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A REMARK ON A THEOREM OF A.E.INGHAM
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ABSTRACT. The theorem of INGHAM $[\text{AEI}]_1$ refered to is:

For all $N \geq N_0$ (an absolute constant) the inequality

$$N^3 \leq p \leq (N+1)^3$$

is solvable in a prime p. (It may be noted that the corresponding theorem for squares is an open question even if we assume RIEMANN HYPOTHESIS). Actually INGHAM proved more, namely

$$\pi(x+h) - \pi(x) \sim h(\log x)^{-1}$$

where $h = x^c$, where $c(> \frac{5}{8})$ is any constant. The purpose of this note is to point out that even this stronger form can be proved without using the functional equation of $\zeta(s)$.

§1.**INTRODUCTION.** The three main ingredients in the proof of INGHAM'S theorems are

(A) I.M.VINOGRADOV's deep result

$$\zeta(s) \neq 0 \ (s = \sigma + it), \ \sigma \geq 1 - K_1(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}}, \tag{1}$$

$t \geq 100$, where $K_1 > 0$ is an absol]ute constant.

(B) Explicit formula for

$$\sum_{p \leq x} \log p \ ([\text{AEI}]_2). \tag{2}$$

(C)

$$N(\sigma,T) < T^{(\frac{8}{3})(1-\sigma)}\ (\log T)^{100} \tag{3}$$

where $\frac{1}{2} \leq \sigma \leq 1,\ T \geq 1000$. The precise power of $\log T$ is unimportant. Any constant in place of 100 will do. $N(\sigma,T)$ denotes the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $\beta \geq \sigma$ and $|\gamma| \leq T$.

1) The toughest part is (A). It follows from the deep result
(due to I.M.VINOGRADOV)

$$|\zeta(\sigma+it)| \leq (t^{(1-\sigma)^{\frac{3}{2}}}\ \log t)^{K_2}\ \ (\frac{1}{2} \leq \sigma \leq 1,\ t \geq 100) \tag{4}$$

where $K_2 > 0$ is an absolute constant and (1) follows from this in a relatively simple way by a method due to E.LANDAU$[KR]_1$. For a proof of (4) without using the functional equation see [KR,AS].

2) Explicit formula uses the functional equation, but an alternative approach is due to $[KR]_2$ by the introduction of HOOLEY-HUXLEY contour.

3) The proof of (3) uses

$$\zeta(\frac{1}{2}+it) = O(t^{\frac{1}{6}} \log t), t \geq 100, \tag{5}$$

where the O-constant is absolute. The main work in the present note is to sketch a proof of this without using the functional equation of $\zeta(s)$.

§2. **<u>SOME REMARKS</u>** In fact we write

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s}\ \ (0<\alpha\leq 1,\ s=\sigma+it,\ \sigma>1), \tag{6}$$

and next if X is any positive integer we have

$$\zeta(s,\alpha) = \alpha^{-s}\ - \sum_{n=1}^{X}(n+\alpha)^{-s}\ + \sum_{n>X}((n+\alpha)^{-s} - \int_n^{n+1} \frac{du}{(u+\alpha)^s}) + \int_{X+1}^{\infty}(u+\alpha)^{-s}\ du. \tag{7}$$

Since the last term in (7) is

$$\frac{(X+1+\alpha)^{1-s}}{s-1} \tag{8}$$

and the rest is analytic in $\sigma > 0$, (7) gives the analytic continuation in $\sigma > 0$ of (6). We prove our main theorem which is as follows.

THEOREM. *We have*

$$\zeta(\frac{1}{2} + it, \alpha) - \alpha^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}} \log t), (t \geq 10) \tag{9}$$

uniformly in the real parameter α. *(Note that* $\zeta(s,1) = \zeta(s)$*).*

§3. **PROOF OF THE THEOREM** We use van-der Corput's theorems (Theorems 5.9 and 5.11 of [ECT]) and after the proof of the theorem we make some comments about the Weyl-Hardy-Littlewood method of proof of (9).

THEOREM 5.9. *If* $f(x)$ *is real and twice continuously differentiable and*

$$0 < \lambda_2 \leq f''(x) \leq h\lambda_2 (\textit{or } 0 < \lambda_2 \leq -f''(x) \leq h\lambda_2)$$

throughout the interval (a, b) *and* $b \geq a + 1$, *then*

$$\sum_{a<n\leq b} e^{2\pi i f(n)} = O(h(b-a)\lambda_2^{\frac{1}{2}}) + O(\lambda_2^{-\frac{1}{2}}). \tag{10}$$

THEOREM 5.11. *If* $f(x)$ *is real and thrice continuously differentiable and*

$$0 < \lambda_3 \leq f'''(x) \leq h\lambda_3 \ (\textit{ or } 0 < \lambda_3 \leq -f'''(x) \leq h\lambda_3)$$

throughout the interval (a,b) and $b \geq a + 1$, *then*

$$\sum_{a<n\leq b} e^{2\pi f(n)} = O(h^{\frac{1}{2}}(b-a)\lambda_3^{\frac{1}{6}}) + O((b-a)^{\frac{1}{2}}\lambda_3^{-\frac{1}{6}}). \tag{11}$$

We now apply these to

$$E \equiv \sum_{a\leq n\leq b(\leq 2a)} (n+\alpha)^{-it} \text{ with } a \geq 10. \tag{12}$$

Here $f(x) = -\frac{t}{2\pi}\log(x+\alpha)$. We have

$$f'(x) = -\frac{t}{2\pi(x+\alpha)}$$

$$f''(x) = \frac{t}{2\pi(x+\alpha)^2}$$
and
$$f'''(x) = -\frac{2t}{2\pi(x+\alpha)^3}.$$

Thus

$$\begin{array}{l} C_1 \leq f''(x)a^2t^{-1} \leq C_2 \\ and\ C_3 \leq f'''(x)a^3t^{-1} \leq C_4 \end{array} \tag{13}$$

where C_1, C_2, C_3 and C_4 are absolute positive constants. Thus we have

$$\sum_{a<n\leq b(\leq 2a)} (n+\alpha)^{-it} = O(t^{\frac{1}{2}}) + O(at^{-\frac{1}{2}}) \tag{14}$$

$$\sum_{a<n\leq b(\leq 2a)} (n+\alpha)^{-it} = O(t^{\frac{1}{6}}a^{\frac{1}{2}}) + O(t^{-\frac{1}{6}}a). \tag{15}$$

Hence by partial summation we have

$$\sum_{a<n\leq b(\leq 2a)} (n+\alpha)^{-\frac{1}{2}-it} = O((\frac{t}{a})^{\frac{1}{2}}) + O((\frac{a}{t})^{\frac{1}{2}}) \tag{16}$$

and

$$t\left(\sum_{a<n\leq b(\leq 2a)} (n+\alpha)^{-\frac{3}{2}-it}\right) = O((\frac{t}{a})^{\frac{3}{2}}) + O((\frac{t}{a})^{\frac{1}{2}}). \tag{17}$$

Also we need

$$\sum_{a<n\leq b(\leq 2a)} (n+\alpha)^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}}) + O(t^{-\frac{1}{6}}a^{\frac{1}{2}}) \tag{18}$$

which follows from (15). From (18) there follows

$$\sum_{1\leq n\leq t^{\frac{2}{3}}} (n+\alpha)^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}}\log t). \tag{19}$$

From (16) there follows

$$\sum_{t^{\frac{2}{3}} \le n \le t^{\frac{4}{3}}} (n+\alpha)^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}} \log t) \tag{20}$$

Thus

$$\sum_{1 \le n \le t^{\frac{4}{3}}} (n+\alpha)^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}} \log t). \tag{21}$$

We now fix $X = [t^{\frac{4}{3}}]$. The term (8) contributes $O(t^{-\frac{1}{3}})$. We note that (with $s = \frac{1}{2} + it$)

$$\begin{aligned} &\sum_{n>X} \left((n+\alpha)^{-s} - \int_n^{n+1} \frac{du}{(u+\alpha)^s} \right) \\ &= \sum_{n>X} \int_n^{n+1} \left((n+\alpha)^{-s} - (u+\alpha)^{-s} \right) du \\ &= \sum_{n>X} s \int_0^1 \left(\int_0^u (n+v+\alpha)^{-s-1} dv \right) du \end{aligned}$$

and so its absolute value is

$$O(\sum_{a>t^{\frac{4}{3}}} \left((\frac{t}{a})^{\frac{3}{2}} + (\frac{t}{a})^{\frac{1}{2}} \right) = O(t^{-\frac{1}{6}}).$$

This proves our main theorem.

REMARK 1 Let X be an arbitrary positive integer $\ge 20(|t|+20)(K+1)$. Then by iteration of the method by which we continued $\zeta(s,\alpha)$ in $\sigma > 0$ (incidentally the method is due to E.LANDAU (Handbuch der primzahlen) we can get the analytic continuation in $|\sigma| \le (K+1)$ (K being arbitrary constant) and also the inequality

$$\zeta(s,\alpha) = \alpha^{-s} + \sum_{n \le X} (n+\alpha)^{-s} + \frac{X^{1-s}}{s-1} + O(X^{-\sigma})$$

where $s = \sigma + it$ (σ arbitrary). (O constant depends on K). For this see $[\mathrm{KR}]_1$.

REMARK 2 A remark on Weyl-Hardy-Littlewood method is necessary here. The proof of Theorem 5.5 of [ECT] goes through to prove

$\sum_{1\leq n\leq t^{\frac{2}{3}}}(n+\alpha)^{-\frac{1}{2}-it}=O(t^{\frac{1}{6}}L)$ except for trival complications arising from the presence of the real parameter α. This uses the integer parameter k to be 2. However if we use the case $k=1$ simple computations show that

$$\sum_{t^{\frac{2}{3}}\leq n\leq Ct}(n+\alpha)^{-\frac{1}{2}-it}=O(t^{\frac{1}{6}}L)$$

whatever the constant $C\geq 10$ be. Here L is some fixed power of log t. These considerations prove the main theorem in view of Remark 1 above. We stress once again that functional equations for $\zeta(s)$ or $\zeta(s,\alpha)$ are not necessary in the proof of INGHAM's theorems. L can be any fixed power of log t and this is enough to prove INGHAM's asymptotic formula mentioned in the abstract.

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