THE NUMBER OF IMAGINARY QUADRATIC FIELDS WITH A GIVEN CLASS NUMBER

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Gauss asked for a list of imaginary quadratic fields with class number one. This problem inspired a great deal of work; some of the prominent milestones being the work of Heilbronn showing that there are only finitely many fields with a given class number, the work of Landau and Siegel providing good (but ineffective) lower bounds for the class number, the work of Heegner and Stark showing that there are exactly nine fields with class number 1, and the effective resolution of the class number problem due to Goldfeld, Gross and Zagier. In this note we investigate the number, $\mathcal{F}(h)$, of imaginary quadratic fields with class number h; thus, $\mathcal{F}(1) = 9$ is the celebrated Heegner-Stark result. From the Landau-Siegel theorem one could compute $\mathcal{F}(h)$ up to an error of 1 relatively easily. The Goldfeld-Gross-Zagier result permits, with great effort, the calculation of $\mathcal{F}(h)$ for any given h, and Watkins [5] has accomplished this for all $h \leq 100$. What is the asymptotic behavior of $\mathcal{F}(h)$ for large h? This question is independent of the Landau-Siegel zero issue; nevertheless it seems difficult to answer. We establish here an asymptotic formula for the average value of $\mathcal{F}(h)$, a modest non-trivial upper bound for $\mathcal{F}(h)$ (together with an application to a question of Rosen and Silverman on odd parts of class numbers), and we speculate on the nature of $\mathcal{F}(h)$.

Throughout we let $-d$ denote a negative fundamental discriminant, χ_{-d} will denote the associated primitive quadratic character (mod $|d|$), and $h(-d)$ will denote the class number of $\mathbb{Q}(\sqrt{-d})$. When $d > 4$ recall that Dirichlet's class number formula gives

$$h(-d) = \sqrt{d}L(1, \chi_{-d})/\pi.$$

Typically $L(1, \chi_{-d})$ has constant size; rarely does it fall outside the range $(1/10, 10)$. Therefore we would expect that class numbers below H arise mainly from fields with discriminants of size about H^2, and the number of such fields should be asymptotically a constant times H^2.

Theorem 1. *As $H \to \infty$ we have*

$$\sum_{h \leq H} \mathcal{F}(h) = \frac{3\zeta(2)}{\zeta(3)} H^2 + O\left(H^2(\log H)^{-\frac{1}{2}+\epsilon}\right).$$

Theorem 2. *For large H we have*

$$\mathcal{F}(H) \ll H^2 \frac{(\log \log H)^4}{\log H}.$$

¿From Watkins [5] we know that there are 42272 fields with class number below 100; the main term of the asymptotic in Theorem 1 is approximately 41053. By modifying our argument one could improve the error term in the asymptotic formula of Theorem 1 to $O(H^2(\log H)^{-1+\epsilon})$. Some new ideas seem needed to improve the power of $\log h$ appearing in Theorem 2.

We expect that $\mathcal{F}(h)$ is of size about h (the average size), although there is some variation. More precisely, we conjecture that

$$(C1) \qquad\qquad \frac{h}{\log h} \ll \mathcal{F}(h) \ll h \log h.$$

Our heuristic reasoning is as follows. Let 2^λ denote the exact power of 2 dividing h. By genus theory, if the class number is h then the fundamental discriminant $-d$ can have at most $(\lambda + 1)$ prime factors if $-d \equiv 1 \pmod 4$, and $-d/4$ can have at most $(\lambda + 1)$ prime factors if $-d \equiv 0 \pmod 4$. By the class number formula we also know that these discriminants are essentially of size h^2. If $\ell \leq \lambda + 1$ then there are $\asymp \frac{h^2}{\log h} \frac{(\log \log h)^{\ell-1}}{(\ell-1)!}$ fundamental discriminants of size h^2 with $-d$ (or $-d/4$) divisible by exactly ℓ primes. For such discriminants the class number is of size about h, and constrained to be a multiple of $2^{\ell-1}$. Thus we may think of the probability of the class number being exactly h as roughly $2^{\ell-1}/h$. In other words we expect that there are $\asymp 2^{\ell-1}h(\log \log h)^{\ell-1}/((\ell-1)! \log h)$ fields with d (or $d/4$) composed of exactly ℓ prime factors, and with class number equal to h. Summing over all $\ell \leq \lambda + 1$ we arrive at

$$(C2) \qquad\qquad \mathcal{F}(h) \asymp \frac{h}{\log h} \sum_{\ell \leq \lambda+1} \frac{2^{\ell-1}(\log \log h)^{\ell-1}}{(\ell-1)!}.$$

The unspecified constant in (C2) seems delicate, and would probably depend on arithmetical properties of h. For example, the Cohen-Lenstra heuristics [1] predict that the probability of class numbers being divisible by 3 is larger than $1/3$. So we would expect $\mathcal{F}(h)$ to be larger when 3 divides h, rather than when $3 \nmid h$. Similar (smaller) biases would exist when 5 divides h etc. Inspecting Watkins' table (page 936 of [5]) we can already see the bias in favor of multiples of 3.

Conjecture (C2) does not lend itself to numerical testing. To provide falsifiable conjectures, we may consider the ratio $\mathcal{F}(h_1)/\mathcal{F}(h_2)$ for various choices of h_1 and h_2. For example, if h_1 and h_2 are primes with $h_1/2 \leq h_2 \leq 2h_1$ (say) then it seems safe to conjecture that

$$(C3) \qquad\qquad \frac{\mathcal{F}(h_1)}{\mathcal{F}(h_2)} \sim \frac{h_1}{h_2}.$$

Also if h is odd, and large then (C2) suggests that $\mathcal{F}(h)\mathcal{F}(4h)/\mathcal{F}(2h)^2$ should tend to $1/2$. It would be interesting to assemble numerical data on these questions.

This note was motivated by the recent paper of Rosen and Silverman [4] where they ask for information on $N(C;X)$ which counts the number of fundamental discriminants $-d$ with $1 \leq d \leq X$ such that $h^{\mathrm{odd}}(-d)$ (the odd part of the class number; in other words, the largest odd number dividing $h(-d)$) lies below a fixed number C. Rosen and Silverman wished to know if $N(C;X) = o(X)$ for large X. We show that such is indeed the case.

Corollary 3. *For a fixed number C, and large X we have*

$$N(C;X) \ll X(\log\log X)^6/\log X.$$

Proof of Theorem 1. We first show that one can restrict attention to discriminants $-d$ with $1 \leq d \leq X = H^2(\log\log H)$. First consider the range $X < d < H^2(\log H)^4$. If $h(-d) < H$ then we must have $L(1, \chi_{-d}) \ll (\log\log H)^{-1/2}$, and by Theorem 4 of [2] there are at most[1] $H^2/(\log H)$ values of $d < H^2(\log H)^4$ with such a small value of $L(1, \chi_{-d})$. To handle the range $d > H^2(\log H)^4$ we invoke some facts from the classical theory of zeros of L-functions; for more information see Chapter 11 of [3]. If $d > H^2(\log H)^4$ then $h(-d)$ can be below H only when $L(1, \chi_{-d}) \ll H/\sqrt{d}$ $(\leq 1/(\log H)^2)$. Siegel's theorem shows that there is at most one such discriminant $-d$ with $d > H^3$, say. In the range $H^2(\log H)^4 < d \leq H^3$, we require $L(1, \chi_{-d}) \ll 1/(\log H)^2$, which happens only if $L(s, \chi_{-d})$ has an exceptional real zero very near 1. Precisely, $L(s, \chi_{-d})$ must have a real zero β with $\beta > 1 - c/\log H$. By an argument of Landau, there is at most one such discriminant. Therefore

$$(1) \qquad \sum_{h \leq H} \mathcal{F}(h) = \sum_{\substack{d \leq X \\ h(-d) \leq H}}^{\flat} 1 + O\left(\frac{H^2}{\log H}\right),$$

where the $\flat$ indicates that the sum is over fundamental discriminants $-d$.

Observe that for any $c > 0$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\left(\frac{(1+\delta)^{s+1}-1}{\delta(s+1)}\right)ds = \begin{cases} 1 & \text{if } x \geq 1 \\ (1+\delta-1/x)/\delta & \text{if } (1+\delta)^{-1} \leq x \leq 1 \\ 0 & \text{if } x \leq (1+\delta)^{-1}. \end{cases}$$

Here $\delta > 0$ is a parameter which we shall choose later. By the class number formula and (1) we get that

$$\sum_{h \leq H} \mathcal{F}(h) \leq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \sum_{d \leq X}^{\flat}\left(\frac{\pi}{\sqrt{d}L(1, \chi_{-d})}\right)^s \frac{H^s}{s}\left(\frac{(1+\delta)^{s+1}-1}{\delta(s+1)}\right)ds + O\left(\frac{H^2}{\log H}\right)$$

$$(2) \qquad \leq \sum_{h \leq H(1+\delta)} \mathcal{F}(h).$$

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[1]In fact Theorem 4 of [2] gives a much better bound, but the bound given above suffices for our purposes.

We now focus on evaluating the integral in (2) which leads naturally to Theorem 1.

We shall take $c = 1/\log H$, and $\delta = (\log H)^{-\frac{1}{2}}$. Set $S = \log X/(10^4(\log\log X)^2)$. The region $|s| > S$ contributes to the integral in (2) an amount

$$
(3) \qquad \ll \frac{X}{\delta}\int_{|s|>S}\frac{1}{|s(s+1)|}|ds| \ll H^2(\log H)^{-\frac{1}{2}+\epsilon}.
$$

In the region $|s| \le S$ we shall use Theorem 2 of [2] in order to evaluate the sum over d. That result evaluates such sums in terms of a probabilistic model for $L(1, \chi_{-d})$.

For primes p let $\mathbb{X}(p)$ denote independent random variables taking the value 1 with probability $p/(2(p+1))$, 0 with probability $1/(p+1)$, and -1 with probability $p/(2(p+1))$. Let $L(1, \mathbb{X}) = \prod_p(1 - \mathbb{X}(p)/p)^{-1}$. This product converges almost surely, and the main results of [2] compare the distribution of $L(1, \chi_{-d})$ with the distribution of such random Euler products. With two caveats that we clarify below, Theorem 2 of [2] gives that for $|z| \le \log x/(500(\log\log x)^2)$ and $\mathrm{Re}(z) > -1$

$$
(4) \qquad \sum_{d\le x}^{\flat} L(1, \chi_{-d})^z = \frac{3}{\pi^2}x\mathbb{E}(L(1, \mathbb{X})^z) + O\Big(x\exp\Big(-\frac{\log x}{5\log\log x}\Big)\Big),
$$

where $\mathbb{E}$ stands for expectation. The first caveat is that Theorem 2 of [2] considers both positive and negative fundamental discriminants, but the arguments given there permit us to restrict to negative fundamental discriminants as above. Secondly, there we omitted a small number ($\ll \log x$) of exceptional Landau-Siegel discriminants. Since $L(1, \chi_{-d}) \gg 1/\sqrt{x}$ and $\mathrm{Re}(z) \ge -1$ the contribution of these exceptional discriminants to our sum is $\ll \sqrt{x}\log x$, and so (4) holds. Using (4) and partial summation we obtain that for $|s| \le S$ and $\mathrm{Re}(s) = 1/\log H$ we have

$$
(5) \qquad \sum_{d\le X}^{\flat}(\sqrt{d}L(1, \chi_{-d}))^{-s} = \frac{3}{\pi^2}\mathbb{E}(L(1, \mathbb{X})^{-s})\int_1^X x^{-s/2}dx + O\Big(X\exp\Big(-\frac{\log X}{5\log\log X}\Big)\Big).
$$

From (3) and (5) we see that the integral in (2) is, with an error $O(H^2(\log H)^{-\frac{1}{2}+\epsilon})$,

$$
(6) \qquad \frac{1}{2\pi i}\int_{|s|\le S}\frac{3}{\pi^2}\mathbb{E}(L(1, \mathbb{X})^{-s})\Big(\int_1^X x^{-s/2}dx\Big)\frac{(\pi H)^s}{s}\Big(\frac{(1+\delta)^{s+1}-1}{\delta(s+1)}\Big)ds.
$$

For $1 \le x \le X$ we may see that

$$
\frac{1}{2\pi i}\int_{|s|\le S}\Big(\frac{\pi H}{\sqrt{x}L(1, \mathbb{X})}\Big)^s\frac{1}{s}\Big(\frac{(1+\delta)^{s+1}-1}{\delta(s+1)}\Big)ds = O\Big(\frac{L(1, \mathbb{X})^{-c}}{(\log H)^{\frac{1}{2}-\epsilon}}\Big)
$$

$$
+ \begin{cases} 1 & \text{if } \sqrt{x}L(1, \mathbb{X}) < \pi H \\ \in [0, 1] & \text{if } \pi H \le \sqrt{x}L(1, \mathbb{X}) \le \pi H(1+\delta) \\ 0 & \text{if } \pi H(1+\delta) < \sqrt{x}L(1, \mathbb{X}). \end{cases}
$$

Integrating this over x from 1 to X we get

$$O\Big(\frac{H^2}{(\log H)^{\frac{1}{2}-\epsilon}}(1+L(1,\mathbb{X})^{-c})\Big) + \min\Big(\frac{\pi^2 H^2}{L(1,\mathbb{X})^2}, X\Big).$$

Therefore the quantity in (6) equals

$$(7) \qquad\qquad \mathbb{E}\Big(\min\Big(\frac{3H^2}{L(1,\mathbb{X})^2}, \frac{3X}{\pi^2}\Big)\Big) + O\Big(\frac{H^2}{(\log H)^{\frac{1}{2}-\epsilon}}\Big),$$

and this is also our integral in (2).

Proposition 1 of [2] reveals that the probability that $L(1,\mathbb{X})$ is less than $\pi^2/(6e^\gamma\tau)$ is $\exp(-e^{\tau-C_1}/\tau + O(e^\tau/\tau^2))$ for some absolute constant C_1. Hence we may see that

$$\mathbb{E}\Big(\min\Big(\frac{3H^2}{L(1,\mathbb{X})^2}, \frac{3X}{\pi^2}\Big)\Big) = \mathbb{E}\Big(\frac{3H^2}{L(1,\mathbb{X})^2}\Big) + O\Big(\frac{H}{\log H}\Big).$$

Finally, by independence of the random variables $\mathbb{X}(p)$ we have

$$\mathbb{E}(L(1,\mathbb{X})^{-2}) = \prod_p \mathbb{E}\Big(\Big(1-\frac{\mathbb{X}(p)}{p}\Big)^2\Big)$$

$$= \prod_p \Big(\frac{p}{2(p+1)}\Big(1-\frac{1}{p}\Big)^2 + \frac{1}{(p+1)} + \frac{p}{2(p+1)}\Big(1+\frac{1}{p}\Big)^2\Big)$$

$$= \prod_p \Big(1-\frac{1}{p^3}\Big)\Big(1-\frac{1}{p^2}\Big)^{-1} = \frac{\zeta(2)}{\zeta(3)}.$$

Using these observations in (7), we conclude that the integral in (2) is

$$\frac{3\zeta(2)}{\zeta(3)}H^2 + O\Big(\frac{H^2}{(\log H)^{\frac{1}{2}-\epsilon}}\Big).$$

This establishes Theorem 1.

Proof of Theorem 2. As before set $X = H^2\log\log H$, and $S = (\log X)/(10^4(\log\log X)^2)$. As in (1) we see that

$$\mathcal{F}(H) = \sum_{\substack{d\le X \\ h(-d)=H}}^{\flat} 1 + O\Big(\frac{H^2}{\log H}\Big).$$

Since

$$\frac{1}{S}\int_{-S}^{S}\Big(1-\frac{|x|}{S}\Big)e^{2\pi i x\xi}dx \qquad \begin{cases} =1 & \text{if } \xi = 0, \\ \ge 0 & \text{always,} \end{cases}$$

we deduce, by the class number formula, that

$$(8) \qquad \mathcal{F}(H) \le O\Big(\frac{H^2}{\log H}\Big) + \frac{1}{S}\int_{-S}^{S}\Big(1-\frac{|x|}{S}\Big)\sum_{d\le X}^{\flat}\Big(\frac{\pi H}{\sqrt{d}L(1,\chi_{-d})}\Big)^{ix}dx.$$

As in (5) we have that

$$\sum_{d \leq X}^{\flat} (\sqrt{d}L(1, \chi_{-d})^{-ix} = \frac{3}{\pi^2} \mathbb{E}(L(1, \mathbb{X})^{-ix}) \int_1^X y^{-ix/2} dy + O\Big(\frac{H}{(\log H)^2}\Big) \ll \frac{X}{1 + |x|} + \frac{H}{(\log H)^2}.$$

Inserting this in (8) we obtain that

$$\mathcal{F}(H) \ll \frac{H^2}{\log H} + X \frac{\log S}{S} \ll H^2 \frac{(\log \log H)^4}{\log H}.$$

Proof of Corollary 3. From Theorem 4 of [2] (with τ there being $\log \log X$) we have that the number of fundamental discriminants $-d$ with $1 \leq d \leq X$ and $h(-d) > \sqrt{X} \log \log X$ is at most $X \exp(-c \log X / \log \log X)$ for some positive constant c. Therefore

$$N(C, X) \leq X \exp\Big(-c \frac{\log X}{\log \log X}\Big) + \sum_{\substack{2^k \ell \leq \sqrt{X} \log \log X \\ \ell \text{ odd} \\ \ell \leq C}} \mathcal{F}(2^k \ell).$$

The Corollary now follows from Theorem 2.

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