0	0	0	-6	0	15	-6	$6x^2+6y^2+6z^2$	$\mathcal{C}_6$
1 2 0 0	$\varphi(q)\varphi^2(q^2)$	-2	1	0	8	0	-4	0	0	$x^2+2y^2+2z^2$	$\mathcal{C}_1$
1 0 2 0	$\varphi(q)\varphi^2(q^3)$	-2	5	-4	-2	10	0	-4	0	$x^2+3y^2+3z^2$	$\mathcal{C}_1$
1 0 0 2	$\varphi(q)\varphi^2(q^6)$	-2	5	0	-2	-4	0	10	-4	$x^2+6y^2+6z^2$	$\mathcal{C}_1$
0 1 2 0	$\varphi(q^2)\varphi^2(q^3)$	0	-2	-4	5	10	-2	-4	0	$2x^2+3y^2+3z^2$	$\mathcal{C}_2$
0 1 0 2	$\varphi(q^2)\varphi^2(q^6)$	0	-2	0	5	-4	-2	10	-4	$2x^2+6y^2+6z^2$	$\mathcal{C}_2$
0 0 1 2	$\varphi(q^3)\varphi^2(q^6)$	0	0	-2	0	1	0	8	-4	$3x^2+6y^2+6z^2$	$\mathcal{C}_3$
2 1 0 0	$\varphi^2(q)\varphi(q^2)$	-4	8	0	1	0	-2	0	0	$x^2+y^2+2z^2$	$\mathcal{C}_2$
2 0 1 0	$\varphi^2(q)\varphi(q^3)$	-4	10	-2	-4	5	0	-2	0	$x^2+y^2+3z^2$	$\mathcal{C}_3$
2 0 0 1	$\varphi^2(q)\varphi(q^6)$	-4	10	0	-4	-2	0	5	-2	$x^2+y^2+6z^2$	$\mathcal{C}_6$
0 2 1 0	$\varphi^2(q^2)\varphi(q^3)$	0	-4	-2	10	5	-4	-2	0	$2x^2+2y^2+3z^2$	$\mathcal{C}_3$
0 2 0 1	$\varphi^2(q^2)\varphi(q^6)$	0	-4	0	10	-2	-4	5	-2	$2x^2+2y^2+6z^2$	$\mathcal{C}_6$
0 0 2 1	$\varphi^2(q^3)\varphi(q^6)$	0	0	-4	0	8	0	1	-2	$3x^2+3y^2+6z^2$	$\mathcal{C}_6$
0 1 1 1	$\varphi(q^2)\varphi(q^3)\varphi(q^6)$	0	-2	-2	5	3	-2	3	-2	$2x^2+3y^2+6z^2$	$\mathcal{C}_1$
1 0 1 1	$\varphi(q)\varphi(q^3)\varphi(q^6)$	-2	5	-2	-2	3	0	3	-2	$x^2+3y^2+6z^2$	$\mathcal{C}_2$
1 1 0 1	$\varphi(q)\varphi(q^2)\varphi(q^6)$	-2	3	0	3	-2	-2	5	-2	$x^2+2y^2+6z^2$	$\mathcal{C}_3$
1 1 1 0	$\varphi(q)\varphi(q^2)\varphi(q^3)$	-2	3	-2	3	5	-2	-2	0	$x^2+2y^2+3z^2$	$\mathcal{C}_6$

Table 2.1: Theta products in $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_6$

Solving this system of linear equations, we obtain

$$c_1 = c_2 = c_3 = c_4 = c_5 = 0,$$

so that $\varphi^3(q)$, $\varphi(q)\varphi^2(q^2)$, $\varphi(q)\varphi^2(q^3)$, $\varphi(q)\varphi^2(q^6)$, $\varphi(q^2)\varphi(q^3)\varphi(q^6)$ are linearly independent. Similarly, we can show that the theta products in (2.4), (2.5), and (2.6) are also linearly independent.

By a result of Cohen and Oesterlé [CoOe77, pp. 75,76], we have

$$\dim M_{\frac{3}{2}}(\Gamma_0(24), \chi) = b(24, f) = 5.$$

Hence, the theta products in (2.3), (2.4), (2.5), (2.6) form a basis for $M_{\frac{3}{2}}(\Gamma_0(24), 1)$, $M_{\frac{3}{2}}\left(\Gamma_0(24), \left(\frac{2}{*}\right)\right)$, $M_{\frac{3}{2}}\left(\Gamma_0(24), \left(\frac{3}{*}\right)\right)$, $M_{\frac{3}{2}}\left(\Gamma_0(24), \left(\frac{6}{*}\right)\right)$, respectively. Thus, if $f(\underline{a}; z) \in \mathcal{C}$, there exist uniquely determined v, w, x, y, $z \in \mathbb{C}$ such that

$$f(\underline{a}; z) = v\varphi^3(q) + w\varphi(q)\varphi^2(q^2) + x\varphi(q)\varphi^2(q^3) + y\varphi(q)\varphi^2(q^6) + z\varphi(q^2)\varphi(q^3)\varphi(q^6) \quad \text{if } f(\underline{a}; z) \in \mathcal{C}_1, \tag{2.8}$$

$$f(\underline{a}; z) = v\varphi^2(q)\varphi(q^2) + w\varphi^3(q^2) + x\varphi(q^2)\varphi^2(q^3) + y\varphi(q^2)\varphi^2(q^6) + z\varphi(q)\varphi(q^3)\varphi(q^6) \quad \text{if } f(\underline{a}; z) \in \mathcal{C}_2, \tag{2.9}$$

$$f(\underline{a}; z) = v\varphi^2(q)\varphi(q^3) + w\varphi^3(q^3) + x\varphi^2(q^2)\varphi(q^3) + y\varphi(q^3)\varphi^2(q^6) + z\varphi(q)\varphi(q^2)\varphi(q^6) \quad \text{if } f(\underline{a}; z) \in \mathcal{C}_3, \tag{2.10}$$

$$f(\underline{a}; z) = v\varphi^3(q^6) + w\varphi^2(q^3)\varphi(q^6) + x\varphi^2(q^2)\varphi(q^6) + y\varphi^2(q)\varphi(q^6) + z\varphi(q)\varphi(q^2)\varphi(q^3) \quad \text{if } f(\underline{a}; z) \in \mathcal{C}_6., \tag{2.11}$$

noting that the order of the theta products in (2.9)–(2.11) differs slightly from those in (2.4)–(2.6). By (1.11) or (1.12) we have $f(\underline{a}; z)|_{q=0} = 1$ and, by (2.1), that $\varphi(0) = 1$. Hence, by (2.8)–(2.11), we have

$$v + w + x + y + z = 1 \tag{2.12}$$

regardless of the class $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ or $\mathcal{C}_6$ to which $f(\underline{a}; z)$ belongs.

For $k_1, k_2, k_3 \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we define the representation number $r(k_1, k_2, k_3; n)$ by

$$r(k_1, k_2, k_3; n) := \text{card}\left\{(x_1, x_2, x_3) \in \mathbb{Z}^3 | k_1x_1^2 + k_2x_2^2 + k_3x_3^2 = n\right\}$$

so that $r(k_1, k_2, k_3; 0) = 1$ and

$$\varphi(q^{k_1})\varphi(q^{k_2})\varphi(q^{k_3}) = \sum_{n=0}^{\infty} r(k_1, k_2, k_3; n)q^n = 1 + \sum_{n=1}^{\infty} r(k_1, k_2, k_3; n)q^n. \tag{2.13}$$

If $m \notin \mathbb{N}_0$ we define $r(k_1, k_2, k_3; m) := 0$. Thus, for example, we have

$$\varphi^2(q^2)\varphi(q^6) = \sum_{n=0}^{\infty} r(2, 2, 6; n)q^n = \sum_{n=0}^{\infty} r\left(1, 1, 3; \frac{n}{2}\right) q^n.$$

For $f(\underline{a}; z) \in \mathcal{C}_1$, equating coefficients of q^n $(n \in \mathbb{N})$ in (1.12) and (2.8), we obtain using (2.13)

$$c(\underline{a}; n) = vr(1, 1, 1; n) + wr(1, 2, 2; n) + xr(1, 3, 3; n) + yr(1, 6, 6; n) + zr(2, 3, 6; n) \quad (n \in \mathbb{N}); \tag{2.14}$$

for $f(\underline{a}; z) \in \mathcal{C}_2$ using (2.9)

$$c(\underline{a}; n) = vr(1, 1, 2; n) + wr\left(1, 1, 1; \frac{n}{2}\right) + xr(2, 3, 3; n) + yr\left(1, 3, 3; \frac{n}{2}\right) + zr(1, 3, 6; n) \quad (n \in \mathbb{N}); \tag{2.15}$$

for $f(\underline{a}; z) \in \mathcal{C}_3$ using (2.10)

$$c(\underline{a}; n) = vr(1, 1, 3; n) + wr\left(1, 1, 1; \frac{n}{3}\right) + xr(2, 2, 3; n) + yr\left(1, 2, 2; \frac{n}{3}\right) + zr(1, 2, 6; n) \quad (n \in \mathbb{N}); \tag{2.16}$$

and for $f(\underline{a}; z) \in \mathcal{C}_6$ using (2.11)

$$c(\underline{a}; n) = vr\left(1, 1, 1; \frac{n}{6}\right) + wr\left(1, 1, 2; \frac{n}{3}\right) + xr\left(1, 1, 3; \frac{n}{2}\right) + yr(1, 1, 6; n) + zr(1, 2, 3; n) \quad (n \in \mathbb{N}). \tag{2.17}$$

The values of $r(1, 1, 1; n), r(1, 2, 2; n), r(1, 3, 3; n), r(1, 6, 6; n)$ and $r(2, 3, 6; n)$ are easily calculated for $n = 1, 2, 3, 4$ and are given in Table 2.2.

n	1	2	3	4
$r(1,1,1;n)$	6	12	8	6
$r(1,2,2;n)$	2	4	8	6
$r(1,3,3;n)$	2	0	4	10
$r(1,6,6;n)$	2	0	0	2
$r(2,3,6;n)$	0	2	2	0

Table 2.2: Values of representation numbers

Taking $n = 1, 2, 3, 4$ in (2.14), we obtain for $f(\underline{a}; z) \in \mathcal{C}_1$

$$\begin{cases} c(\underline{a}; 1) = 6v + 2w + 2x + 2y, \\ c(\underline{a}; 2) = 12v + 4w + 2z, \\ c(\underline{a}; 3) = 8v + 8w + 4x + 2z, \\ c(\underline{a}; 4) = 6v + 6w + 10x + 2y. \end{cases} \tag{2.18}$$

Solving the system (2.18) together with (2.12) for v, w, x, y, and z, we obtain

$$\begin{cases} v = -\frac{1}{12} + \frac{1}{8}c(\underline{a}; 2) - \frac{1}{12}c(\underline{a}; 3) + \frac{1}{24}c(\underline{a}; 4), \\ w = -\frac{1}{6} + \frac{1}{4}c(\underline{a}; 1) - \frac{1}{4}c(\underline{a}; 2) + \frac{1}{3}c(\underline{a}; 3) - \frac{1}{6}c(\underline{a}; 4), \\ x = \frac{1}{12} - \frac{1}{4}c(\underline{a}; 1) + \frac{1}{8}c(\underline{a}; 2) - \frac{1}{6}c(\underline{a}; 3) + \frac{5}{24}c(\underline{a}; 4), \\ y = \frac{1}{3} + \frac{1}{2}c(\underline{a}; 1) - \frac{1}{4}c(\underline{a}; 2) + \frac{1}{12}c(\underline{a}; 3) - \frac{1}{6}c(\underline{a}; 4), \\ z = \frac{5}{6} - \frac{1}{2}c(\underline{a}; 1) + \frac{1}{4}c(\underline{a}; 2) - \frac{1}{6}c(\underline{a}; 3) + \frac{1}{12}c(\underline{a}; 4). \end{cases} \tag{2.19}$$

We have shown for $f(\underline{a}; z) = 1 + \sum_{n=1}^{\infty} c(\underline{a}; n)q^n \in \mathcal{C}_1$ that $c(\underline{a}; n)$ is given by (2.14), where v, w, x, y, z are given in (2.19).

We now turn to the class $\mathcal{C}_2$. The relevant representation numbers are given in Table 2.3.

n	1	2	3	4
$r(1, 1, 2; n)$	4	6	8	12
$r(1, 1, 1; n/2)$	0	6	0	12
$r(2, 3, 3; n)$	0	2	4	0
$r(1, 3, 3; n/2)$	0	2	0	0
$r(1, 3, 6; n)$	2	0	2	6

Table 2.3: Values of representation numbers

Taking $n = 1, 2, 3, 4$ in (2.15), we obtain for $f(\underline{a}; z) \in \mathcal{C}_2$

$$\begin{cases} c(\underline{a}; 1) = 4v + 2z, \\ c(\underline{a}; 2) = 6v + 6w + 2x + 2y, \\ c(\underline{a}; 3) = 8v + 4x + 2z, \\ c(\underline{a}; 4) = 12v + 12w + 6z. \end{cases} \tag{2.20}$$

Solving the system (2.20) together with (2.12) for v, w, x, y, and z, we obtain

$$\begin{cases} v = -\frac{1}{4} + \frac{1}{4}c(\underline{a}; 1) + \frac{1}{8}c(\underline{a}; 2) - \frac{1}{24}c(\underline{a}; 4), \\ w = -\frac{1}{4}c(\underline{a}; 1) + \frac{1}{12}c(\underline{a}; 4), \\ x = \frac{1}{4} - \frac{1}{2}c(\underline{a}; 1) - \frac{1}{8}c(\underline{a}; 2) + \frac{1}{4}c(\underline{a}; 3) + \frac{1}{24}c(\underline{a}; 4), \\ y = \frac{1}{2} + \frac{1}{2}c(\underline{a}; 1) + \frac{1}{4}c(\underline{a}; 2) - \frac{1}{4}c(\underline{a}; 3) - \frac{1}{6}c(\underline{a}; 4), \\ z = \frac{1}{2} - \frac{1}{4}c(\underline{a}; 2) + \frac{1}{12}c(\underline{a}; 4). \end{cases} \tag{2.21}$$

We have shown for $f(\underline{a}; z) = 1 + \sum_{n=1}^{\infty} c(\underline{a}; n)q^n \in \mathcal{C}_2$ that $c(\underline{a}; n)$ is given by (2.15), where v, w, x, y, z are given in (2.21).

Next we consider the class $\mathcal{C}_3$. The values of the relevant representation numbers are given in Table 2.4.

n	1	2	3	4
$r(1,1,3;n)$	4	4	2	12
$r(1,1,1;n/3)$	0	0	6	0
$r(2,2,3;n)$	0	4	2	4
$r(1,2,2;n/3)$	0	0	2	0
$r(1,2,6;n)$	2	2	4	2

Table 2.4: Values of representation numbers

Taking $n = 1, 2, 3, 4$ in (2.16), we deduce

$$\begin{cases} c(\underline{a};1) = 4v + 2z, \\ c(\underline{a};2) = 4v + 4x + 2z, \\ c(\underline{a};3) = 2v + 6w + 2x + 2y + 4z, \\ c(\underline{a};4) = 12v + 4x + 2z. \end{cases} \tag{2.22}$$

Solving the system (2.22) together with (2.12) for v, w, x, y, and z, we obtain

$$\begin{cases} v = -\frac{1}{8}c(\underline{a};2) + \frac{1}{8}c(\underline{a};4), \\ w = -\frac{1}{2} - \frac{1}{4}c(\underline{a};1) - \frac{1}{8}c(\underline{a};2) + \frac{1}{4}c(\underline{a};3) + \frac{1}{8}c(\underline{a};4), \\ x = -\frac{1}{4}c(\underline{a};1) + \frac{1}{4}c(\underline{a};2), \\ y = \frac{3}{2} - \frac{1}{4}c(\underline{a};2) - \frac{1}{4}c(\underline{a};3), \\ z = \frac{1}{2}c(\underline{a};1) + \frac{1}{4}c(\underline{a};2) - \frac{1}{4}c(\underline{a};4). \end{cases} \tag{2.23}$$

We have shown for $f(\underline{a};z) = 1 + \sum_{n=1}^{\infty} c(\underline{a};n)q^n \in \mathcal{C}_3$ that $c(\underline{a};n)$ is given by (2.16), where v, w, x, y, z are given in (2.23).

Finally, we consider the class $\mathcal{C}_6$. Table 2.5 gives the values of the relevant representation numbers.

n	1	2	3	4
$r(1,1,1;n/6)$	0	0	0	0
$r(1,1,2;n/3)$	0	0	4	0
$r(1,1,3;n/2)$	0	4	0	4
$r(1,1,6;n)$	4	4	0	4
$r(1,2,3;n)$	2	2	6	6

Table 2.5: Values of representation numbers

For the class $\mathcal{C}_6$ we take $n = 1, 2, 3, 4$ to obtain

$$\begin{cases} c(\underline{a};1) = 4y + 2z, \\ c(\underline{a};2) = 4x + 4y + 2z, \\ c(\underline{a};3) = 4w + 6z, \\ c(\underline{a};4) = 4x + 4y + 6z. \end{cases} \tag{2.24}$$

Solving the system (2.24) together with (2.12) for v, w, x, y, and z, we obtain

$$\begin{cases} v = 1 - \frac{1}{2}c(\underline{a};2) - \frac{1}{4}c(\underline{a};3) + \frac{1}{4}c(\underline{a};4), \\ w = \frac{3}{8}c(\underline{a};2) + \frac{1}{4}c(\underline{a};3) - \frac{3}{8}c(\underline{a};4), \\ x = -\frac{1}{4}c(\underline{a};1) + \frac{1}{4}c(\underline{a};2), \\ y = \frac{1}{4}c(\underline{a};1) + \frac{1}{8}c(\underline{a};2) - \frac{1}{8}c(\underline{a};4), \\ z = -\frac{1}{4}c(\underline{a};2) + \frac{1}{4}c(\underline{a};4). \end{cases} \tag{2.25}$$

We have shown for $f(\underline{a};z) = 1 + \sum_{n=1}^{\infty} c(\underline{a};n)q^n \in \mathcal{C}_6$ that $c(\underline{a};n)$ is given by (2.17), where v, w, x, y, z are given in (2.25).

Our final step in the evaluation of $c(\underline{a};n)$ from (2.14)–(2.17) is the determination of $r(1,1,1;n)$, $r(1,2,2;n), \ldots$. This is carried out in Section 3.

3. Ternary quadratic forms

In the previous section we showed that if the eta quotient $f(\underline{a};z) = 1 + \sum_{n=1}^{\infty} c(\underline{a};n)q^n$ belongs to the class $\mathcal{C}_i$ ($i \in \{1,2,3,6\}$), then $c(\underline{a};n)$ is given as a linear combination of representation numbers of five ternary quadratic forms $ax^2 + by^2 + cz^z$ of discriminant $\Delta = 4abc$ (depending on the subclass $\mathcal{C}_i$ to which $f(\underline{a};z)$ belongs) with the coefficients in the linear combination depending on $\underline{a}$. The ternaries in question are

$$\begin{aligned} &x^2+y^2+z^2\ (\Delta=2^2),\quad x^2+2y^2+2z^2\ (\Delta=2^4),\quad x^2+3y^2+3z^2\ (\Delta=2^23^2),\\ &\qquad x^2+6y^2+6z^2\ (\Delta=2^43^2),\quad 2x^2+3y^2+6z^2\ (\Delta=2^43^2)\quad \text{if } f(\underline{a};z)\in\mathcal{C}_1,\\ &x^2+y^2+2z^2\ (\Delta=2^3),\quad x^2+y^2+z^2\ (\Delta=2^2),\quad 2x^2+3y^2+3z^2\ (\Delta=2^33^2),\\ &\qquad x^2+3y^2+3z^2\ (\Delta=2^23^2),\quad x^2+3y^2+6z^2\ (\Delta=2^33^2)\quad \text{if } f(\underline{a};z)\in\mathcal{C}_2,\\ &x^2+y^2+3z^2\ (\Delta=2^23),\quad x^2+y^2+z^2\ (\Delta=2^2),\quad 2x^2+2y^2+3z^2\ (\Delta=2^43),\\ &\qquad x^2+2y^2+2z^2\ (\Delta=2^4),\quad x^2+2y^2+6z^2\ (\Delta=2^43)\quad \text{if } f(\underline{a};z)\in\mathcal{C}_3,\\ &x^2+y^2+z^2\ (\Delta=2^2),\quad x^2+y^2+2z^2\ (\Delta=2^3),\quad x^2+y^2+3z^2\ (\Delta=2^23),\\ &\qquad x^2+y^2+6z^2\ (\Delta=2^33),\quad x^2+2y^2+3z^2\ (\Delta=2^23)\quad \text{if } f(\underline{a};z)\in\mathcal{C}_6. \end{aligned}$$

Each of theses ternaries is alone in its genus [JKS97] so its representation number can be determined by Siegel's formula [Sie35] (see for example [Cas78, pp. 377–8]). This was carried out in 1971 by Lomadze [Lom71] for all primitive, positive, diagonal, integral, ternary quadratic forms that belong to one-class genera. Using these formulas for the thirteen ternaries of interest to us, and after considerable simplification using Dirichlet's class number formula for an imaginary quadratic field, we obtain all the thirteen formulas in the form

$$r(a,b,c;n) = k(a,b,c;n)l(n,\Delta)h\left(\mathbb{Q}\left(\sqrt{-s(n,\Delta)}\right)\right), \tag{3.1}$$

where $\Delta = 4abc$, $s(n,\Delta)$ is defined in (1.15), $l(n,\Delta)$ is defined in (1.16), $h\left(\mathbb{Q}\left(\sqrt{-s(n,\Delta)}\right)\right)$ denotes the class number of the imaginary quadratic field $\mathbb{Q}\left(\sqrt{-s(n,\Delta)}\right)$, and the values of $k(a,b,c;n)$ ($n \in \mathbb{N}$) are given in Tables 3.1–3.4 with n in the form (1.13).

Using the results obtained in Sections 2 and 3, we show in Section 4 that for $f(\underline{a};z) = 1 + \sum_{n=1}^{\infty} c(\underline{a};n)q^n \in \mathcal{C}$ we have

$$c(\underline{a};n) = \left(R2^{[\alpha/2]} + S3^{[\beta/2]} + T\right) l(n,\Delta)h\left(\mathbb{Q}\left(\sqrt{-s(n,\Delta)}\right)\right),$$

where n is expressed in the form (1.13), the values of R, S, and T are given in terms of α, β, g, $c(\underline{a},1), \ldots, c(\underline{a},4)$, and $\Delta = 4, 8, 12$ or 24 according as $f(\underline{a};z) \in \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ or $\mathcal{C}_6$.

α,β	g	$s(n,4)$	$k(1,1,1;n)$	$k(1,2,2;n)$	$k(1,3,3;n)$	$k(1,6,6;n)$	$k(2,3,6;n)$
			α(even) β(even)				
$\alpha=0,\beta\geq 0$	$g=1$	g	$12\cdot 3^{\beta/2}-6$	$4\cdot 3^{\beta/2}-2$	2	2	$2\cdot 3^{\beta/2}-2$
	$g\equiv 1\pmod{24}, g\neq 1$		$24\cdot 3^{\beta/2}-12$	$8\cdot 3^{\beta/2}-4$	4	4	$4\cdot 3^{\beta/2}-4$
	$g\equiv 5\pmod{24}$		$12\cdot 3^{\beta/2}$	$4\cdot 3^{\beta/2}$	0	0	$2\cdot 3^{\beta/2}$
	$g\equiv 7\pmod{24}$		0	0	16	8	0
	$g\equiv 11\pmod{24}$		$24\cdot 3^{\beta/2}$	$24\cdot 3^{\beta/2}$	0	0	$12\cdot 3^{\beta/2}$
	$g\equiv 13\pmod{24}$		$24\cdot 3^{\beta/2}-12$	$8\cdot 3^{\beta/2}-4$	4	4	$4\cdot 3^{\beta/2}-4$
	$g\equiv 17\pmod{24}$		$12\cdot 3^{\beta/2}$	$4\cdot 3^{\beta/2}$	0	0	$2\cdot 3^{\beta/2}$
	$g\equiv 19\pmod{24}$		$48\cdot 3^{\beta/2}-24$	$48\cdot 3^{\beta/2}-24$	24	0	$24\cdot 3^{\beta/2}-24$
	$g\equiv 23\pmod{24}$		0	0	0	0	0
$\alpha\geq 2,\beta\geq 0$	$g=1$	g	$12\cdot 3^{\beta/2}-6$	$12\cdot 3^{\beta/2}-6$	$8\cdot 2^{\alpha/2}-6$	$4\cdot 2^{\alpha/2}-6$	$6\cdot 3^{\beta/2}-6$
	$g\equiv 1\pmod{24}, g\neq 1$		$24\cdot 3^{\beta/2}-12$	$24\cdot 3^{\beta/2}-12$	$16\cdot 2^{\alpha/2}-12$	$8\cdot 2^{\alpha/2}-12$	$12\cdot 3^{\beta/2}-12$
	$g\equiv 5\pmod{24}$		$12\cdot 3^{\beta/2}$	$12\cdot 3^{\beta/2}$	0	0	$6\cdot 3^{\beta/2}$
	$g\equiv 7\pmod{24}$		0	0	$16\cdot 2^{\alpha/2}$	$8\cdot 2^{\alpha/2}$	0
	$g\equiv 11\pmod{24}$		$24\cdot 3^{\beta/2}$	$24\cdot 3^{\beta/2}$	0	0	$12\cdot 3^{\beta/2}$
	$g\equiv 13\pmod{24}$		$24\cdot 3^{\beta/2}-12$	$24\cdot 3^{\beta/2}-12$	$16\cdot 2^{\alpha/2}-12$	$8\cdot 2^{\alpha/2}-12$	$12\cdot 3^{\beta/2}-12$
	$g\equiv 17\pmod{24}$		$12\cdot 3^{\beta/2}$	$12\cdot 3^{\beta/2}$	0	0	$6\cdot 3^{\beta/2}$
	$g\equiv 19\pmod{24}$		$48\cdot 3^{\beta/2}-24$	$48\cdot 3^{\beta/2}-24$	$48\cdot 2^{\alpha/2}-24$	$24\cdot 2^{\alpha/2}-24$	$24\cdot 3^{\beta/2}-24$
	$g\equiv 23\pmod{24}$		0	0	0	0	0
			α(even) β(odd)				
$\alpha=0,\beta\geq 1$	$g=1$	$3g$	$12\cdot 3^{(\beta-1)/2}-4$	$12\cdot 3^{(\beta-1)/2}-4$	4	0	$6\cdot 3^{(\beta-1)/2}-4$
	$g\equiv 1\pmod{8}, g\neq 1$		$36\cdot 3^{(\beta-1)/2}-12$	$36\cdot 3^{(\beta-1)/2}-12$	12	0	$18\cdot 3^{(\beta-1)/2}-12$
	$g\equiv 5\pmod{8}$		0	0	8	4	0
	$g\equiv 3\pmod{4}$		$18\cdot 3^{(\beta-1)/2}-6$	$6\cdot 3^{(\beta-1)/2}-2$	2	2	$3\cdot 3^{(\beta-1)/2}-2$
$\alpha\geq 2,\beta\geq 1$	$g=1$	$3g$	$12\cdot 3^{(\beta-1)/2}-4$	$12\cdot 3^{(\beta-1)/2}-4$	$8\cdot 2^{\alpha/2}-4$	$4\cdot 2^{\alpha/2}-4$	$6\cdot 3^{(\beta-1)/2}-4$
	$g\equiv 1\pmod{8}, g\neq 1$		$36\cdot 3^{(\beta-1)/2}-12$	$36\cdot 3^{(\beta-1)/2}-12$	$24\cdot 2^{\alpha/2}-12$	$12\cdot 2^{\alpha/2}-12$	$18\cdot 3^{(\beta-1)/2}-12$
	$g\equiv 5\pmod{8}$		0	0	$8\cdot 2^{\alpha/2}$	$4\cdot 2^{\alpha/2}$	0
	$g\equiv 3\pmod{4}$		$18\cdot 3^{(\beta-1)/2}-6$	$18\cdot 3^{(\beta-1)/2}-6$	$8\cdot 2^{\alpha/2}-6$	$4\cdot 2^{\alpha/2}-6$	$9\cdot 3^{(\beta-1)/2}-6$

Continued on next page

α, β	g	$s(n,4)$	$k(1,1,1;n)$	$k(1,2,2;n)$	$k(1,3,3;n)$	$k(1,6,6;n)$	$k(2,3,6;n)$
α(odd) β(even)							
$\alpha = 1, \beta \geq 0$	$g \equiv 1 \pmod 3$	$2g$	$12 \cdot 3^{\beta/2}$	$4 \cdot 3^{\beta/2}$	0	0	$2 \cdot 3^{\beta/2}$
	$g \equiv 2 \pmod 3$		$24 \cdot 3^{\beta/2} - 12$	$8 \cdot 3^{\beta/2} - 4$	4	4	$4 \cdot 3^{\beta/2} - 4$
$\alpha \geq 3, \beta \geq 0$	$g \equiv 1 \pmod 3$	$2g$	$12 \cdot 3^{\beta/2}$	$12 \cdot 3^{\beta/2}$	0	0	$6 \cdot 3^{\beta/2}$
	$g \equiv 2 \pmod 3$		$24 \cdot 3^{\beta/2} - 12$	$24 \cdot 3^{\beta/2} - 12$	$16 \cdot 2^{(\alpha-1)/2} - 12$	$8 \cdot 2^{(\alpha-1)/2} - 12$	$12 \cdot 3^{\beta/2} - 12$
α(odd) β(odd)							
$\alpha = 1, \beta \geq 1$		$6g$	$18 \cdot 3^{(\beta-1)/2} - 6$	$6 \cdot 3^{(\beta-1)/2} - 2$	2	2	$3 \cdot 3^{(\beta-1)/2} - 2$
$\alpha \geq 3, \beta \geq 1$		$6g$	$18 \cdot 3^{(\beta-1)/2} - 6$	$18 \cdot 3^{(\beta-1)/2} - 6$	$8 \cdot 2^{(\alpha-1)/2} - 6$	$4 \cdot 2^{(\alpha-1)/2} - 6$	$9 \cdot 3^{(\beta-1)/2} - 6$

Table 3.1: Values of $k(1,1,1;n)$, $k(1,2,2;n)$, $k(1,3,3;n)$, $k(1,6,6;n)$, $k(2,3,6;n)$. Here $\Delta = 4$ and $s(n,\Delta) = 2^{\alpha-2[\alpha/2]}3^{\beta-2[\beta/2]}g$.

α,β	g	$s(n,8)$	$k(1,1,2;n)$	$k(1,1,1;n/2)$	$k(2,3,3;n)$	$k(1,3,3;n/2)$	$k(1,3,6;n)$
α(even) β(even)							
$\alpha=0,\beta\geq 0$	$g\equiv 1 \pmod 3$	$2g$	$4\cdot 3^{\beta/2}$	0	0	0	$2\cdot 3^{\beta/2}$
	$g\equiv 2 \pmod 3$		$8\cdot 3^{\beta/2}-4$	0	4	0	$4\cdot 3^{\beta/2}-4$
$\alpha\geq 2,\beta\geq 0$	$g\equiv 1 \pmod 3$	$2g$	$12\cdot 3^{\beta/2}$	$12\cdot 3^{\beta/2}$	0	0	$6\cdot 3^{\beta/2}$
	$g\equiv 2 \pmod 3$		$24\cdot 3^{\beta/2}-12$	$24\cdot 3^{\beta/2}-12$	$8\cdot 2^{\alpha/2}-12$	$8\cdot 2^{\alpha/2}-12$	$12\cdot 3^{\beta/2}-12$
α(even) β(odd)							
$\alpha=0,\beta\geq 1$		$6g$	$6\cdot 3^{(\beta-1)/2}-2$	0	2	0	$3\cdot 3^{(\beta-1)/2}-2$
$\alpha\geq 2,\beta\geq 1$		$6g$	$18\cdot 3^{(\beta-1)/2}-6$	$18\cdot 3^{(\beta-1)/2}-6$	$4\cdot 2^{\alpha/2}-6$	$4\cdot 2^{\alpha/2}-6$	$9\cdot 3^{(\beta-1)/2}-6$
α(odd) β(even)							
$\alpha\geq 1,\beta\geq 0$	$g=1$	g	$12\cdot 3^{\beta/2}-6$	$12\cdot 3^{\beta/2}-6$	$8\cdot 2^{(\alpha-1)/2}-6$	$8\cdot 2^{(\alpha-1)/2}-6$	$6\cdot 3^{\beta/2}-6$
	$g\equiv 1 \pmod{24}, g\neq 1$		$24\cdot 3^{\beta/2}-12$	$24\cdot 3^{\beta/2}-12$	$16\cdot 2^{(\alpha-1)/2}-12$	$16\cdot 2^{(\alpha-1)/2}-12$	$12\cdot 3^{\beta/2}-12$
	$g\equiv 5 \pmod{24}$		$12\cdot 3^{\beta/2}$	$12\cdot 3^{\beta/2}$	0	0	$6\cdot 3^{\beta/2}$
	$g\equiv 7 \pmod{24}$		0	0	$16\cdot 2^{(\alpha-1)/2}$	$16\cdot 2^{(\alpha-1)/2}$	0
	$g\equiv 11 \pmod{24}$		$24\cdot 3^{\beta/2}$	$24\cdot 3^{\beta/2}$	0	0	$12\cdot 3^{\beta/2}$
	$g\equiv 13 \pmod{24}$		$24\cdot 3^{\beta/2}-12$	$24\cdot 3^{\beta/2}-12$	$16\cdot 2^{(\alpha-1)/2}-12$	$16\cdot 2^{(\alpha-1)/2}-12$	$12\cdot 3^{\beta/2}-12$
	$g\equiv 17 \pmod{24}$		$12\cdot 3^{\beta/2}$	$12\cdot 3^{\beta/2}$	0	0	$6\cdot 3^{\beta/2}$
	$g\equiv 19 \pmod{24}$		$48\cdot 3^{\beta/2}-24$	$48\cdot 3^{\beta/2}-24$	$48\cdot 2^{(\alpha-1)/2}-24$	$48\cdot 2^{(\alpha-1)/2}-24$	$24\cdot 3^{\beta/2}-24$
	$g\equiv 23 \pmod{24}$		0	0	0	0	0
α(odd) β(odd)							
$\alpha\geq 1,\beta\geq 1$	$g=1$	$3g$	$12\cdot 3^{(\beta-1)/2}-4$	$12\cdot 3^{(\beta-1)/2}-4$	$8\cdot 2^{(\alpha-1)/2}-4$	$8\cdot 2^{(\alpha-1)/2}-4$	$6\cdot 3^{(\beta-1)/2}-4$
	$g\equiv 1 \pmod 8, g\neq 1$		$36\cdot 3^{(\beta-1)/2}-12$	$36\cdot 3^{(\beta-1)/2}-12$	$24\cdot 2^{(\alpha-1)/2}-12$	$24\cdot 2^{(\alpha-1)/2}-12$	$18\cdot 3^{(\beta-1)/2}-12$
	$g\equiv 5 \pmod 8$		0	0	$8\cdot 2^{(\alpha-1)/2}$	$8\cdot 2^{(\alpha-1)/2}$	0
	$g\equiv 3 \pmod 4$		$18\cdot 3^{(\beta-1)/2}-6$	$18\cdot 3^{(\beta-1)/2}-6$	$8\cdot 2^{(\alpha-1)/2}-6$	$8\cdot 2^{(\alpha-1)/2}-6$	$9\cdot 3^{(\beta-1)/2}-6$

Table 3.2: Values of $k(1,1,2;n)$, $k(1,1,1;n/2)$, $k(2,3,3;n)$, $k(1,3,3;n/2)$, $k(1,3,6;n)$. Here $\Delta=8$ and $s(n,\Delta)=2^{\alpha+1-2[(\alpha+1)/2]}3^{\beta-2[\beta/2]}g$.

α, β	g	$s(n,12)$	$k(1,1,3;n)$	$k(1,1,1;n/3)$	$k(2,2,3;n)$	$k(1,2,2;n/3)$	$k(1,2,6;n)$
			α(even) β(even)				
$\alpha=0,\beta=0$	$g=1$	$3g$	4	0	0	0	2
	$g\equiv 1 \pmod 8, g\neq 1$		12	0	0	0	6
	$g\equiv 5 \pmod 8$		8	0	4	0	0
	$g\equiv 3 \pmod 4$		2	0	2	0	1
$\alpha\geq 2,\beta=0$	$g=1$	$3g$	$8\cdot 2^{\alpha/2}-4$	0	$4\cdot 2^{\alpha/2}-4$	0	2
	$g\equiv 1 \pmod 8, g\neq 1$		$24\cdot 2^{\alpha/2}-12$	0	$12\cdot 2^{\alpha/2}-12$	0	6
	$g\equiv 5 \pmod 8$		$8\cdot 2^{\alpha/2}$	0	$4\cdot 2^{\alpha/2}$	0	0
	$g\equiv 3 \pmod 4$		$8\cdot 2^{\alpha/2}-6$	0	$4\cdot 2^{\alpha/2}-6$	0	3
$\alpha=0,\beta\geq 2$	$g=1$	$3g$	4	$4\cdot 3^{\beta/2}-4$	0	$4\cdot 3^{\beta/2}-4$	$6\cdot 3^{\beta/2}-4$
	$g\equiv 1 \pmod 8, g\neq 1$		12	$12\cdot 3^{\beta/2}-12$	0	$12\cdot 3^{\beta/2}-12$	$18\cdot 3^{\beta/2}-12$
	$g\equiv 5 \pmod 8$		8	0	4	0	0
	$g\equiv 3 \pmod 4$		2	$6\cdot 3^{\beta/2}-6$	2	$2\cdot 3^{\beta/2}-2$	$3\cdot 3^{\beta/2}-2$
$\alpha\geq 2,\beta\geq 2$	$g=1$	$3g$	$8\cdot 2^{\alpha/2}-4$	$4\cdot 3^{\beta/2}-4$	$4\cdot 2^{\alpha/2}-4$	$4\cdot 3^{\beta/2}-4$	$6\cdot 3^{\beta/2}-4$
	$g\equiv 1 \pmod 8, g\neq 1$		$24\cdot 2^{\alpha/2}-12$	$12\cdot 3^{\beta/2}-12$	$12\cdot 2^{\alpha/2}-12$	$12\cdot 3^{\beta/2}-12$	$18\cdot 3^{\beta/2}-12$
	$g\equiv 5 \pmod 8$		$8\cdot 2^{\alpha/2}$	0	$4\cdot 2^{\alpha/2}$	0	0
	$g\equiv 3 \pmod 4$		$8\cdot 2^{\alpha/2}-6$	$6\cdot 3^{\beta/2}-6$	$4\cdot 2^{\alpha/2}-6$	$6\cdot 3^{\beta/2}-6$	$9\cdot 3^{\beta/2}-6$

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α,β	g	$s(n,12)$	$k(1,1,3;n)$	$k(1,1,1;n/3)$	$k(2,2,3;n)$	$k(1,2,2;n/3)$	$k(1,2,6;n)$
α(even) β(odd)							
$\alpha=0,\beta\geq 1$	$g=1$	g	2	$12\cdot 3^{(\beta-1)/2}-6$	2	$4\cdot 3^{(\beta-1)/2}-2$	$6\cdot 3^{(\beta-1)/2}-2$
	$g\equiv 1 \pmod{24}, g\neq 1$		4	$24\cdot 3^{(\beta-1)/2}-12$	4	$8\cdot 3^{(\beta-1)/2}-4$	$12\cdot 3^{(\beta-1)/2}-4$
	$g\equiv 5 \pmod{24}$		0	$12\cdot 3^{(\beta-1)/2}$	0	$4\cdot 3^{(\beta-1)/2}$	$6\cdot 3^{(\beta-1)/2}$
	$g\equiv 7 \pmod{24}$		16	0	8	0	0
	$g\equiv 11 \pmod{24}$		0	$24\cdot 3^{(\beta-1)/2}$	0	$24\cdot 3^{(\beta-1)/2}$	$36\cdot 3^{(\beta-1)/2}$
	$g\equiv 13 \pmod{24}$		4	$24\cdot 3^{(\beta-1)/2}-12$	4	$8\cdot 3^{(\beta-1)/2}-4$	$12\cdot 3^{(\beta-1)/2}-4$
	$g\equiv 17 \pmod{24}$		0	$12\cdot 3^{(\beta-1)/2}$	0	$4\cdot 3^{(\beta-1)/2}$	$6\cdot 3^{(\beta-1)/2}$
	$g\equiv 19 \pmod{24}$		24	$48\cdot 3^{(\beta-1)/2}-24$	0	$48\cdot 3^{(\beta-1)/2}-24$	$72\cdot 3^{(\beta-1)/2}-24$
	$g\equiv 23 \pmod{24}$		0	0	0	0	0
$\alpha\geq 2,\beta\geq 1$	$g=1$	g	$8\cdot 2^{\alpha/2}-6$	$12\cdot 3^{(\beta-1)/2}-6$	$4\cdot 2^{\alpha/2}-6$	$12\cdot 3^{(\beta-1)/2}-6$	$18\cdot 3^{(\beta-1)/2}-6$
	$g\equiv 1 \pmod{24}, g\neq 1$		$16\cdot 2^{\alpha/2}-12$	$24\cdot 3^{(\beta-1)/2}-12$	$8\cdot 2^{\alpha/2}-12$	$24\cdot 3^{(\beta-1)/2}-12$	$36\cdot 3^{(\beta-1)/2}-12$
	$g\equiv 5 \pmod{24}$		0	$12\cdot 3^{(\beta-1)/2}$	0	$12\cdot 3^{(\beta-1)/2}$	$18\cdot 3^{(\beta-1)/2}$
	$g\equiv 7 \pmod{24}$		$16\cdot 2^{\alpha/2}$	0	$8\cdot 2^{\alpha/2}$	0	0
	$g\equiv 11 \pmod{24}$		0	$24\cdot 3^{(\beta-1)/2}$	0	$24\cdot 3^{(\beta-1)/2}$	$36\cdot 3^{(\beta-1)/2}$
	$g\equiv 13 \pmod{24}$		$16\cdot 2^{\alpha/2}-12$	$24\cdot 3^{(\beta-1)/2}-12$	$8\cdot 2^{\alpha/2}-12$	$24\cdot 3^{(\beta-1)/2}-12$	$36\cdot 3^{(\beta-1)/2}-12$
	$g\equiv 17 \pmod{24}$		0	$12\cdot 3^{(\beta-1)/2}$	0	$12\cdot 3^{(\beta-1)/2}$	$18\cdot 3^{(\beta-1)/2}$
	$g\equiv 19 \pmod{24}$		$48\cdot 2^{\alpha/2}-24$	$48\cdot 3^{(\beta-1)/2}-24$	$24\cdot 2^{\alpha/2}-24$	$48\cdot 3^{(\beta-1)/2}-24$	$72\cdot 3^{(\beta-1)/2}-24$
	$g\equiv 23 \pmod{24}$		0	0	0	0	0
α(odd) β(even)							
$\alpha=1,\beta=0$		$6g$	2	0	2	0	1
$\alpha\geq 3,\beta=0$		$6g$	$8\cdot 2^{(\alpha-1)/2}-6$	0	$4\cdot 2^{(\alpha-1)/2}-6$	0	3
$\alpha=1,\beta\geq 2$		$6g$	2	$6\cdot 3^{\beta/2}-6$	2	$2\cdot 3^{\beta/2}-2$	$3\cdot 3^{\beta/2}-2$
$\alpha\geq 3,\beta\geq 2$		$6g$	$8\cdot 2^{(\alpha-1)/2}-6$	$6\cdot 3^{\beta/2}-6$	$4\cdot 2^{(\alpha-1)/2}-6$	$6\cdot 3^{\beta/2}-6$	$9\cdot 3^{\beta/2}-6$
α(odd) β(odd)							
$\alpha=1,\beta\geq 1$	$g\equiv 1 \pmod 3$	$2g$	0	$12\cdot 3^{(\beta-1)/2}$	0	$4\cdot 3^{(\beta-1)/2}$	$6\cdot 3^{(\beta-1)/2}$
	$g\equiv 2 \pmod 3$		4	$24\cdot 3^{(\beta-1)/2}-12$	4	$8\cdot 3^{(\beta-1)/2}-4$	$12\cdot 3^{(\beta-1)/2}-4$
$\alpha\geq 3,\beta\geq 1$	$g\equiv 1 \pmod 3$	$2g$	0	$12\cdot 3^{(\beta-1)/2}$	0	$12\cdot 3^{(\beta-1)/2}$	$18\cdot 3^{(\beta-1)/2}$
	$g\equiv 2 \pmod 3$		$16\cdot 2^{(\alpha-1)/2}-12$	$24\cdot 3^{(\beta-1)/2}-12$	$8\cdot 2^{(\alpha-1)/2}-12$	$24\cdot 3^{(\beta-1)/2}-12$	$36\cdot 3^{(\beta-1)/2}-12$

Table 3.3: Values of $k(1,1,3;n)$, $k(1,1,1;n/3)$, $k(2,2,3;n)$, $k(1,2,2;n/3)$, $k(1,2,6;n)$. Here $\Delta=12$ and $s(n,\Delta)=2^{\alpha-2[\alpha/2]}3^{\beta+1-2[(\beta+1)/2]}g$.

α,β	g	$s(n,24)$	$k(1,1,1;n/6)$	$k(1,1,2;n/3)$	$k(1,1,3;n/2)$	$k(1,1,6;n)$	$k(1,2,3;n)$
α(even) β(even)							
$\alpha=0,\beta=0$		$6g$	0	0	0	2	1
$\alpha=0,\beta\geq 2$		$6g$	0	$2\cdot 3^{\beta/2}-2$	0	2	$3\cdot 3^{\beta/2}-2$
$\alpha\geq 2,\beta=0$		$6g$	0	0	$4\cdot 2^{\alpha/2}-6$	$4\cdot 2^{\alpha/2}-6$	3
$\alpha\geq 2,\beta\geq 2$		$6g$	$6\cdot 3^{\beta/2}-6$	$6\cdot 3^{\beta/2}-6$	$4\cdot 2^{\alpha/2}-6$	$4\cdot 2^{\alpha/2}-6$	$9\cdot 3^{\beta/2}-6$
α(even) β(odd)							
$\alpha=0,\beta\geq 1$	$g\equiv 1 \pmod 3$	$2g$	0	$4\cdot 3^{(\beta-1)/2}$	0	0	$6\cdot 3^{(\beta-1)/2}$
	$g\equiv 2 \pmod 3$		0	$8\cdot 3^{(\beta-1)/2}-4$	0	4	$12\cdot 3^{(\beta-1)/2}-4$
$\alpha\geq 2,\beta\geq 1$	$g\equiv 1 \pmod 3$	$2g$	$12\cdot 3^{(\beta-1)/2}$	$12\cdot 3^{(\beta-1)/2}$	0	0	$18\cdot 3^{(\beta-1)/2}$
	$g\equiv 2 \pmod 3$		$24\cdot 3^{(\beta-1)/2}-12$	$24\cdot 3^{(\beta-1)/2}-12$	$8\cdot 2^{\alpha/2}-12$	$8\cdot 2^{\alpha/2}-12$	$36\cdot 3^{(\beta-1)/2}-12$
α(odd) β(even)							
$\alpha\geq 1,\beta=0$	$g=1$	$3g$	0	0	$8\cdot 2^{(\alpha-1)/2}-4$	$8\cdot 2^{(\alpha-1)/2}-4$	2
	$g\equiv 1 \pmod 8, g\neq 1$		0	0	$24\cdot 2^{(\alpha-1)/2}-12$	$24\cdot 2^{(\alpha-1)/2}-12$	6
	$g\equiv 5 \pmod 8$		0	0	$8\cdot 2^{(\alpha-1)/2}$	$8\cdot 2^{(\alpha-1)/2}$	0
	$g\equiv 3 \pmod 4$		0	0	$8\cdot 2^{(\alpha-1)/2}-6$	$8\cdot 2^{(\alpha-1)/2}-6$	3
$\alpha\geq 1,\beta\geq 2$	$g=1$	$3g$	$4\cdot 3^{\beta/2}-4$	$4\cdot 3^{\beta/2}-4$	$8\cdot 2^{(\alpha-1)/2}-4$	$8\cdot 2^{(\alpha-1)/2}-4$	$6\cdot 3^{\beta/2}-4$
	$g\equiv 1 \pmod 8, g\neq 1$		$12\cdot 3^{\beta/2}-12$	$12\cdot 3^{\beta/2}-12$	$24\cdot 2^{(\alpha-1)/2}-12$	$24\cdot 2^{(\alpha-1)/2}-12$	$18\cdot 3^{\beta/2}-12$
	$g\equiv 5 \pmod 8$		0	0	$8\cdot 2^{(\alpha-1)/2}$	$8\cdot 2^{(\alpha-1)/2}$	0
	$g\equiv 3 \pmod 4$		$6\cdot 3^{\beta/2}-6$	$6\cdot 3^{\beta/2}-6$	$8\cdot 2^{(\alpha-1)/2}-6$	$8\cdot 2^{(\alpha-1)/2}-6$	$9\cdot 3^{\beta/2}-6$
α(odd) β(odd)							
$\alpha\geq 1,\beta\geq 1$	$g=1$	g	$12\cdot 3^{(\beta-1)/2}-6$	$12\cdot 3^{(\beta-1)/2}-6$	$8\cdot 2^{(\alpha-1)/2}-6$	$8\cdot 2^{(\alpha-1)/2}-6$	$18\cdot 3^{(\beta-1)/2}-6$
	$g\equiv 1 \pmod{24}, g\neq 1$		$24\cdot 3^{(\beta-1)/2}-12$	$24\cdot 3^{(\beta-1)/2}-12$	$16\cdot 2^{(\alpha-1)/2}-12$	$16\cdot 2^{(\alpha-1)/2}-12$	$36\cdot 3^{(\beta-1)/2}-12$
	$g\equiv 5 \pmod{24}$		$12\cdot 3^{(\beta-1)/2}$	$12\cdot 3^{(\beta-1)/2}$	0	0	$18\cdot 3^{(\beta-1)/2}$
	$g\equiv 7 \pmod{24}$		0	0	$16\cdot 2^{(\alpha-1)/2}$	$16\cdot 2^{(\alpha-1)/2}$	0
	$g\equiv 11 \pmod{24}$		$24\cdot 3^{(\beta-1)/2}$	$24\cdot 3^{(\beta-1)/2}$	0	0	$36\cdot 3^{(\beta-1)/2}$
	$g\equiv 13 \pmod{24}$		$24\cdot 3^{(\beta-1)/2}-12$	$24\cdot 3^{(\beta-1)/2}-12$	$16\cdot 2^{(\alpha-1)/2}-12$	$16\cdot 2^{(\alpha-1)/2}-12$	$36\cdot 3^{(\beta-1)/2}-12$
	$g\equiv 17 \pmod{24}$		$12\cdot 3^{(\beta-1)/2}$	$12\cdot 3^{(\beta-1)/2}$	0	0	$18\cdot 3^{(\beta-1)/2}$
	$g\equiv 19 \pmod{24}$		$48\cdot 3^{(\beta-1)/2}-24$	$48\cdot 3^{(\beta-1)/2}-24$	$48\cdot 2^{(\alpha-1)/2}-24$	$48\cdot 2^{(\alpha-1)/2}-24$	$72\cdot 3^{(\beta-1)/2}-24$
	$g\equiv 23 \pmod{24}$		0	0	0	0	0

Table 3.4: Values of $k(1,1,1;n/6)$, $k(1,1,2;n/3)$, $k(1,1,3;n/2)$, $k(1,1,6;n)$, $k(1,2,3;n)$. Here $\Delta = 24$ and $s(n,\Delta) = 2^{\alpha+1-2[(\alpha+1)/2]}3^{\beta+1-2[(\beta+1)/2]}g$.

4. Proof of Theorem 1.1

Suppose that $f(\underline{a};z) \in \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, or $\mathcal{C}_6$. Then $c(\underline{a};n)$ is given by (2.14), (2.15), (2.16), or (2.17), respectively. The values of v, w, x, y, z are given in terms of $c(\underline{a};t)$ ($t \in \{1,2,3,4\}$) in (2.19), (2.21), (2.23), (2.25), respectively. The required thirteen formulas for $r(a,b,c;n)$ ($(a,b,c) = (1,1,1)$, $(1,1,2)$, $(1,1,3)$, $(1,1,6)$, $(1,2,2)$, $(1,2,3)$, $(1,2,6)$, $(1,3,3)$, $(1,3,6)$, $(1,6,6)$, $(2,2,3)$, $(2,3,3)$, $(2,3,6)$) are given by (3.1) together with the values of $k(a,b,c;n)$ given in Tables 3.1–3.4. Then, for all $n \in \mathbb{N}$, we have, making use of (1.17) and (1.18),

$$c\left(\underline{a};n\right) = \left(R2^{[\alpha/2]} + S3^{[\beta/2]} + T\right) l(n,\Delta) h\left(\mathbb{Q}\left(\sqrt{-s(n,\Delta)}\right)\right),$$

where the values of R, S, T are given in Tables 4.1–4.4 and $\Delta = 4, 8, 12, 24$ according as $f(\underline{a};z) \in \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, or $\mathcal{C}_6$.

α, β	g	R	S	T
		α(even) β(even) $s(n,4) = g$		
$\alpha = 0, \beta \geq 0$	$g = 1$	0	$12v + 4w + 2z$	$-6v - 2w + 2x + 2y - 2z$
	$g \equiv 1 \pmod{24}, g \neq 1$	0	$24v + 8w + 4z$	$-12v - 4w + 4x + 4y - 4z$
	$g \equiv 5 \pmod{24}$	0	$12v + 4w + 2z$	0
	$g \equiv 7 \pmod{24}$	0	0	$16x + 8y$
	$g \equiv 11 \pmod{24}$	0	$24v + 24w + 12z$	0
	$g \equiv 13 \pmod{24}$	0	$24v + 8w + 4z$	$-12v - 4w + 4x + 4y - 4z$
	$g \equiv 17 \pmod{24}$	0	$12v + 4w + 2z$	0
	$g \equiv 19 \pmod{24}$	0	$48v + 48w + 24z$	$-24v - 24w + 24x - 24z$
	$g \equiv 23 \pmod{24}$	0	0	0
$\alpha \geq 2, \beta \geq 0$	$g = 1$	$8x + 4y$	$12v + 12w + 6z$	-6
	$g \equiv 1 \pmod{24}, g \neq 1$	$16x + 8y$	$24v + 24w + 12z$	-12
	$g \equiv 5 \pmod{24}$	0	$12v + 12w + 6z$	0
	$g \equiv 7 \pmod{24}$	$16x + 8y$	0	0
	$g \equiv 11 \pmod{24}$	0	$24v + 24w + 12z$	0
	$g \equiv 13 \pmod{24}$	$16x + 8y$	$24v + 24w + 12z$	-12
	$g \equiv 17 \pmod{24}$	0	$12v + 12w + 6z$	0
	$g \equiv 19 \pmod{24}$	$48x + 24y$	$48v + 48w + 24z$	-24
	$g \equiv 23 \pmod{24}$	0	0	0
		α(even) β(odd) $s(n,4) = 3g$		
$\alpha = 0, \beta \geq 1$	$g = 1$	0	$12v + 12w + 6z$	$-4v - 4w + 4x - 4z$
	$g \equiv 1 \pmod{8}, g \neq 1$	0	$36v + 36w + 18z$	$-12v - 12w + 12x - 12z$
	$g \equiv 5 \pmod{8}$	0	0	$8x + 4y$
	$g \equiv 3 \pmod{4}$	0	$18v + 6w + 3z$	$-6v - 2w + 2x + 2y - 2z$
$\alpha \geq 2, \beta \geq 1$	$g = 1$	$8x + 4y$	$12v + 12w + 6z$	-4
	$g \equiv 1 \pmod{8}, g \neq 1$	$24x + 12y$	$36v + 36w + 18z$	-12
	$g \equiv 5 \pmod{8}$	$8x + 4y$	0	0
	$g \equiv 3 \pmod{4}$	$8x + 4y$	$18v + 18w + 9z$	-6
		α(odd) β(even) $s(n,4) = 2g$		
$\alpha = 1, \beta \geq 0$	$g \equiv 1 \pmod{3}$	0	$12v + 4w + 2z$	0
	$g \equiv 2 \pmod{3}$	0	$24v + 8w + 4z$	$-12v - 4w + 4x + 4y - 4z$
$\alpha \geq 3, \beta \geq 0$	$g \equiv 1 \pmod{3}$	0	$12v + 12w + 6z$	0
	$g \equiv 2 \pmod{3}$	$16x + 8y$	$24v + 24w + 12z$	-12

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α, β	g	R	S	T
		α(odd) β(odd) $s(n,4) = 6g$		
$\alpha = 1, \beta \geq 1$		0	$18v + 6w + 3z$	$-6v - 2w + 2x + 2y - 2z$
$\alpha \geq 3, \beta \geq 1$		$8x + 4y$	$18v + 18w + 9z$	-6

Table 4.1: Values of R, S, T for $f(\underline{a}; z) \in \mathcal{C}_1$

α, β	g	R	S	T
		α(even) β(even) $s(n,8) = 2g$		
$\alpha = 0, \beta \geq 0$	$g \equiv 1 \pmod 3$	0	$4v + 2z$	0
	$g \equiv 2 \pmod 3$	0	$8v + 4z$	$-4v + 4x - 4z$
$\alpha \geq 2, \beta \geq 0$	$g \equiv 1 \pmod 3$	0	$12v + 12w + 6z$	0
	$g \equiv 2 \pmod 3$	$8x + 8y$	$24v + 24w + 12z$	-12
		α(even) β(odd) $s(n,8) = 6g$		
$\alpha = 0, \beta \geq 1$		0	$6v + 3z$	$-2v + 2x - 2z$
$\alpha \geq 2, \beta \geq 1$		$4x + 4y$	$18v + 18w + 9z$	-6
		α(odd) β(even) $s(n,8) = g$		
$\alpha \geq 1, \beta \geq 0$	$g = 1$	$8x + 8y$	$12v + 12w + 6z$	-6
	$g \equiv 1 \pmod{24}, g \neq 1$	$16x + 16y$	$24v + 24w + 12z$	-12
	$g \equiv 5 \pmod{24}$	0	$12v + 12w + 6z$	0
	$g \equiv 7 \pmod{24}$	$16x + 16y$	0	0
	$g \equiv 11 \pmod{24}$	0	$24v + 24w + 12z$	0
	$g \equiv 13 \pmod{24}$	$16x + 16y$	$24v + 24w + 12z$	-12
	$g \equiv 17 \pmod{24}$	0	$12v + 12w + 6z$	0
	$g \equiv 19 \pmod{24}$	$48x + 48y$	$48v + 48w + 24z$	-24
	$g \equiv 23 \pmod{24}$	0	0	0
		α(odd) β(odd) $s(n,8) = 3g$		
$\alpha \geq 1, \beta \geq 1$	$g = 1$	$8x + 8y$	$12v + 12w + 6z$	-4
	$g \equiv 1 \pmod 8, g \neq 1$	$24x + 24y$	$36v + 36w + 18z$	-12
	$g \equiv 5 \pmod 8$	$8x + 8y$	0	0
	$g \equiv 3 \pmod 4$	$8x + 8y$	$18v + 18w + 9z$	-6

Table 4.2: Values of R, S, T for $f(\underline{a}; z) \in \mathcal{C}_2$

α, β	g	R	S	T
	α(even) β(even) $s(n,12) = 3g$			
$\alpha = 0, \beta = 0$	$g = 1$	0	0	$4v + 2z$
	$g \equiv 1 \pmod 8, g \neq 1$	0	0	$12v + 6z$
	$g \equiv 5 \pmod 8$	0	0	$8v + 4x$
	$g \equiv 3 \pmod 4$	0	0	$2v + 2x + z$
$\alpha \geq 2, \beta = 0$	$g = 1$	$8v + 4x$	0	$-4v - 4x + 2z$
	$g \equiv 1 \pmod 8, g \neq 1$	$24v + 12x$	0	$-12v - 12x + 6z$
	$g \equiv 5 \pmod 8$	$8v + 4x$	0	0
	$g \equiv 3 \pmod 4$	$8v + 4x$	0	$-6v - 6x + 3z$
$\alpha = 0, \beta \geq 2$	$g = 1$	0	$4w + 4y + 6z$	$4v - 4w - 4y - 4z$
	$g \equiv 1 \pmod 8, g \neq 1$	0	$12w + 12y + 18z$	$12v - 12w - 12y - 12z$
	$g \equiv 5 \pmod 8$	0	0	$8v + 4x$
	$g \equiv 3 \pmod 4$	0	$6w + 2y + 3z$	$2v - 6w + 2x - 2y - 2z$
$\alpha \geq 2, \beta \geq 2$	$g = 1$	$8v + 4x$	$4w + 4y + 6z$	-4
	$g \equiv 1 \pmod 8, g \neq 1$	$24v + 12x$	$12w + 12y + 18z$	-12
	$g \equiv 5 \pmod 8$	$8v + 4x$	0	0
	$g \equiv 3 \pmod 4$	$8v + 4x$	$6w + 6y + 9z$	-6
	α(even) β(odd) $s(n,12) = g$			
$\alpha = 0, \beta \geq 1$	$g = 1$	0	$12w + 4y + 6z$	$2v - 6w + 2x - 2y - 2z$
	$g \equiv 1 \pmod{24}, g \neq 1$	0	$24w + 8y + 12z$	$4v - 12w + 4x - 4y - 4z$
	$g \equiv 5 \pmod{24}$	0	$12w + 4y + 6z$	0
	$g \equiv 7 \pmod{24}$	0	0	$16v + 8x$
	$g \equiv 11 \pmod{24}$	0	$24w + 24y + 36z$	0
	$g \equiv 13 \pmod{24}$	0	$24w + 8y + 12z$	$4v - 12w + 4x - 4y - 4z$
	$g \equiv 17 \pmod{24}$	0	$12w + 4y + 6z$	0
	$g \equiv 19 \pmod{24}$	0	$48w + 48y + 72z$	$24v - 24w - 24y - 24z$
	$g \equiv 23 \pmod{24}$	0	0	0
$\alpha \geq 2, \beta \geq 1$	$g = 1$	$8v + 4x$	$12w + 12y + 18z$	-6
	$g \equiv 1 \pmod{24}, g \neq 1$	$16v + 8x$	$24w + 24y + 36z$	-12
	$g \equiv 5 \pmod{24}$	0	$12w + 12y + 18z$	0
	$g \equiv 7 \pmod{24}$	$16v + 8x$	0	0
	$g \equiv 11 \pmod{24}$	0	$24w + 24y + 36z$	0
	$g \equiv 13 \pmod{24}$	$16v + 8x$	$24w + 24y + 36z$	-12
	$g \equiv 17 \pmod{24}$	0	$12w + 12y + 18z$	0
	$g \equiv 19 \pmod{24}$	$48v + 24x$	$48w + 48y + 72z$	-24
	$g \equiv 23 \pmod{24}$	0	0	0
	α(odd) β(even) $s(n,12) = 6g$			
$\alpha = 1, \beta = 0$		0	0	$2v + 2x + z$
$\alpha \geq 3, \beta = 0$		$8v + 4x$	0	$-6v - 6x + 3z$
$\alpha = 1, \beta \geq 2$		0	$6w + 2y + 3z$	$2v - 6w + 2x - 2y - 2z$
$\alpha \geq 3, \beta \geq 2$		$8v + 4x$	$6w + 6y + 9z$	-6
	α(odd) β(odd) $s(n,12) = 2g$			
$\alpha = 1, \beta \geq 1$	$g \equiv 1 \pmod 3$	0	$12w + 4y + 6z$	0
	$g \equiv 2 \pmod 3$	0	$24w + 8y + 12z$	$4v - 12w + 4x - 4y - 4z$
$\alpha \geq 3, \beta \geq 1$	$g \equiv 1 \pmod 3$	0	$12w + 12y + 18z$	0
	$g \equiv 2 \pmod 3$	$16v + 8x$	$24w + 24y + 36z$	-12

Table 4.3: Values of R, S, T for $f(\underline{a}; z) \in \mathcal{C}_3$

α, β	g	R	S	T
α(even) β(even) $s(n, 24) = 6g$				
$\alpha = 0, \beta = 0$		0	0	$2y + z$
$\alpha = 0, \beta \geq 2$		0	$2w + 3z$	$-2w + 2y - 2z$
$\alpha \geq 2, \beta = 0$		$4x + 4y$	0	$-6x - 6y + 3z$
$\alpha \geq 2, \beta \geq 2$		$4x + 4y$	$6v + 6w + 9z$	-6
α(even) β(odd) $s(n, 24) = 2g$				
$\alpha = 0, \beta \geq 1$	$g \equiv 1 \pmod 3$	0	$4w + 6z$	0
	$g \equiv 2 \pmod 3$	0	$8w + 12z$	$-4w + 4y - 4z$
$\alpha \geq 2, \beta \geq 1$	$g \equiv 1 \pmod 3$	0	$12v + 12w + 18z$	0
	$g \equiv 2 \pmod 3$	$8x + 8y$	$24v + 24w + 36z$	-12
α(odd) β(even) $s(n, 24) = 3g$				
$\alpha \geq 1, \beta = 0$	$g = 1$	$8x + 8y$	0	$-4x - 4y + 2z$
	$g \equiv 1 \pmod 8, g \neq 1$	$24x + 24y$	0	$-12x - 12y + 6z$
	$g \equiv 5 \pmod 8$	$8x + 8y$	0	0
	$g \equiv 3 \pmod 4$	$8x + 8y$	0	$-6x - 6y + 3z$
$\alpha \geq 1, \beta \geq 2$	$g = 1$	$8x + 8y$	$4v + 4w + 6z$	-4
	$g \equiv 1 \pmod 8, g \neq 1$	$24x + 24y$	$12v + 12w + 18z$	-12
	$g \equiv 5 \pmod 8$	$8x + 8y$	0	0
	$g \equiv 3 \pmod 4$	$8x + 8y$	$6v + 6w + 9z$	-6
α(odd) β(odd) $s(n, 24) = g$				
$\alpha \geq 1, \beta \geq 1$	$g = 1$	$8x + 8y$	$12v + 12w + 18z$	-6
	$g \equiv 1 \pmod{24}, g \neq 1$	$16x + 16y$	$24v + 24w + 36z$	-12
	$g \equiv 5 \pmod{24}$	0	$12v + 12w + 18z$	0
	$g \equiv 7 \pmod{24}$	$16x + 16y$	0	0
	$g \equiv 11 \pmod{24}$	0	$24v + 24w + 36z$	0
	$g \equiv 13 \pmod{24}$	$16x + 16y$	$24v + 24w + 36z$	-12
	$g \equiv 17 \pmod{24}$	0	$12v + 12w + 18z$	0
	$g \equiv 19 \pmod{24}$	$48x + 48y$	$48v + 48w + 72z$	-24
	$g \equiv 23 \pmod{24}$	0	0	0

Table 4.4: Values of R, S, T for $f(\underline{a}; z) \in \mathcal{C}_6$

5. Examples

A search of $(a_1, a_2, a_3, a_4, a_6, a_8, a_{12}, a_{24}) \in \mathbb{Z}^8$ satisfying (1.2)–(1.5) in the range $-28 \leq a_i \leq 28$, $i \in \{1, 2, 3, 4, 6, 8, 12, 24\}$ yielded 34 eta quotients in $\mathcal{C}_1$, 30 in $\mathcal{C}_2$, 30 in $\mathcal{C}_3$, and 34 in $\mathcal{C}_6$, see Tables 5.1, 5.2, 5.3, and 5.4, respectively. These tables also contain the values of $(v, w, x, y, z) \in \mathbb{Q}^5$ computed using (2.19), (2.21), (2.23), and (2.25), respectively.

We illustrate Theorem 1.1 with four examples, one each from Tables 5.1, 5.2, 5.3, and 5.4. The four examples show how different the formulas for $c(\underline{a}; n)$ for different sequences of n can be.

ID	$(a_1, a_2, a_3, a_4, a_6, a_8, a_{12}, a_{24})$	(v, w, x, y, z)
1	$(-6, 15, 0, -6, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0)$
2	$(-5, 13, 1, -7, -4, 2, 5, -2)$	$\frac{1}{4}(3, -2, -3, 6, 0)$
3	$(-4, 6, 2, 4, -1, -3, -2, 1)$	$\frac{1}{2}(1, 1, 3, -3, 0)$
4	$(-4, 8, 2, -3, 1, 0, -1, 0)$	$\frac{1}{2}(1, 0, 1, 0, 0)$
5	$(-4, 9, 2, -5, -2, 3, 1, -1)$	$\frac{1}{4}(2, -1, 0, 3, 0)$
6	$(-4, 10, 2, -6, -5, 1, 8, -3)$	$\frac{1}{2}(1, -1, -1, 3, 0)$
7	$(-3, 6, -1, 0, 3, 0, -2, 0)$	$\frac{1}{4}(1, 0, 3, 0, 0)$
8	$(-3, 6, 3, -4, -3, 2, 4, -2)$	$\frac{1}{4}(1, 0, -1, 4, 0)$
9	$(-2, -1, 4, 7, 0, -3, -3, 1)$	$\frac{1}{2}(1, -1, 3, -3, 2)$
10	$(-2, 1, 0, 8, 0, -4, 0, 0)$	$(0, 1, 0, 0, 0)$
11	$(-2, 1, 4, 0, 2, 0, -2, 0)$	$\frac{1}{3}(1, -1, 2, -1, 2)$
12	$(-2, 2, 4, -2, -1, 3, 0, -1)$	$\frac{1}{4}(1, -1, 1, 1, 2)$
13	$(-2, 3, 0, 1, 2, -1, 1, -1)$	$\frac{1}{2}(0, 1, 0, 1, 0)$
14	$(-2, 3, 4, -3, -4, 1, 7, -3)$	$\frac{1}{6}(1, -1, -1, 5, 2)$
15	$(-2, 4, 0, -1, -1, 2, 3, -2)$	$\frac{1}{4}(0, 1, 0, 3, 0)$
16	$(-2, 5, -4, -2, 10, 0, -4, 0)$	$(0, 0, 1, 0, 0)$
17	$(-2, 5, 0, -2, -4, 0, 10, -4)$	$(0, 0, 0, 1, 0)$
18	$(-1, -3, 1, 10, 2, -3, -4, 1)$	$\frac{1}{2}(1, -2, 3, -3, 3)$
19	$(-1, -1, 1, 3, 4, 0, -3, 0)$	$\frac{1}{4}(1, -2, 3, -2, 4)$
20	$(-1, 0, 1, 1, 1, 3, -1, -1)$	$\frac{1}{8}(1, -2, 3, 0, 6)$
21	$(-1, 1, 1, 0, -2, 1, 6, -3)$	$\frac{1}{2}(0, 0, 0, 1, 1)$
22	$(-1, 3, -3, -3, 6, 2, 1, -2)$	$\frac{1}{4}(-1, 2, 1, 2, 0)$
23	$(0, -6, -2, 15, 7, -6, -7, 2)$	$(1, -3, 3, -3, 3)$
24	$(0, -4, -2, 8, 9, -3, -6, 1)$	$\frac{1}{2}(1, -3, 3, -3, 4)$
25	$(0, -3, -2, 6, 6, 0, -4, 0)$	$\frac{1}{4}(1, -3, 3, -3, 6)$
26	$(0, -2, -2, 1, 11, 0, -5, 0)$	$\frac{1}{6}(1, -4, 5, -4, 8)$
27	$(0, -2, -2, 5, 3, -2, 3, -2)$	$(0, 0, 0, 0, 1)$
28	$(0, -1, -2, -1, 8, 3, -3, -1)$	$\frac{1}{4}(0, -1, 2, -1, 4)$
29	$(0, 0, -2, -3, 5, 6, -1, -2)$	$\frac{1}{8}(-1, 0, 3, 0, 6)$
30	$(0, 0, -2, -2, 5, 1, 4, -3)$	$\frac{1}{6}(-1, 1, 1, 1, 4)$
31	$(0, 1, -2, -4, 2, 4, 6, -4)$	$\frac{1}{4}(-1, 1, 1, 1, 2)$
32	$(0, 2, -2, -5, -1, 2, 13, -6)$	$\frac{1}{3}(-1, 1, 1, 1, 1)$
33	$(1, -4, -5, 4, 13, 0, -6, 0)$	$\frac{1}{4}(1, -4, 3, -4, 8)$
34	$(2, -5, -8, 2, 20, 0, -8, 0)$	$\frac{1}{3}(1, -4, 2, -4, 8)$

Table 5.1: Eta quotients $\prod_{d|24} \eta^{a_d}(dz) \in \mathcal{C}_1$

ID	$(a_1, a_2, a_3, a_4, a_6, a_8, a_{12}, a_{24})$	(v, w, x, y, z)
1	$(-6, 15, 2, -6, -7, 0, 7, -2)$	$(3, -2, -3, 6, -3)$
2	$(-5, 11, 3, -4, -5, 1, 3, -1)$	$\frac{1}{2}(4, -2, -3, 6, -3)$
3	$(-4, 8, 0, 1, 0, -2, 0, 0)$	$(1, 0, 0, 0, 0)$
4	$(-4, 8, 4, -3, -6, 0, 6, -2)$	$\frac{1}{2}(3, -2, -3, 6, -2)$
5	$(-3, 4, 5, -1, -4, 1, 2, -1)$	$\frac{1}{2}(2, -1, -1, 3, -1)$
6	$(-3, 6, 1, 0, -4, 0, 5, -2)$	$\frac{1}{4}(3, -2, -3, 6, 0)$
7	$(-2, -1, 2, 11, -1, -5, -2, 1)$	$\frac{1}{2}(1, 1, 3, -3, 0)$
8	$(-2, 1, 2, 4, 1, -2, -1, 0)$	$\frac{1}{2}(1, 0, 1, 0, 0)$
9	$(-2, 1, 6, 0, -5, 0, 5, -2)$	$\frac{1}{3}(2, -1, -2, 5, -1)$
10	$(-2, 2, 2, 2, -2, 1, 1, -1)$	$\frac{1}{4}(2, -1, 0, 3, 0)$
11	$(-2, 3, 2, 1, -5, -1, 8, -3)$	$\frac{1}{2}(1, -1, -1, 3, 0)$
12	$(-2, 5, -2, -2, 3, 0, 3, -2)$	$(0, 0, 0, 0, 1)$
13	$(-1, -1, -1, 7, 3, -2, -2, 0)$	$\frac{1}{4}(1, 0, 3, 0, 0)$
14	$(-1, -1, 3, 3, -3, 0, 4, -2)$	$\frac{1}{4}(1, 0, -1, 4, 0)$
15	$(-1, 1, -1, 0, 5, 1, -1, -1)$	$\frac{1}{2}(0, 0, 1, 0, 1)$
16	$(-1, 2, -1, -2, 2, 4, 1, -2)$	$\frac{1}{8}(-1, 0, 3, 0, 6)$
17	$(-1, 3, -1, -3, -1, 2, 8, -4)$	$\frac{1}{4}(-1, 0, 1, 0, 4)$
18	$(0, -6, 0, 15, 0, -6, 0, 0)$	$(0, 1, 0, 0, 0)$
19	$(0, -4, 0, 8, 2, -3, 1, -1)$	$\frac{1}{2}(0, 1, 0, 1, 0)$
20	$(0, -3, 0, 6, -1, 0, 3, -2)$	$\frac{1}{4}(0, 1, 0, 3, 0)$
21	$(0, -2, -4, 5, 10, -2, -4, 0)$	$(0, 0, 1, 0, 0)$
22	$(0, -2, 0, 1, 4, 0, 2, -2)$	$\frac{1}{6}(-1, 2, 1, 2, 2)$
23	$(0, -2, 0, 5, -4, -2, 10, -4)$	$(0, 0, 0, 1, 0)$
24	$(0, -1, 0, -1, 1, 3, 4, -3)$	$\frac{1}{4}(-1, 1, 1, 1, 2)$
25	$(0, 0, 0, -3, -2, 6, 6, -4)$	$\frac{1}{8}(-3, 2, 3, 0, 6)$
26	$(0, 0, 0, -2, -2, 1, 11, -5)$	$\frac{1}{6}(-2, 1, 2, 1, 4)$
27	$(0, 1, 0, -4, -5, 4, 13, -6)$	$\frac{1}{4}(-2, 1, 2, -1, 4)$
28	$(0, 2, 0, -5, -8, 2, 20, -8)$	$\frac{1}{3}(-2, 1, 2, -2, 4)$
29	$(1, -4, -3, 4, 6, 0, 1, -2)$	$\frac{1}{4}(-1, 2, 1, 2, 0)$
30	$(2, -5, -6, 2, 13, 0, -1, -2)$	$\frac{1}{3}(-1, 2, 1, 2, -1)$

Table 5.2: Eta quotients $\prod_{d|24} \eta^{a_d}(dz) \in \mathcal{C}_2$

ID	$(a_1, a_2, a_3, a_4, a_6, a_8, a_{12}, a_{24})$	(v, w, x, y, z)
1	$(-8, 20, 2, -8, -5, 0, 2, 0)$	$(-2, 3, 4, -12, 8)$
2	$(-6, 13, 4, -5, -4, 0, 1, 0)$	$\frac{1}{2}(-1, 3, 4, -12, 8)$
3	$(-5, 11, 1, -2, -2, 0, 0, 0)$	$\frac{1}{4}(1, 3, 4, -12, 8)$
4	$(-4, 6, 6, -2, -3, 0, 0, 0)$	$(0, 1, 1, -3, 2)$
5	$(-4, 8, 2, -1, -3, -1, 3, -1)$	$\frac{1}{2}(0, 0, 1, -3, 4)$
6	$(-4, 10, -2, -4, 5, 0, -2, 0)$	$(1, 0, 0, 0, 0)$
7	$(-3, 4, 3, 1, -1, 0, -1, 0)$	$\frac{1}{4}(1, 3, 2, -6, 4)$
8	$(-3, 8, -1, -5, 1, 2, 3, -2)$	$\frac{1}{4}(3, -3, -2, 6, 0)$
9	$(-2, -1, 0, 13, 2, -6, -5, 2)$	$(1, 3, 1, -3, -1)$
10	$(-2, 1, 0, 6, 4, -3, -4, 1)$	$\frac{1}{2}(1, 3, 1, -3, 0)$
11	$(-2, 1, 4, 2, -2, -1, 2, -1)$	$\frac{1}{2}(0, 0, 1, -1, 2)$
12	$(-2, 2, 0, 4, 1, 0, -2, 0)$	$\frac{1}{4}(1, 3, 1, -3, 2)$
13	$(-2, 3, 0, -1, 6, 0, -3, 0)$	$\frac{1}{2}(1, 1, 0, 0, 0)$
14	$(-2, 3, 0, 3, -2, -2, 5, -2)$	$(0, 0, 0, 0, 1)$
15	$(-2, 4, 0, -3, 3, 3, -1, -1)$	$\frac{1}{4}(2, 0, -1, 3, 0)$
16	$(-2, 5, 0, -5, 0, 6, 1, -2)$	$\frac{1}{8}(5, -3, -4, 12, -2)$
17	$(-2, 5, 0, -4, 0, 1, 6, -3)$	$\frac{1}{2}(1, -1, -1, 3, 0)$
18	$(-2, 6, 0, -6, -3, 4, 8, -4)$	$\frac{1}{4}(3, -3, -3, 9, -2)$
19	$(-2, 7, 0, -7, -6, 2, 15, -6)$	$(1, -1, -1, 3, -1)$
20	$(-1, -1, 1, 5, 0, -1, 1, -1)$	$\frac{1}{2}(0, 0, 1, 0, 1)$
21	$(-1, 1, -3, 2, 8, 0, -4, 0)$	$\frac{1}{4}(1, 3, 0, 0, 0)$
22	$(-1, 1, 1, -2, 2, 2, 2, -2)$	$\frac{1}{4}(1, -1, 0, 4, 0)$
23	$(-1, 2, 1, -4, -1, 5, 4, -3)$	$\frac{1}{8}(3, -3, -2, 12, -2)$
24	$(-1, 3, 1, -5, -4, 3, 11, -5)$	$\frac{1}{2}(1, -1, -1, 4, -1)$
25	$(0, -4, -2, 10, 5, -4, -2, 0)$	$(0, 0, 1, 0, 0)$
26	$(0, -2, -2, 3, 7, -1, -1, -1)$	$\frac{1}{2}(0, 0, 1, 1, 0)$
27	$(0, -1, -2, 1, 4, 2, 1, -2)$	$\frac{1}{4}(0, 0, 1, 3, 0)$
28	$(0, 0, -6, 0, 15, 0, -6, 0)$	$(0, 1, 0, 0, 0)$
29	$(0, 0, -2, 0, 1, 0, 8, -4)$	$(0, 0, 0, 1, 0)$
30	$(1, -2, -5, -1, 11, 2, -1, -2)$	$\frac{1}{4}(-1, 1, 2, 2, 0)$

Table 5.3: Eta quotients $\prod_{d\mid 24} \eta^{a_d}(dz) \in \mathcal{C}_3$

ID	$(a_1, a_2, a_3, a_4, a_6, a_8, a_{12}, a_{24})$	(v, w, x, y, z)
1	$(-6, 13, 2, -1, -5, -2, 2, 0)$	$(-6, 3, 2, 1, 1)$
2	$(-4, 6, 4, 2, -4, -2, 1, 0)$	$\frac{1}{2}(-6, 3, 2, 1, 2)$
3	$(-4, 10, 0, -4, -2, 0, 5, -2)$	$(0, 0, 0, 1, 0)$
4	$(-3, 4, 1, 5, -2, -2, 0, 0)$	$\frac{1}{4}(-6, 3, 2, 1, 4)$
5	$(-3, 6, 1, -2, 0, 1, 1, -1)$	$\frac{1}{2}(0, 0, 0, 1, 1)$
6	$(-3, 7, 1, -4, -3, 4, 3, -2)$	$\frac{1}{8}(6, -3, -2, 5, 2)$
7	$(-3, 8, 1, -5, -6, 2, 10, -4)$	$\frac{1}{4}(6, -3, -2, 3, 0)$
8	$(-2, -1, 6, 5, -3, -2, 0, 0)$	$(-1, 0, 1, 0, 1)$
9	$(-2, 1, 2, 6, -3, -3, 3, -1)$	$\frac{1}{2}(-3, 3, 1, 1, 0)$
10	$(-2, 3, -2, 3, 5, -2, -2, 0)$	$(0, 0, 0, 0, 1)$
11	$(-2, 3, 2, -1, -1, 0, 4, -2)$	$\frac{1}{2}(0, 1, 0, 1, 0)$
12	$(-2, 4, 2, -3, -4, 3, 6, -3)$	$\frac{1}{4}(3, 0, -1, 2, 0)$
13	$(-2, 5, 2, -4, -7, 1, 13, -5)$	$\frac{1}{2}(3, -1, -1, 1, 0)$
14	$(-1, -3, 3, 8, -1, -2, -1, 0)$	$\frac{1}{4}(0, -3, 4, -1, 4)$
15	$(-1, -1, 3, 1, 1, 1, 0, -1)$	$\frac{1}{2}(1, -1, 1, 0, 1)$
16	$(-1, 0, 3, -1, -2, 4, 2, -2)$	$\frac{1}{8}(6, -3, 2, 1, 2)$
17	$(-1, 1, -1, 2, 1, 0, 3, -2)$	$\frac{1}{4}(0, 3, 0, 1, 0)$
18	$(-1, 1, 3, -2, -5, 2, 9, -4)$	$\frac{1}{4}(4, -1, 0, 1, 0)$
19	$(0, -8, 0, 20, 2, -8, -5, 2)$	$(3, -6, 2, -2, 4)$
20	$(0, -6, 0, 13, 4, -5, -4, 1)$	$\frac{1}{2}(3, -6, 3, -2, 4)$
21	$(0, -5, 0, 11, 1, -2, -2, 0)$	$\frac{1}{4}(3, -6, 5, -2, 4)$
22	$(0, -4, 0, 6, 6, -2, -3, 0)$	$\frac{1}{2}(2, -3, 2, -1, 2)$
23	$(0, -4, 0, 10, -2, -4, 5, -2)$	$(0, 0, 1, 0, 0)$
24	$(0, -3, 0, 4, 3, 1, -1, -1)$	$\frac{1}{4}(3, -3, 3, -1, 2)$
25	$(0, -2, 0, 2, 0, 4, 1, -2)$	$\frac{1}{8}(6, -3, 4, -1, 2)$
26	$(0, -2, 0, 3, 0, -1, 6, -3)$	$\frac{1}{2}(1, 0, 1, 0, 0)$
27	$(0, -1, 0, 1, -3, 2, 8, -4)$	$\frac{1}{4}(3, 0, 1, 0, 0)$
28	$(0, 0, -4, 0, 8, 0, 1, -2)$	$(0, 1, 0, 0, 0)$
29	$(0, 0, 0, 0, -6, 0, 15, -6)$	$(1, 0, 0, 0, 0)$
30	$(1, -6, -3, 9, 8, -2, -4, 0)$	$\frac{1}{4}(6, -9, 6, -3, 4)$
31	$(1, -4, -3, 2, 10, 1, -3, -1)$	$\frac{1}{2}(2, -2, 2, -1, 1)$
32	$(1, -3, -3, 0, 7, 4, -1, -2)$	$\frac{1}{8}(6, -3, 6, -3, 2)$
33	$(1, -2, -3, -1, 4, 2, 6, -4)$	$\frac{1}{4}(2, 1, 2, -1, 0)$
34	$(2, -7, -6, 7, 15, -2, -6, 0)$	$(2, -3, 2, -1, 1)$

Table 5.4: Eta quotients $\prod_{d|24} \eta^{a_d}(dz) \in \mathcal{C}_6$

Example 5.1. *For our first illustration of Theorem 1.1 we consider eta quotient no. 34 in Table 5.1, which has* $\underline{a} := (2, -5, -8, 2, 20, 0, -8, 0)$, *namely*

$$f(\underline{a}; z) := \prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-q^{4n})^2(1-q^{6n})^{20}}{(1-q^{2n})^5(1-q^{3n})^8(1-q^{12n})^8} \in \mathcal{C}_1. \tag{5.1}$$

This eta quotient has

$$(v, w, x, y, z) = \left(\frac{1}{3}, -\frac{4}{3}, \frac{2}{3}, -\frac{4}{3}, \frac{8}{3}\right).$$

We now use Theorem 1.1 to determine the coefficient of q^n in the Fourier expansion of $f(\underline{a}; z)$ for all $n \in \mathbb{N}$ of the form

$$n = 2^{2r}3^{2s}13, \quad r \in \mathbb{N}, s \in \mathbb{N}_0.$$

By (1.13) *we have* $\alpha = 2r$, $\beta = 2s$, $g = 13$ *and* $h = 1$. *The relevant ternary quadratic form is* $x^2 + y^2 + z^2$, *which has discriminant* $\Delta = 4$, *so by* (1.15) *we have* $s(n, \Delta) = \mathrm{sqf}(2^{2r}3^{2s}13 \cdot 2^2) = 13$ *and thus the classnumber* $h\left(\mathbb{Q}\left(\sqrt{-s(n,\Delta)}\right)\right) = h\left(\mathbb{Q}\left(\sqrt{-13}\right)\right) = 2$. *By* (1.16), *as* $h = 1$, *we have* $l(n, \Delta) = l(2^{2r}3^{2s}13, 4) = 1$. *Appealing to Table 4.1 we see that*

$$R = 16x + 8y = 0,$$
$$S = 24v + 24w + 12z = 8,$$
$$T = -12.$$

Hence, by Theorem 1.1, we obtain

$$c\left(\underline{a}; 2^{2r}3^{2s}13\right) = \left(R2^r + S3^s + T\right)2 = 16 \cdot 3^s - 24, \quad r \in \mathbb{N}, s \in \mathbb{N}_0.$$

Thus, for example, the coefficient of $q^{9937354752}$ $(r = 10, s = 3)$ *in the expansion of the eta quotient* (5.1) *is* 408. *To the authors' knowledge, the computation of this coefficient is beyond the capabilities of software packages such as Maple.*

Example 5.2. *Our second example is no. 4 in Table 5.2, namely*

$$f(\underline{a}; z) := \prod_{n=1}^{\infty} \frac{(1-q^{2n})^8(1-q^{3n})^4(1-q^{12n})^6}{(1-q^n)^4(1-q^{4n})^3(1-q^{6n})^6(1-q^{24n})^2} \in \mathcal{C}_2,$$

where $\underline{a} := (-4, 8, 4, -3, -6, 0, 6, -2)$ *and*

$$(v, w, x, y, z) = \left(\frac{3}{2}, -1, -\frac{3}{2}, 3, -1\right).$$

We use Theorem 1.1 to determine the coefficients of q^n *in the Fourier expansion of* $f(\underline{a}; z)$ *for all* $n \equiv 10 \pmod{12}$. *We set* $n = 12m - 2$, *where* $m \in \mathbb{N}$. *Appealing to* (1.13) *we have*

$$2(6m - 1) = n = 2^\alpha 3^\beta g h^2,$$

from which we deduce as $h^2 \equiv 1 \pmod{12}$

$$\alpha = 1, \beta = 0, g \equiv 5 \pmod{6}.$$

Then from Table 4.2 we have as $12v + 12w + 6z = 12\left(\frac{3}{2}\right) + 12(-1) + 6(-1) = 18 - 12 - 6 = 0$ *that* $R = S = T = 0$ *for* $g \equiv 5, 11, 17, 23 \pmod{24}$. *Hence, by Theorem 1.1 we have*

$$c(\underline{a}; n) = 0, \quad n \equiv 10 \pmod{12}.$$

Example 5.3. *For our third example, we consider*

$$f(\underline{a}; z) := \prod_{n=1}^{\infty} \frac{(1-q^{2n})^6(1-q^{3n})^6}{(1-q^n)^4(1-q^{4n})^2(1-q^{6n})^3} \in \mathcal{C}_3,$$

where

$$\underline{a} := (-4, 6, 6, -2, -3, 0, 0, 0),$$

which is no. 4 in Table 5.3. From the table we have

$$(v, w, x, y, z) = (0, 1, 1, -3, 2).$$

We determine the coefficient of q^n *in the Fourier expansion of* $f(\underline{a}; z)$ *for all* $n \in \mathbb{N}$ *of the form* $n = 3p^2$, *where* p *is a prime not equal to* 2 *or* 3. *By* (1.13) *we have* $\alpha = 0$, $\beta = 1$, $g = 1$, *and*

$h = p$. The relevant ternary quadratic form is $x^2+y^2+3z^2$ of discriminant $\Delta = 12$. Hence, $s(n,\Delta) = s(3p^2,12) = \mathrm{sqf}(36p^2) = 1$. As $h = p$, by (1.16) we have

$$l(n,\Delta) = l(3p^2,12) = \sigma(p) - \left(\frac{-1}{p}\right)\sigma(1) = p+1-(-1)^{\frac{p-1}{2}}.$$

Appealing to Table 4.3 we see that

$$\begin{aligned} R &= 0, \\ S &= 12w+4y+6z = 12, \\ T &= 2v-6w+2x-2y-2z = -2, \end{aligned}$$

so, by Theorem 1.1, we have

$$\begin{aligned} c\left(\underline{a};3p^2\right) &= \left(R2^{[\alpha/2]}+S3^{[\beta/2]}+T\right) l(n,\Delta) h\left(\mathbb{Q}\left(\sqrt{-s(n,\Delta)}\right)\right) \\ &= (0+12-2)\left(p+1-(-1)^{\frac{p-1}{2}}\right) h\left(\mathbb{Q}\left(\sqrt{-1}\right)\right) \end{aligned}$$

that is

$$c\left(\underline{a};3p^2\right) = 10\left(p+1-(-1)^{\frac{p-1}{2}}\right), \quad p \text{ (prime)} \neq 2,3.$$

Example 5.4. *In our fourth and final application of Theorem 1.1, we consider the eta quotient*

$$f(\underline{a};z) := \prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-q^{4n})^7(1-q^{6n})^{15}}{(1-q^{2n})^7(1-q^{3n})^6(1-q^{8n})^2(1-q^{12n})^6} \in \mathcal{C}_6,$$

where

$$\underline{a} := (2,-7,-6,7,15,-2,-6,0),$$

which is no. 34 in Table 5.4. From the table we see that

$$(v,w,x,y,z) = (2,-3,2,-1,1).$$

We use Theorem 1.1 to determine the coefficient of q^n in the Fourier expansion of $f(\underline{a};z)$ for all $n \in \mathbb{N}$ of the form $n = p$, where p is a prime not equal to 2 or 3. By (1.13) we have $\alpha = 0$, $\beta = 0$, $g = p$, and $h = 1$. The relevant ternary quadratic form is $x^2+y^2+6z^2$ so that $\Delta = 24$. Hence, we have $s(n,\Delta) = s(p,24) = \mathrm{sqf}(p24) = 6p$. As $h = 1$, by (1.16) we have

$$l(n,\Delta) = l(p,24) = 1.$$

Appealing to Table 4.4 we see that

$$R = 0, \quad S = 0, \quad T = 2y+z = -1.$$

Hence, by Theorem 1.1, we have

$$c(\underline{a};p) = (R+S+T)l(p,24)h\left(\mathbb{Q}\left(\sqrt{-s(p,24)}\right)\right),$$

that is

$$c(\underline{a};p) = -h\left(\mathbb{Q}\left(\sqrt{-6p}\right)\right), \quad p \text{ (prime)} \neq 2,3.$$

We conclude by remarking that our work has shown how modular forms can benefit from classical results about ternary quadratic forms.

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Florian Münkel
Saint Mary's University
Department of Finance, Information Systems, and Management Science
923 Robie Street
Halifax, B3H 3C3, NS, Canada
e-mail: Florian.Muenkel@smu.ca

Kenneth S. Williams
Carleton University
Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics
1125 Colonel By Drive
Ottawa, K1S 5B6, ON, Canada
e-mail: KennethWilliams@cunet.carleton.ca