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Rational triangles with the same perimeter and the same area

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Abstract

This paper is concerned with the problem of finding two or more rational triangles with the same perimeter and the same area. As the problem of finding two isosceles rational triangles with the same perimeter and same area has already been solved, in this paper we obtain several parametric solutions to the problem of finding a pair of rational triangles with the same area and perimeter when at least one of the triangles is scalene. We also show that, with certain exceptions, given an arbitrary scalene rational triangle, a second scalene rational triangle with the same perimeter and same area may be constructed, and by repeated application of this process, we may obtain an arbitrarily large number of scalene rational triangles with the same perimeter and same area.

Introduction

A triangle with rational sides and rational area is called a rational triangle or Heron triangle. This paper is concerned with the problem of finding two or more rational triangles all of which have the same perimeter and the same area. A special case of this problem requiring the determination of two isosceles rational triangles with equal perimeters and equal areas has been considered by several authors ([2, p. 201], [4]). One infinite family of two rational triangles, one isosceles and the other scalene, with equal perimeters and equal areas, is also known [4]. Aassila [1] has given a parametric solution to the problem of finding two scalene rational triangles with equal perimeters and equal areas in terms of univariate polynomials of degree 10, and has stated the following open problem:

"Prove or disprove that for any positive integer $k \geq 2$ there exist k mutually incongruent Heron triangles having the same area and semiperimeter"

While Aassila defines a Heron triangle as having integer sides and integer area, this does not make any essential difference to the present problem since rational triangles with equal perimeters and equal areas yield, on appropriate scaling, triangles with integer sides and integer areas, and having the same properties.

Guy [3, p. 295] gives a set of 8 rational triangles, obtained by Rathbun, with the same perimeter and same area. He also mentions that Ronald van Luijk had "outlined" a proof that there are arbitrarily large sets of Heron triangles with common perimeter and area. However, no proof of such a result has been published till now.

In this paper we first obtain a two-parameter solution to the problem of finding two scalene rational triangles with equal perimeters and equal areas. This solution is much more general than that given in [1] and yields simple parametric solutions of degrees 2 and 4. We shall also show how additional parametric solutions of this problem can be obtained. If in our method of solution, we impose the further condition that two sides of one of the triangles become equal, we obtain two new parametric solutions to the problem of finding two rational triangles, one isosceles and one scalene, with equal perimeters and equal areas. We also show how more such solutions can be obtained. Next, we describe a method of constructing a new rational triangle having the same perimeter and the same area as an arbitrary scalene rational triangle with certain exceptions. We use this method to obtain arbitrarily many rational triangles with equal perimeters and the equal areas, and thus provide an affirmative answer to the open problem stated above.

Section 2 deals with the problem of pairs of rational triangles while Section 3 deals with the case of arbitrarily many rational triangles with the same perimeter and the same area.

2 Pairs of rational triangles with the same perimeter and same area

2.1 Both the triangles are scalene

Theorem 1: There exist infinitely many pairs of distinct scalene rational triangles with equal perimeters and equal areas. Specifically, the two rational triangles T_1 and T_2 whose sides a_1, b_1, c_1 and a_2, b_2, c_2 are defined respectively by

$$\begin{array}{rcl} a_1 & = & (r^2+1)(s^2+s+1) \\ b_1 & = & s(s+1)(r^2+s^2+s+1), \\ c_1 & = & r^2+(s^2+s+1)^2, \end{array} \tag{1}$$

and

$$\begin{array}{rcl} a_2 &=& (r^2+s^2)(s^2+s+1), \\ b_2 &=& (s+1)(r^2+s^2+s+1), \\ c_2 &=& r^2s^2+(s^2+s+1)^2, \end{array} \tag{2}$$

where r, s are arbitrary positive rational numbers, have the same perimeter, namely $2(r^2+s^2+s+1)(s^2+s+1)$, and the same area namely, $rs(s+1)(r^2+s^2+s+1)(s^2+s+1)$.

Proof: The area A of a triangle is given in terms of its sides a, b, c by the following formula:

$$A^2 = \{(a+b+c)(a+b-c)(b+c-a)(c+a-b)\}/16. \tag{3}$$

Thus two triangles T_1 and T_2, with sides a_1, b_1, c_1 and a_2, b_2, c_2 respectively, will have the same perimeter and the same area if and only if the following two conditions are satisfied:

$$a_1+b_1+c_1 = a_2+b_2+c_2, \tag{4}$$

and

$$\begin{array}{l} (a_1+b_1+c_1)(a_1+b_1-c_1)(a_1-b_1+c_1)(-a_1+b_1+c_1) \\ \quad = (a_2+b_2+c_2)(a_2+b_2-c_2)(a_2-b_2+c_2)(-a_2+b_2+c_2). \end{array} \tag{5}$$

In view of (4), it is easily seen that equation (5) is equivalent to the following three equations:

$$\begin{array}{rcl} a_1+b_1-c_1 &=& s(a_2+b_2-c_2), \\ t(a_1-b_1+c_1) &=& a_2-b_2+c_2, \\ s(-a_1+b_1+c_1) &=& t(-a_2+b_2+c_2), \end{array} \tag{6}$$

where s and t are arbitrary nonzero parameters. Equations (4) and (6), considered as four linear equations in the variables c_1, a_2, b_2, c_2, readily yield the following solution:

$$\begin{array}{rcl} c_1 &=& [\{s^2-(t^2-t)s-t\}a_1-\{s^2-(t^2+t)s+t\}b_1] \\ && \quad \times\{s^2+(t^2-t)s-t\}^{-1}, \\ a_2 &=& \{t(s^2-1)a_1-(st-1)(s-t)b_1\}\{s^2+(t^2-t)s-t\}^{-1}, \\ b_2 &=& \{(t-1)(t-s^2)a_1+t(s^2-1)b_1\}\{s^2+(t^2-t)s-t\}^{-1}, \\ c_2 &=& \{(s^2-t^2)a_1+s(t^2-1)b_1\}\{s^2+(t^2-t)s-t\}^{-1}, \end{array} \tag{7}$$

where a_1, b_1, s and t are arbitrary parameters. Thus, if two sides a_1 and b_1 of the triangle T_1 are chosen arbitrarily, and its third side c_1 and the three sides of the triangle T_2 are given by (7), then T_1 and T_2 will have the same perimeter and the same area A which, however, is not necessarily rational.

Using (3), we find that the common area A of the two triangles is given by

$$\begin{aligned} A^2 &= st\{(s^2-t)a_1+t(st-1)b_1\}\{t(t-1)a_1+(s-t)b_1\} \\ &\quad \times\{(t-s^2)a_1+s(s-t)b_1\}\{s(t-1)a_1-(st-1)b_1\} \\ &\quad \times\{s^2+(t^2-t)s-t\}^{-4}. \end{aligned} \tag{8}$$

For A to be rational, we must make the righthand side of (8) a perfect square. This may be done by taking $t = s^2$ when the righthand side of (8) reduces to

$$b_1^2(s^2+s+1)\{s(s+1)a_1-b_1)\}\{-s(s+1)a_1+(s^2+s+1)b_1\}\{s(s+1)\}^{-4}, \tag{9}$$

and would thus be a perfect square if a_1, b_1, s and t are rational numbers such that

$$(s^2+s+1)\{s(s+1)a_1-b_1)\} = r^2\{-s(s+1)a_1+(s^2+s+1)b_1\}, \tag{10}$$

where r is some rational number. This leads to the values of a_1 and b_1 as stated in (1). Substituting $t = s^2$ and the values of a_1, b_1 in (7), we get the third side c_1 of T_1 and the sides of T_2 as given by (1) and (2).

We note that when both r and s are arbitrary positive rational numbers, the rational numbers a_1, b_1, c_1 and a_2, b_2, c_2 defined by (1) and (2) are positive and both sets of numbers a_i, b_i, c_i, $i = 1, 2$ satisfy the three triangle inequalities so that T_1 and T_2 are indeed triangles with the desired properties. We also note that, in general, both the triangles are scalene. The conditions under which one or both the triangles become isosceles are easily worked out, and are omitted. It is easily verified that the common perimeter and common area of the two triangles are as stated in the theorem.

The solution given in the theorem is of degree 2 in the rational parameter r and degree 4 in the rational parameter s, and hence is much more general and far simpler than the solution given in [1]. In fact, by assigning fixed numerical values to s, we get solutions of degree 2 in one parameter, and by fixing numerical values of r, we get solutions of degree 4 in one parameter.

Finally we note that we can obtain more parametric solutions of our problem by suitably choosing s, t, a_1 and b_1 such that (8) becomes a perfect square. A straightforward way of doing this, using a result of Subsection **2.2**, will be outlined at the end of that Subsection. These solutions are, however, much more cumbersome than the solution obtained above, and are hence omitted.

2.2 One triangle is isosceles and one is scalene

Theorem 2: There exist infinitely many pairs of rational triangles, one isosceles and one scalene, which have equal perimeters and equal areas. Specifically, the isosceles rational triangle T_1 and the scalene rational triangle T_2, whose sides a_1, b_1, c_1 and a_2, b_2, c_2 are defined respectively by

$$\begin{aligned} a_1 = b_1 &= 2t^4 - 6t^3 + 9t^2 - 4t + 4, \\ c_1 &= 2(2t^2 - 3t + 2)(t + 2), \end{aligned} \tag{11}$$

and

$$\begin{aligned} &a_2 = 2t(t + 2)(t^2 - 2t + 2), \quad b_2 = (2 - t)(5t^2 - 4t + 4), \\ &c_2 = (t^2 + 4)(2t^2 - 3t + 2), \end{aligned} \tag{12}$$

where t is an arbitrary positive rational number such that $t < 2$ and $t \notin \{1, 2/3\}$, have the same perimeter $4(t^2 - t + 2)^2$ and the same area $2t(t - 2)(t + 2)(t^2 - t + 2)(2t^2 - 3t + 2)$. Further, the isosceles rational triangle T_1 and the scalene rational triangle T_2 whose sides a_1, b_1, c_1 and a_2, b_2, c_2 are defined respectively by

$$\begin{aligned} a_1 = b_1 &= 4t^6 - 8t^5 + 29t^4 - 64t^3 + 104t^2 - 64t + 16, \\ c_1 &= 8t(3t - 2)(2 - t)(t^2 - t + 2), \end{aligned} \tag{13}$$

and

$$\begin{aligned} a_2 &= 4(t^4 - 4t^3 + 13t^2 - 12t + 4)(t^2 - t + 2), \\ b_2 &= 16t^6 - 64t^5 + 145t^4 - 216t^3 + 216t^2 - 96t + 16, \\ c_2 &= (3t - 2)(2 - t)(4t^4 - 4t^3 + 9t^2 + 4t + 4), \end{aligned} \tag{14}$$

where t is an arbitrary rational number such that $2/3 < t < 2$, $t \neq 1$, have the same perimeter $2(2t^3 - 5t^2 + 12t - 4)^2$ and the same area

$$4t(t - 2)(t + 2)(3t - 2)(t^2 - t + 2)(2t^2 - 3t + 2)(2t^3 - 5t^2 + 12t - 4).$$

Proof: We have already seen in Subsection **2.1** that (7) gives the sides c_1, a_2, b_2, c_2 of two triangles T_1 and T_2 with equal perimeters and equal areas in terms of the sides a_1, b_1 of the triangle T_1 and the arbitrary parameters s and t. As a_1 and b_1 may be chosen arbitrarily, we ensure that the triangle T_1 is isosceles by simply choosing $b_1 = a_1$, and substituting this value of b_1 in (9), we find that the common area A of the two triangles is given by

$$\begin{aligned} A^2 = &\{s^4 + 2t(t - 1)s^3 + t(t^3 - 2t^2 - 2)s^2 - 2t^2(t - 2)s\} \\ &\times a^4t^2(s - 1)^2\{s^2 + (t^2 - t)s - t\}^{-4}. \end{aligned} \tag{15}$$

The area A will be rational if the quartic function

$$s^4 + 2t(t - 1)s^3 + t(t^3 - 2t^2 - 2)s^2 - 2t^2(t - 2)s \tag{16}$$

is made a perfect square. This can be done quite readily following the standard procedure described by Dickson [2, p. 639], and two values of s that make this quartic function a perfect square are easily found to be $s = 2t(2-t)/(2+t)$ and $s = (t^2+4t+4)/\{4(t^2-t+2)\}$ which lead to two distinct solutions. The first solution is given by (11) and (12), and the second solution is given by (13) and (14). In both cases the parameter t is an arbitrary rational number satisfying the specified constraints that are necessary to ensure that a_1, b_1, c_1 and a_2, b_2, c_2 are positive rational numbers and both sets of numbers a_i, b_i, c_i, $i = 1, 2$ satisfy the three triangle inequalities, and also that the second triangle is not isosceles, so that we actually get the two desired triangles. The common perimeter and area of the two triangles is readily verified in both cases to be as stated in the theorem.

We also note that by following the aforesaid standard procedure, we can find other values of s that make the quartic function (16) a perfect square, and thus we can find more parametric solutions of our problem.

Finally, reverting to the last remark of Subsection **2.1**, we observe that by fixing $s = 2t(2-t)/(2+t)$ and b_1 as given by (11) in (8), one value of a_1 that makes the quartic function (8) a perfect square is given by (11). Thus, following the standard procedure, we can find other values of a_1 that make the quartic function (8) a perfect square, and hence we can obtain more parametric solutions of the problem discussed in Subsection **2.1**.

2.3 Finding a rational triangle having the same perimeter and same area as a given rational triangle

We will now show that, in general, given an arbitrary scalene rational triangle, a second scalene rational triangle with the same perimeter and same area may be constructed. Exceptions occur when the sides of the given triangle satisfy, in some order, a certain diophantine equation. We first prove a few preliminary lemmas.

Lemma 1: Any three arbitrary rational numbers a, b, c that satisfy the three triangle inequalities

$$a+b-c > 0, \quad b+c-a > 0, \quad c+a-b > 0, \tag{17}$$

must be all positive.

Proof: Adding the inequalities in pairs proves the lemma.

Lemma 2: The sides a, b, c of a rational triangle satisfy the following inequalities:

$$\begin{aligned} a^2+b^2+c^2-ab-bc-ca &> 0, \qquad (18)\\ a^3+b^3+c^3-2a^2b-2b^2c-2c^2a+3abc &> 0, \qquad (19)\\ a^3+b^3+c^3-2ab^2-2bc^2-2ca^2+3abc &> 0. \qquad (20) \end{aligned}$$

Proof: The first inequality holds for any rational numbers a, b, c since

$$a^2+b^2+c^2-ab-bc-ca=\{(a-b)^2+(b-c)^2+(c-a)^2\}/2>0.$$

Further, since the rational numbers a, b, c satisfy the triangle inequalities, it follows that the three rational numbers $(a+b-c)^2(b+c-a)$, $(b+c-a)^2(c+a-b)$, $(c+a-b)^2(a+b-c)$, are all positive, and hence their arithmetic mean is greater than their geometric mean, that is,

$$\begin{aligned} &\{(a+b-c)^2(b+c-a)+(b+c-a)^2(c+a-b)+(c+a-b)^2(a+b-c)\}/3\\ &\quad > (a+b-c)(b+c-a)(c+a-b), \end{aligned}$$

from which the inequality (19) follows immediately. The inequality (20) is similarly proved.

Lemma 3: The complete solution in rational numbers of the homogeneous diophantine equation

$$\begin{aligned} &x^4-x^3y-2x^3z+3x^2yz-xy^3+3xy^2z\\ &\quad -6xyz^2+3xz^3+y^4-2y^3z+3yz^3-2z^4=0, \end{aligned} \qquad (21)$$

is given by

$$\begin{aligned} x &= \rho(m+n)(2m^3+m^2n+2mn^2-n^3),\\ y &= \rho(m-n)(2m^3-m^2n+2mn^2+n^3),\\ z &= 2\rho(m^2+3n^2)m^2, \end{aligned} \qquad (22)$$

where m, n are arbitrary integers and ρ is an arbitrary rational number.

Proof: Without loss of generality, we may substitute $y=x-2m, z=x+n-m$ in (21) which then reduces to a linear equation in x, and is hence readily solved to get a rational solution. As equation (21) is homogeneous, we may multiply this solution through by a suitable constant when we get the solution (22).

Theorem 3: Given any scalene rational triangle T_1, whose sides a_1, b_1, c_1 do not, in any order, satisfy the diophantine equation (21), there exists a second rational triangle T_2 whose perimeter and area are equal respectively to the perimeter and area of the triangle T_1, and whose sides a_2, b_2, c_2 are defined by

$$
\begin{aligned}
a_2 \;=\; & [a_1^5(-b_1+2c_1)+b_1^5(-c_1+2a_1)+c_1^5(-a_1+2b_1) \\
& \quad +a_1^4(2b_1^2-3c_1^2)+b_1^4(2c_1^2-3a_1^2)+c_1^4(2a_1^2-3b_1^2) \\
& \quad -a_1b_1c_1\{3(a_1^3+b_1^3+c_1^3)-8(a_1b_1^2+b_1c_1^2+c_1a_1^2) \\
& \quad +2(a_1^2b_1+b_1^2c_1+c_1^2a_1)+9a_1b_1c_1\}] \\
& \times(a_1^2+b_1^2+c_1^2-a_1b_1-b_1c_1-c_1a_1)^{-1} \\
& \times(a_1^3+b_1^3+c_1^3-2a_1^2b_1-2b_1^2c_1-2c_1^2a_1+3a_1b_1c_1)^{-1}, \\
b_2 \;=\; & [a_1^5(2b_1-c_1)+b_1^5(2c_1-a_1)+c_1^5(2a_1-b_1) \\
& \quad +a_1^4(-3b_1^2+2c_1^2)+b_1^4(-3c_1^2+2a_1^2)+c_1^4(-3a_1^2+2b_1^2) \\
& \quad -a_1b_1c_1\{3(a_1^3+b_1^3+c_1^3)-8(a_1^2b_1+b_1^2c_1+c_1^2a_1) \\
& \quad +2(a_1b_1^2+b_1c_1^2+c_1a_1^2)+9a_1b_1c_1\}] \\
& \times(a_1^2+b_1^2+c_1^2-a_1b_1-b_1c_1-c_1a_1)^{-1} \\
& \times(a_1^3+b_1^3+c_1^3-2a_1b_1^2-2b_1c_1^2-2c_1a_1^2+3a_1b_1c_1)^{-1}, \\
c_2 \;=\; & \{(a_1^3+b_1^3+c_1^3-a_1^2(b_1+c_1)-b_1^2(c_1+a_1) \\
& \quad -c_1^2(a_1+b_1)+3a_1b_1c_1\} \\
& \times\{(a_1^4+b_1^4+c_1^4-a_1^3(b_1+c_1)-b_1^3(c_1+a_1) \\
& \qquad -c_1^3(a_1+b_1)+a_1b_1c_1(a_1+b_1+c_1))\} \\
& \times(a_1^3+b_1^3+c_1^3-2a_1^2b_1-2b_1^2c_1-2c_1^2a_1+3a_1b_1c_1)^{-1} \\
& \times(a_1^3+b_1^3+c_1^3-2a_1b_1^2-2b_1c_1^2-2c_1a_1^2+3a_1b_1c_1)^{-1}.
\end{aligned} \tag{23}
$$

Proof: It can be verified by direct computation that (23) is a solution of equations (4) and (5). We shall now show that a_2, b_2, c_2 are, in fact, the sides of a second rational triangle. Using the relations (23), we find that

$$
\begin{aligned}
a_2+b_2-c_2 \;=\; & (a_1+b_1-c_1)(b_1+c_1-a_1)(c_1+a_1-b_1) \\
& \times(a_1^2+b_1^2+c_1^2-a_1b_1-b_1c_1-c_1a_1)^2 \\
& \times(a_1^3+b_1^3+c_1^3-2a_1^2b_1-2b_1^2c_1-2c_1^2a_1+3a_1b_1c_1)^{-1} \\
& \times(a_1^3+b_1^3+c_1^3-2a_1b_1^2-2b_1c_1^2-2c_1a_1^2+3a_1b_1c_1)^{-1}, \\
b_2+c_2-a_2 \;=\; & (a_1^3+b_1^3+c_1^3-2a_1b_1^2-2b_1c_1^2-2c_1a_1^2+3a_1b_1c_1)^2 \\
& \times(a_1^2+b_1^2+c_1^2-a_1b_1-b_1c_1-c_1a_1)^{-1} \\
& \times(a_1^3+b_1^3+c_1^3-2a_1^2b_1-2b_1^2c_1-2c_1^2a_1+3a_1b_1c_1)^{-1}, \\
c_2+a_2-b_2 \;=\; & (a_1^3+b_1^3+c_1^3-2a_1^2b_1-2b_1^2c_1-2c_1^2a_1+3a_1b_1c_1)^2 \\
& \times(a_1^2+b_1^2+c_1^2-a_1b_1-b_1c_1-c_1a_1)^{-1} \\
& \times(a_1^3+b_1^3+c_1^3-2a_1b_1^2-2b_1c_1^2-2c_1a_1^2+3a_1b_1c_1)^{-1}.
\end{aligned} \tag{24}
$$

Since a_1, b_1, c_1 are the sides of a triangle, they satisfy the three triangle inequalities, and also, by Lemma 2, the inequalities (18), (19) and (20). Therefore it follows from (24) that the rational numbers a_2, b_2, c_2 satisfy the triangle inequalities

$$a_2+b_2-c_2>0, \quad b_2+c_2-a_2>0, \quad c_2+a_2-b_2>0, \tag{25}$$

and hence, by Lemma 1, it also follows that a_2, b_2, c_2 are all positive. Thus, a_2, b_2, c_2 are the sides of a second triangle T_2 whose area is necessarily rational, being equal to the area of a given rational triangle.

It would thus appear that given an arbitrary scalene rational triangle T_1, we can always construct a second rational triangle T_2 whose perimeter and area are equal respectively to the perimeter and area of the given triangle. The triangle T_2, however, turns out to be the same as the triangle T_1 in certain exceptional situations which we now consider. Since a_1, b_1, c_1 are all distinct rational numbers being the sides of a scalene triangle, it follows from (23) that $a_2 = a_1$ if and only if

$$\begin{aligned} &a_1^4 - a_1^3b_1 - 2a_1^3c_1 + 3a_1^2b_1c_1 - a_1b_1^3 + 3a_1b_1^2c_1 \\ &\quad -6a_1b_1c_1^2 + 3a_1c_1^3 + b_1^4 - 2b_1^3c_1 + 3b_1c_1^3 - 2c_1^4 = 0, \end{aligned} \tag{26}$$

and when (26) holds, we also have $b_2 = b_1$ and $c_2 = c_1$. Similarly, it follows from (23) that $a_2 = b_1$ if and only if

$$\begin{aligned} &b_1^4 - b_1^3c_1 - 2b_1^3a_1 + 3b_1^2c_1a_1 - b_1c_1^3 + 3b_1c_1^2a_1 \\ &\quad -6b_1c_1a_1^2 + 3b_1a_1^3 + c_1^4 - 2c_1^3a_1 + 3c_1a_1^3 - 2a_1^4 = 0, \end{aligned} \tag{27}$$

and when (27) holds, we also have $b_2 = c_1$ and $c_2 = a_1$. Finally $a_2 = c_1$ if and only if

$$\begin{aligned} &c_1^4 - c_1^3a_1 - 2c_1^3b_1 + 3c_1^2a_1b_1 - c_1a_1^3 + 3c_1a_1^2b_1 \\ &\quad -6c_1a_1b_1^2 + 3c_1b_1^3 + a_1^4 - 2a_1^3b_1 + 3a_1b_1^3 - 2b_1^4 = 0, \end{aligned} \tag{28}$$

and when (27) holds, we also have $b_2 = a_1$ and $c_2 = b_1$. On substituting $a = x$, $b = y$, $c = z$, in (26), $a = z, b = x, c = y$, in (27) and $a = y, b = z, c = x$, in (28), each of these equations reduces to (21). It now follows that the triangle T_2 will be identical to the given triangle T_1 if and only if the sides of the triangle T_1 satisfy, in some order, the diophantine equation (21). Thus, using Lemma 4, it follows that when a_1, b_1, c_1 are given, in some order, by

$$\begin{gathered} \rho(m+n)(2m^3 + m^2n + 2mn^2 - n^3), \\ \rho(m-n)(2m^3 - m^2n + 2mn^2 + n^3),\ 2\rho(m^2 + 3n^2)m^2 \end{gathered} \tag{29}$$

where m, n are arbitrary integers such that $m > n$ and ρ is an arbitrary positive rational number, the triangle T_2 is identical to the given triangle T_1, and in all other cases, the triangle T_2 is distinct. This completes the proof of the theorem.

3 Arbitrarily many rational triangles with the same perimeter and the same area

It has been shown in [4] that for every isosceles triangle, there exists a unique non-congruent isosceles triangle that has the same perimeter and the same area. Thus, if there exist more than two rational triangles with the same area and the same perimeter, at most two of these triangles can be isosceles. We will now prove that there are arbitrarily many scalene rational triangles with the same perimeter and same area.

Theorem 4: Given an arbitrary positive integer $k \geq 2$, there exist k scalene rational triangles with the same perimeter and the same area.

Proof: We take a scalene rational triangle T_1 whose sides a_1, b_1, c_1 do not, in any order, satisfy equation (21). We apply Theorem 3 to obtain a second scalene rational triangle T_2 whose sides a_2, b_2, c_2 are given by (23), and repeat the process, taking a_2, b_2, c_2 as the sides of a new given triangle and thus get a third triangle T_3 with sides a_3, b_3, c_3. Continuing this process, we get a sequence of k rational triangles $T_1, T_2, \ldots, T_k$ with sides $a_i, b_i, c_i,\ i = 1, 2, \ldots, k$. As these rational triangles have been obtained by repeated application of Theorem 3, all these triangles have the same perimeter and the same area.

Next we will show that we can actually choose a rational triangle T_1 whose sides a_1, b_1, c_1 have such numerical values that the above process generates k distinct scalene rational triangles $T_i,\ i = 1, 2, \ldots, k$, with the same perimeter and the same area.

We will first show that no two of the triangles $T_i,\ i = 1, 2, \ldots, k$, are identical for all values of a_1, b_1, c_1. We note $a_i, b_i, c_i,\ i = 1, 2, \ldots, k$, are all rational functions of a_1, b_1, c_1 and we will show that the two sets of functions a_u, b_u, c_u and a_v, b_v, c_v, where $1 \leq u \leq k,\ 1 \leq v \leq k,\ u \neq v$ are distinct. It is easily seen by induction that if we assign the values $1, -1, 0$ respectively to a_1, b_1, c_1, then we get

$$a_u = 1/(-3)^u,\ b_u = -1/(-3)^u,\ c_u = 0, \quad u = 1, 2, \ldots, k. \tag{30}$$

As the the two sets of functions $\{a_u, b_u, c_u\}$ and $\{a_v, b_v, c_v\}$, where $1 \leq u \leq k,\ 1 \leq v \leq k,\ u \neq v$ attain distinct values in this special case, they must necessarily be distinct. It similarly follows that for any $u,\ 1 \leq u \leq k$, the functions a_u, b_u, c_u are also necessarily distinct. Thus the triangles $T_i,\ i = 1, 2, \ldots, k$, are, in general, scalene and no two of them are identical for all values of a_1, b_1, c_1.

Next we note that, as stated by Brahmagupta [2, p. 191], we may take the sides of the rational triangle T_1 as follows:

$$\begin{aligned} a_1 &= \tfrac{1}{2}\left(\tfrac{a^2}{b}+b\right), \\ b_1 &= \tfrac{1}{2}\left(\tfrac{a^2}{c}+c\right), \\ c_1 &= \tfrac{1}{2}\left(\tfrac{a^2}{b}-b\right)+\tfrac{1}{2}\left(\tfrac{a^2}{c}-c\right), \end{aligned} \tag{31}$$

where a, b, c are arbitrary positive rational numbers such that $a^2 > bc$. The sides of all the rational triangles T_i, $i = 1, 2, \ldots, k$, are now rational functions of the variables a, b and c. We observe that one side of the triangle T_u, $1 \leq u \leq k$ will be equal to one side of the triangle T_v, $1 \leq v \leq k$ where $u \neq v$ if any one of 9 equations of the type $a_u = b_v$ is satisfied. Similarly the triangle T_u, $1 \leq u \leq k$ will not be scalene if any one of the 3 equations $a_u = b_u$, $b_u = c_u$, $c_u = a_u$ is satisfied. We assign arbitrary positive numerical rational values to b and c when all these 12 equations may be written as polynomial equations in the single variable a. Moreover, for any given value of k, there are only a finite number values of u and v, and hence only finitely many such polynomial equations, and these equations in the variable a have a finite number of roots. Similarly, for the assigned numerical rational values of b and c, the sides a_1, b_1, c_1 of the triangle T_1 will satisfy equation (21) in some order only for finitely many values of the variable a. Thus by assigning to the variable a a positive rational value so as to avoid all these finite number of possible values of a, and also such that $a^2 > bc$, we are assured that all the triangles T_i, $i = 1, 2, \ldots, k$, are distinct as well as scalene. This proves the theorem.

As a numerical example, if we take the sides of the initial triangle T_1 as $(a_1, b_1, c_1) = (3, 4, 5)$, and apply the Theorem 3 two times, we get the triangles T_2 and T_3 whose sides are given by

$$(a_2, b_2, c_2) = \left(\frac{101}{21}, \frac{41}{15}, \frac{156}{35}\right),$$

and

$$(a_3, b_3, c_3) = \left(\frac{147311847839141}{29415245790105}, \frac{91856397607001}{23156698179195}, \frac{112134202236876}{37065988023371}\right).$$

The three triangles T_1, T_2 and T_3 have the same perimeter 12 and the same area 6. Multiplying the sides of these three triangles by a suitable integer, we can obtain three triangles with integer sides and with the same area and perimeter.

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