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ON THE BARBAN-DAVENPORT HALBERSTAM THEOREM : XIX

C. Hooley

1. Introduction

We pursue one of the themes of the previous article [1]XVIII of this series, which as in III, IX, X, XIV (and also [2]) was concerned with strictly increasing sequences of positive integers s that are postulated to adhere to the following

CRITERION V, *For any positive constant A,*

$$S(x;a,k) = \sum_{\substack{s\leq x\\ s\equiv a, \bmod k}} 1 = f(a,k)x + O\left(\frac{x}{\log^A x}\right),$$

where

$$f(0,1) = C > 0.$$

Referring the reader to the previous papers for a full description of what they covered in order to avoid unnecessary repetition, we shall once again be concerned as in XVIII with the dispersion

$$G(x,Q) = \sum_{k\leq Q}\sum_{0<a\leq k}\{S(x;a,k) - f(a,k)x\}^2$$

for large Q and the associated asymptotic formula

$$G(x,Q) = \{D + o(1)\}Qx + O(x\log^{-A} x) \tag{1}$$

that is analogous to the Barban-Montgomery theorem for prime numbers mentioned in other papers of this series. As in XVIII and also IX,X and XIV, the sequences considered fall into two classes as follows:

Class 1; $D > 0$, there being a genuine asymptotic formula,

Class 2; $D = 0$, there only being an upper bound presented in the first place in the absence of further data.

Possibilities abound for $G(x,Q)$ in the second case; on the one hand there are sequences such as that of the square-free numbers for which $G(x,Q)$ is asymptotic to a multiple of $Q^{1+\alpha}x^{1-\alpha}$, while on the other we found sequences in XIV for which $G(x,Q)$ oscillated in size between $Q^{1+\varepsilon}x^{1-\varepsilon}$ and $Q^{2-\varepsilon}x^{\varepsilon}$. Therefore, having concluded in XVIII that the only credible

[1]We refer to each article by the Roman numeral indicating its position in the series.

goal in the study of the size of $G(x,Q)$ was the determination of a good universal lower bound, we proved the inequality

$$G(x,Q) > \frac{1}{12}\{C_3 + o(1)\}Q^2 + O(x\log^{-A} x),$$

where

$$C_3 = \begin{cases} 3C - 3C^2, & \text{if } C \le \frac{1}{2}, \\ 5C - 3C^2 - 1, & \text{if } C > \frac{1}{2}. \end{cases}$$

This was seen to be best possible for $C = \frac{1}{2}$ or 1 but it was suggested that it might be susceptible to betterment for other values of C.

To make such an improvement is the purpose of the present paper, the result obtained being stated in the theorem at the end.

2. Resumé of material from Paper XVIII

Assuming throughout that Criterion V is observed, we define $w(a,l)$ by

$$f(a,k) = \frac{1}{k}\sum_{l|k} w(a,l) \tag{2}$$

and the Möbius inversion formula; this function is periodic, mod l, because $f(a,k)$ is periodic, mod k. We then let

$$ma_m = \frac{1}{m}\sum_{0\le a<m} w^2(a,m) \tag{3}$$

and form the series

$$\sum_{m=1}^{\infty} ma_m,$$

which, being convergent, is assigned the value C_1. Since in formula (1) $D = C - C_1$, the sequence belongs to the first class when $C_1 < C$ and $G(x,Q)/Q^2 \to \infty$ when Q is large and $x/Q \to \infty$. Therefore, in determining lower bounds for $G(x,Q)$ of the type sought, we may restrict attention to sequences of the second class for which $C_1 = C$. In this situation we consider the series

$$\sum_{m=1}^{\infty} m^2 a_m;$$

if it be divergent, then $G(x,Q)/Q^2 \to \infty$ but, if it be convergent, then

$$G(x,Q) > \frac{1}{12}\left(C_2 + o(1)\right)Q^2 + O\left(x^2\log^{-A} x\right), \tag{4}$$

where

$$C_2 = \sum_{m=1}^{\infty} m^2 a_m.$$

It is therefore only in the latter case that we need examine our problem, which is tantamount to finding a good lower bound for C_2

We also note the identity

$$f(a,d) = \sum_{0<\lambda\leq k/d} f(a,+\lambda d,k) \tag{5}$$

that will be needed for some small moduli k

3.Conjugate sequences and further properties of $w(a,m)$ and a_m

To confine our discussion to the case where the density C of the sequence s does not exceed $\frac{1}{2}$ we introduce when necessary the conjugate sequences of numbers s^* that is the complement of the sequence s within the set of natural numbers. To this there correspond the counting functions

$$\begin{aligned} S^*(x;a,k) &= \sum_{\substack{s^*\leq x \\ s^*\equiv a,\ \mathrm{mod}\ k}} 1 = \sum_{\substack{n\leq x \\ n\equiv a,\ \mathrm{mod}\ k}} 1 - S(x;a,k) \\ &= \frac{x}{k} + O(1) - xf(a,k) + O\left(\frac{x}{\log^A x}\right) \\ &= x\left(\frac{1}{k} - f(a,k)\right) + O\left(\frac{x}{\log^A x}\right) \\ &= xf^*(a,k) + O\left(\frac{x}{\log^A x}\right), \text{ say,} \end{aligned}$$

so that it also satisfies Criterion V but with density $C^* = 1 - C$. Also, still attaching asterisk superscripts to the notation of §2 when it is to appertain to the conjugate sequence, we find from (2)
that

$$w^*(0,1) = f^*(0,1) = C^* = 1 - C$$

and

$$\begin{aligned} w^*(a,m) = \sum_{d|m} \mu\left(\frac{m}{d}\right) df^*(a,d) &= \sum_{d|m} \mu\left(\frac{m}{d}\right) - \sum_{d|m} df(a,d) \\ &= -w(a,m) \end{aligned}$$

for $m > 1$, whence

$$ma_m^* = \begin{cases} C^{*2}, & \text{if } m = 1, \\ ma_m, & \text{if } m > 1. \end{cases}$$

by (3). In particular therefore, if the sequence s belong to Class 2 so that

$$C(1-C) = \sum_{m=2}^{\infty} ma_m,$$

then so does the sequence s^* because $C(1-C) = C^*(1-C^*)$, while also the convergency of the series

$$\sum_{m=2}^{\infty} m^2 a_m$$

implies that of the equal series

$$\sum_{m=2}^{\infty} m^2 a_m^*.$$

Furthermore

$$C_2 - C_2^* = C^2 - C^{*2} = C^2 - (1-C)^2. \tag{6}$$

Consequently, save when formulating our final conclusion, it may be assumed that $C \leq \frac{1}{2}$.

Our method leans on some properties for smaller moduli l of the function $w(a,l)$ that we now enumerate. First, since

$$0 \leq f(a,2) \leq C \text{ and } f(0,2) + f(1,2) = C,$$

we deduce from the special case

$$f(a,2) = \frac{1}{2}\{C + w(a,2)\} \tag{7}$$

of (2) that the values of $w(0,2)$ and $w(1,2)$ can be denoted by b and $-b$ in some order where $0 \leq b \leq C$, wherefore

$$2a_2 \leq \frac{1}{2}(b^2 + b^2) = b^2. \tag{8}$$

Similarly, from

$$f(0,1) + f(1,3) + f(2,3) = C$$

and the counterpart of (7) for modulus 3, we have the relations

$$w(a,3) \geq -C \tag{9}$$

and

$$w(0,3) + w(1,3) + w(2,3) = 0,$$

the latter of which implies that one of the $w(a,3)$ is negative and another positive unless all three of them be zero. Temporarily excluding the last possibility, let $-d$ be the least $w(a,3)$ occuring, d_1 the (positive) greatest, and $\pm d_2$ the intermediate one, where $d_2 \geq 0$ and the positive sign is used when $d_2 = 0$. If the negative sign be apposite, then $d_2 \leq d$ and $d_1 = d + d_2 \leq 2d$ so that

$$3a_3 = \frac{1}{3}(d^2 + d_2^2 + d_1^2) \leq 2d^2, \tag{10}$$

whereas, if the positive sign be apposite, $d = d_1 + d_2$ and

$$3a_3 = \frac{1}{3}(d^2 + d_2^2 + d_1^2) \leq \frac{1}{3}\{d^2 + (d_1+d_2)^2\} = \frac{2d^2}{3}.$$

Thus, in all cases including the one that was put aside, the inequality (10) is valid with the particular implication that

$$3a_3 \leq 2C^2 \tag{11}$$

since

$$d \leq C \tag{12}$$

by (9).

As for the modulus 4, we see that (2) and the definition of b above supply the two equations

$$C \pm b + w(a,4) = 4f(a,4), C \pm b + w(a+2,4) = 4f(a+2,4)$$

with a constant interpretation of the sign attached to b for a given value of a, the combination of which implies that

$$\begin{aligned} 2C \pm 2b + w(a,4) + w(a+2,4) &= 4\{f(a,4) + f(a+2,4)\} \\ &= 4f(a,2) = 2C \pm 2b \end{aligned}$$

and hence that

$$w(a,4) = -w(a+2,4).$$

Also,

$$w(a,4), w(a+2,4) \geq -(C \pm b),$$

which inequalities imply that

$$|w(a,4)| \leq C \pm b$$

because $w(a,4)$ and $w(a+2,4)$ do not share the same sign. In all, therefore,

$$4a_4 \leq \frac{1}{4}\{2(C-b)^2 + 2(C+b)^2\} = C^2 + b^2 \leq 2C^2. \tag{13}$$

Finally, moving on to the modulus 6, we deduce from (2) that

$$\begin{aligned} w(a,6) + w(a+3,6) = 2C - 2\{f(a,2) + f(a+3,2)\} - 3\{f(a,3) + f(a+3,3)\} \\ +f(a,6) + f(a+3,6) \\ = 2C - 2\{f(a,2) + f(a+1,2)\} - 6f(a,3) + 6f(a,3) \\ = 2C - 2C = 0 \end{aligned} \tag{14}$$

for each value of a. Also, in somewhat similar manner,

$$\begin{aligned} w(a,6) + w(a+2,6) + w(a+4,6) \\ = 3C - 2\{f(a,2) + f(a+2,2) + f(a+4,2)\} - 3\{f(a,3) + f(a+2,3) + f(a+f,3)\} \\ +6\{f(a,6) + f(a+2,6) + f(a+4,6)\} \\ = 3C - 6f(a,2) - 3C + 6f(a,2) = 0. \end{aligned} \tag{15}$$

Some of these preparatory results lead easily to the first lower bound for C_2, which we derive in the next section before going on to the more difficult ones.

4. First lower bound for C_2

It being given that

$$C = C_1 = \sum_{m=1}^{\infty} m a_m = C^2 + 2a_2 + 3a_3 + 4a_4 + \sum_{m>4} m a_m \tag{16}$$

by the remarks in §2, we have

$$\sum_{m>4} m^2 a_m \geq 5 \sum_{m>4} m a_m$$

and then

$$\sum_{m>4} m^2 a_m \geq 5(C - C^2 - 2a_2 - 3a_2 - 4a_4).$$

Hence

$$\begin{aligned} C_2 = \sum_{m=1}^{\infty} m^2 a_m &\geq C^2 + 4a_2 + 9a_3 + 16a_4 \\ &+5(C - C^2 - 2a_2 - 3a_3 - 4a_4) \\ &= 5C - 4C^2 - 6a_2 - 6a_3 - 4a_4, \end{aligned} \tag{17}$$

from which through (8), (11), and (13) we derive the *first bound*

$$C_2 \geq 5C - 10C^2 - 3b^2 \geq 5C - 13C^2. \tag{18}$$

This is only better than the bound

$$C_2 \geq 3(C - C^2) \tag{19}$$

of our previous paper XVIII for $C < \frac{1}{5}$ but will be supplemented by a bound that is stronger when C is not too small.

5. The second lower bound for C_2

In deriving the second lower bound for C_2 we examine separately the two cases

$$\text{a)} \quad d \leq C - b \tag{20}$$

and

$$\text{b)} \quad d > C - b.$$

In case a) the restatement of (16) as

$$C = C^2 + 2a_2 + 3a_3 + \sum_{m>3} m a_m$$

implies that

$$\sum_{m>3} ma_m = C - C^2 - 2a_2 - 3a_3$$

and hence that

$$\sum_{m>3} m^2 a_m \geq 4(C - C^2 - 2a_2 - 3a_3).$$

Therefore

$$\begin{aligned} C_2 = \sum_{m=1}^{\infty} m^2 a_m &\geq C^2 + 4a_2 + 9a_3 + 4(C - C^2 - 2a_2 - 3a_3) \\ &= 4C - 3C^2 - 4a_2 - 3a_3 \\ &\geq 4C - 3C^2 - 2b^2 - 2(C-b)^2 \end{aligned} \tag{21}$$

by (8), (10), and (20). The maximum of $b^2 + (C-b)^2$ for $0 \leq b \leq C$ being $2C^2$, the bound

$$C_2 \geq 4C - 5C^2 \tag{22}$$

follows in case a).

In case b) let us consider the situation for values of b, d for which

$$d = \alpha(C - b) > 0 \tag{23}$$

when α is some number exceeding 1, where

$$b > \left(1 - \frac{1}{\alpha}\right) C$$

by (12). For each α we derive two estimates, which will then be compared to elicit the more favourable one.

To obtain the first estimate we merely vary the procedure of deriving (21). In the new circumstances, by (8) and (23), we now have

$$C_2 \geq 4C - 3C^2 - 2b^2 - 2\alpha^2(C-b)^2$$

in place of (21). But, the minimum of $b^2 + \alpha^2(C-b)^2$ being at

$$b = \frac{\alpha^2 C}{1+\alpha^2} = \left(1 - \frac{1}{1+\alpha^2}\right) C > \left(1 - \frac{1}{2\alpha}\right) C$$

because $1 + \alpha^2 - 2\alpha = (1-\alpha)^2 > 0$, the inequality

$$b^2 + \alpha^2(C-b)^2 < \left\{ \left(1 - \frac{1}{\alpha}\right)^2 + 1 \right\} C^2$$

holds for $b > \left(1 - \frac{1}{\alpha}\right) C$. We thus infer that

$$C_2 \geq 4C - \left\{ 5 + 2\left(1 - \frac{1}{\alpha}\right)^2 \right\} C^2 = 4C - (5 + 9u)C^2 \tag{24}$$

on setting

$$u = \frac{1}{4}\left(1 - \frac{1}{\alpha}\right)^2 < \frac{1}{4}. \tag{25}$$

The production of the second estimate takes slightly longer. Recalling the meaning attached to d in §3, let us choose a_1 and a_2 so that $w(a_1, 2) = -b$ and $w(a, 3) = -d$. Then, if

$$a \equiv a_1, \text{ mod } 2, \text{ and } a \equiv a_2, \text{ mod } 3,$$

it follows from (2) that

$$C - b - d + w(a, 6) = 6f(a, 6) \geq 0$$

with the implication that

$$w(a, 6) \geq \left(1 - \frac{1}{\alpha}\right) d$$

by (23). Also, by both (14) and (15),

$$|w(a+3, 6)| = |w(a, 6)|,$$

$$w^2(a+2, 6) + w^2(a+4, 6) \geq \frac{1}{2}\{w(a+2, 6) + w(a+4, 6)\}^2 = \frac{1}{2}w^2(a, 6),$$

and

$$w^2(a+5, 6) + w^2(a+1, 6) \geq \frac{1}{2}w^2(a+3, 6) = \frac{1}{2}w^2(a, 6).$$

Therefore,

$$6a_6 = \frac{1}{6}\sum_{j=0}^{5} w^2(a+j, 6) \geq \frac{1}{2}w^2(a, 6) \geq \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)^2 d^2 = 2ud^2$$

and thus, by (10)

$$6a_6 \geq u.3a_3,$$

which inequality enables us to contrast the constituents $3a_3 + 6a_6$ and $9a_3 + 36a_6$ of the series $\sum ma_m$ and $\sum m^2 a_m$. Indeed

$$9a_3 + 36a_6 \geq v(3a_3 + 6a_6)$$

for some multiplier v exceeding 3, if

$$3a_3(3 - v) + 6a_6(6 - v) \geq 0$$

and therefore certainly if (even when $a_3 = 0$)

$$u(6 - v) \geq v - 3$$

and thus, in particular, if

$$v = \frac{3(1 + 2u)}{1 + u},$$

which lies between 3 and 18/5 by (25).

Consequently

$$\sum_{m>2} m^2 a_m \geq v \sum_{m>2} m a_m = v(C - C^2 - b^2)$$

and we obtain the estimate

$$C_2 \geq C^2 + 2b^2 + v(C - C^2 - b^2) = vC - (v-1)C^2 - (v-2)b^2$$
$$\geq vC - (2v-3)C^2$$
$$= \frac{3}{1+u}\{(1+2u)C - (1+3u)C^2\}.$$

in parallel with (24).

The right-hand sides of (24) and (26) are, respectively, decreasing and increasing functions of u, since

$$\frac{1+2u-(1+3u)C}{1+u} = 2 - 3C - \frac{(1-2C)}{1+u}.$$

Therefore in case b) a bound is supplied by using the value of u that answers to the equality of these expressions provided that it lie in the range $(0, \frac{1}{4}$, it then being clear that this lower limit is universal because it is implied by (22) of case a); moreover the range of u can be extended to include 0 because of case a). Accordingly, equating these expressions, we obtain the condition that

$$3(1+2u) - 3(1+3u)C = \{4 - (5+8u)C\}(1+u)$$

that is tantamount to

$$C = \frac{1-2u}{8u^2+4u+2}, \tag{26}$$

the right-hand side of which is a decreasing function of u that runs from $\frac{1}{2}$ to $\frac{1}{7}$ as u runs from 0 to $\frac{1}{4}$. This process provides the *second lower bound* for C_2 when $\frac{1}{7} < C \leq \frac{1}{2}$. For smaller values of C it is clear we have $C_2 \geq 4C - 7C^2$ by letting $u \to \frac{1}{4}$ in (24); this bound, however, will be eclipsed in the coming section.

6. The final bound

To confirm that our second bound for $\frac{1}{7} < C \leq \frac{1}{2}$ is superior to the previous bound in XVIII it must be shewn that

$$4 - (5+8u)C > 3 - 3C$$

when u is determined by (27) or, in other words, that

$$1 > 2(1+4u)C = \frac{(1+4u)(1-2u)}{4u^2+2u+1} = \frac{-8u^2+2u+1}{4u^2+2u+1},$$

which inequality is trivial for $u > 0$. Also, we have already seen in §4 that the first bound (18) is better than the bound in XVIII for $C \leq \frac{1}{7}$.

In like manner we determine when the second bound is superseded by the first bound. This happens at the point where

$$4-(5+8u)=5-13C,$$

that is where

$$1=8(1-u)C$$

and therefore, by (27), where

$$4(1-u)(1-2u)-4u^2-2u-1=4u^2-14u+3=0,$$

the lesser root of which, being equal to

$$\frac{1}{4}(7-\sqrt{37})<\frac{1}{4}(7-6)=\frac{1}{4},$$

falls within the requisite range of u. The change-over for the bounds thus occurs when

$$C=\frac{1}{2(\sqrt{37-3})}=\frac{1}{56}(3+\sqrt{37})>\frac{1}{7}.$$

To summarize our findings let us define the function $\Gamma(C)$ as follows:

(i) for $(3+\sqrt{37})/56<C\leq\frac{1}{2}$,

$$\Gamma(C)=4C-(5+8u)C^2,$$

where u is the least positive root of (27);

(ii) for $C\leq(3+\sqrt{37})/56$,

$$\Gamma(C)=5C-13C^2,$$

(iii) for $\frac{1}{2}\leq C\leq 1$,

$$\Gamma(C)=2C-1+\Gamma(1-C).$$

Then $\Gamma(C)\leq C_2$ and we have the

THEOREM. *Suppose that the sequence of numbers s satisfy* Criterion V *and let* $G(x,Q)$ *be the dispersion defined by* (1). *Then*

$$G(x,Q)>\frac{1}{12}\{\Gamma(C)+o(1)\}Q^2+O(x\log^{-A}x)$$

when $x/Q\to\infty$

The truth of the theorem is clear from (4) and the discussion in this section. But, for $C>\frac{1}{2}$, we consider the conjugate sequence s^* of density $C^*=1-C<\frac{1}{2}$, deducing from (6) that

$$C_2=C^2-(1-C)^2+C_2^*=2C-1+\Gamma(C^*)=2C-1+\Gamma(1-C),$$

as asserted.

7. Final remarks

It is illuminating to consider what our theorem yields for particular values of C. For example, if $C = \frac{1}{4}$ as in the example tested in XVIII, we find the value of u given by (27) is

$$\frac{1}{4}(\sqrt{13} - 3)$$

and then that

$$C_2/C \quad > \quad 4 - \frac{1}{4}(2\sqrt{13} - 1) > \frac{313}{128},$$

a palpable improvement on the previous bound $C_2/C > \frac{9}{4}$ being thus obtained

Our theorem does not represent the ultimate that can be obtained, especially as sharper results can be easily obtained as C becomes very small. As a step in this direction, we can shew by the methods of §3 that $5a_5 \leq 4C^2$ and then utilize the analogue

$$C_2 \geq C^2 + 4a_2 + 9a_3 + 16a_4 + 25a_5 + 6(C - C^2 - 2a_2 - 3a_3 - 4a_4 - 5a_5)$$

of (17) to provide the bound

$$C_2 \geq 6C - 5C^2 - 8a_2 - 9a_3 - 8a_4 - 5a_5 \geq 6C - 23C^2.$$

This is better than what is latent in our theorem when $C < \frac{1}{10}$.

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Address

Cardiff School of Mathematics,
Cardiff University,
Senghennydd Road,
Cardiff CF24 4AG,
United Kingdom

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