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On Kronecker's limit formula and the hypergeometric function

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1 Introduction

Let k be an imaginary quadratic field and G be its ideal class group. For any character χ of G, the L-series

$$L(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{(N\mathfrak{a})^s}, \ \sigma = \Re s > 1$$

decomposes into

$$L(s,\chi) = \sum_{A \in G} \chi(A)\zeta(s,A), \quad \sigma > 1,$$

where $\zeta(s,A) = \sum_{\mathfrak{a}\in A}(N\mathfrak{a})^{-s}$ is the ideal class zeta-function so that the evaluation of $L(1,\chi)$ leads to that of the constant term $\rho(A)$ of $\zeta(s,A)$. The closed form of $\rho(A)$ is called Kronecker's (first) limit formula, permits one to get a closed form for $L(1,\chi)$, which in turn leads to a closed form for the residue of the Dedekind zeta-function of k.

As is well-known (cf. e.g. Siegel [7]), the norm-form can be expressed by a positive definite quadratic form $Q(x,y)$, and therefore it suffices to consider the Epstein zeta-function $\zeta_Q(s)$ associated to Q:

$$(1.1) \quad \zeta_Q(s) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{Q(m,n)^s} = \sum \frac{1}{(am^2+bmn+cn^2)^s}, \quad \sigma > 1.$$

We write $(a,b>0)$

$$(1.2) \qquad Q(x,y) = a(x+\omega y)(x+\bar{\omega}y),$$

where

$$\omega = \frac{b + i\sqrt{\Delta}}{2a}, \quad \bar{\omega} = \frac{b - i\sqrt{\Delta}}{2a}, \tag{1.3}$$

and

$$\Delta = -(b^2 - 4ac) > 0. \tag{1.4}$$

We transform (1.1) slightly to obtain

$$\zeta_Q(s) = \frac{2}{a^s}\zeta(2s) + \frac{2}{c^s}\zeta(2s) + S(s), \tag{1.5}$$

where

$$\begin{aligned} S(s) &= \sum_{\substack{m,n\in\mathbb{Z} \\ m\neq 0, n\neq 0}} \frac{1}{(am^2 + bmn + cn^2)^s} \\ &= \frac{2}{a^s}\sum_{n=1}^{\infty}\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{(m+\omega n)^s(m+\bar{\omega}n)^s}. \end{aligned} \tag{1.6}$$

It therefore suffices to consider the limit of $S(s)$ as $s \to 1$. Koshlyakov [3] uses the integral

$$\int \frac{x^{s-1}}{(x+\alpha)^s(x+\beta)^s}, \quad 0 < \arg\alpha, \arg\beta < \pi$$

first with s a positive integer and then considers s as a continuous real variable and takes the limit $s \to 1$. This process cannot be easily justified. In this paper, we propose a new intrinsic method of using the (connection formula of) hypergeometric function in the spirit of Koshlyakov. The hypergeometric function has been used by Novikov [5] in deriving the Kronecker limit formula for a real quadratic field. We hope to develop our method further to treat Novikov's case subsequently.

2 The hypergeometric functional argument

We are to express $S(s)$ in terms of hypergeometric functions:

$$S(s) = \frac{2}{a^s}\sum_{n=1}^{\infty}\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{(m+\omega n)^s(m+\bar{\omega}n)^s}, \tag{2.1}$$

the branch of the power function being principal, i.e. $\arg z \in [-\pi, \pi)$. We express $S(s)$ as

$$S(s) = \frac{2}{a^s}\{J_1(s) + J_2(s)\}, \tag{2.2}$$

where

$$\begin{aligned} J_1(s) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(m+\omega n)^s(m+\bar{\omega} n)^s} \\ &= \sum_{n=1}^{\infty} I(s, \omega n, \bar{\omega} n), \end{aligned} \tag{2.3}$$

say, and

$$\begin{aligned} J_2(s) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(-m+\omega n)^s(-m+\bar{\omega} n)^s} \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(m-\omega n)^s(m-\bar{\omega} n)^s} \\ &= \sum_{n=1}^{\infty} I(s, -\bar{\omega} n, -\omega n), \end{aligned} \tag{2.4}$$

and where for $\alpha, \beta \in \mathbb{C}\backslash\mathbb{R}$,

$$I(s;\alpha,\beta) = \sum_{m=1}^{\infty} \frac{1}{(m+\alpha)^s(m+\beta)^s}. \tag{2.5}$$

We shall study this function $I(s;\alpha,\beta)$ in detail. The starting point is the formula [2, p.314, 3.197.1] (with $\mu = \rho = s$):

$$\begin{aligned} &\int_0^{\infty} x^{\nu-1}(x+\alpha)^{-s}(x+\beta)^{-s}dx \\ &= \beta^{-s}\alpha^{\nu-s}B(\nu, 2s-\nu)F\left(s, \nu; 2s; 1-\frac{\alpha}{\beta}\right), \end{aligned} \tag{2.6}$$

where $0 < \Re\nu < 2\Re s$, $B(\alpha,\beta)$ is the beta function and $F(a,b;c;z)$ is the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ (cf. [1, p.56]). The above formula is valid only if

$$|\arg\alpha - \arg\beta| < \pi \tag{2.7}$$

(note that the argument is taken in $[-\pi, \pi)$), though in [2] the condition on the argument of α and β is stated wrongly as

$$|\arg \alpha| < \pi, \quad |\arg \beta| < \pi.$$

Under condition (2.7), we may deduce (2.6) by rotating from $\mathbb{R}_+$ to $\mathbb{R}_+ e^{i \arg \alpha}$ and apply [1, p.115, (5)]. Similarly, we may deduce

$$\begin{aligned} \text{(2.8)} \qquad & \int_0^\infty x^{\nu-1}(x+\alpha)^{-s}(x+\beta)^{-s}\, dx \\ &= \alpha^{-s}\beta^{\nu-s}B(\nu, 2s-\nu)F\left(s, \nu; 2s; 1-\frac{\beta}{\alpha}\right). \end{aligned}$$

We may confirm the consistency of (2.6) and (2.8) by Kummer's relation ([1, p.105, (1), (3)]).

By the Mellin inversion, (2.6), respectively (2.8) becomes

$$\begin{aligned} \text{(2.9)} \qquad & (x+\alpha)^{-s}(x+\beta)^{-s} \\ &= \frac{\alpha^{-s}\beta^{-s}}{2\pi i}\int_{(c)} \alpha^{\nu}B(\nu, 2s-\nu)F\left(s, \nu; 2s; 1-\frac{\alpha}{\beta}\right)x^{-\nu}d\nu, \end{aligned}$$

respectively

$$\begin{aligned} \text{(2.10)} \qquad & (x+\alpha)^{-s}(x+\beta)^{-s} \\ &= \frac{\alpha^{-s}\beta^{-s}}{2\pi i}\int_{(c)} \beta^{\nu}B(\nu, 2s-\nu)F\left(s, \nu; 2s; 1-\frac{\beta}{\alpha}\right)x^{-\nu}\, d\nu, \end{aligned}$$

where $1 < c < 2\sigma$.

Of these we shall use (2.10) (see below) to treat J_1 with $\alpha = \omega y$, $\beta = \bar{\omega} y$, α, β being in the right hand-side of the imaginary axis. However, we cannot apply (2.9) directly to the treatment of J_2, since $\alpha = -\bar{\omega} n$ and $\beta = -\omega n$ do not satisfy the condition (2.7). To overcome this difficulty, we start from the initial domain of α and β lying in the first and the fourth quadrant, respectively, and continue into another region

$$\text{(2.11)} \qquad \frac{\pi}{2} \le \arg \alpha < \pi, \quad -\pi < \arg \beta \le -\frac{\pi}{2}.$$

For this purpose we recall the following connection formula of the hypergeometric function:

(2.12)

$$\begin{aligned} F(a, b; c; z) &= (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}F\left(a, c-b; 1+a-b; \frac{1}{1-z}\right) \\ &+ (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}F\left(c-a, b; 1-a+b; \frac{1}{1-z}\right) \end{aligned}$$

valid for $|\arg(-z)| < \pi$, $|\arg(1-z)| < \pi$, $a - b \notin \mathbb{Z}$ ([1, p.109 (3)]).

This gives, in the right-hand side of (2.10), that

$$
\begin{aligned}
(2.13) \qquad & B(s, 2s-\nu) F\left(s, \nu; 2s; 1-\frac{\beta}{\alpha}\right) \\
&= \left(\frac{\beta}{\alpha}\right)^{-s} \frac{\Gamma(2s-\nu)\Gamma(\nu-s)}{\Gamma(s)} F\left(s, 2s-\nu; 1+s-\nu; \frac{\alpha}{\beta}\right) \\
&+ \left(\frac{\beta}{\alpha}\right)^{-\nu} \frac{\Gamma(\nu)\Gamma(s-\nu)}{\Gamma(s)} F\left(s, \nu; 1-s+\nu; \frac{\alpha}{\beta}\right)
\end{aligned}
$$

for α and β in the first and the fourth quadrant, respectively.

We recall our previous statement that we use (2.10) to treat J_1, but what we apply is not exactly (2.10) but the following which is obtained by substituting (2.13) into (2.10):

$$
\begin{aligned}
(2.14) \quad & (x+\alpha)^{-s}(x+\beta)^{-s} \\
&= \frac{\beta^{-2s}}{2\pi i} \int_{(c)} \beta^{\nu} \frac{\Gamma(2s-\nu)\Gamma(\nu-s)}{\Gamma(s)} F\left(s, 2s-\nu; 1+s-\nu; \frac{\alpha}{\beta}\right) x^{-\nu} d\nu \\
&+ \frac{\alpha^{-s}\beta^{-s}}{2\pi i} \int_{(c)} \alpha^{\nu} \frac{\Gamma(\nu)\Gamma(s-\nu)}{\Gamma(s)} F\left(s, \nu; 1-s+\nu; \frac{\alpha}{\beta}\right) x^{-\nu} d\nu
\end{aligned}
$$

for $0 < c < 2\sigma$.

In deducing (2.14), we made use of the exponential law

$$
(2.15) \qquad \left(\frac{\alpha}{\beta}\right)^{s} = \alpha^{s}\beta^{-s}
$$

valid for any $s \in \mathbb{C}$ and α, β in their original region. However, in the region (2.11), the argument of β/α should be $-2\pi < \arg \beta/\alpha \le -\pi$ after the analytic continuation, so that (2.13) does not hold as it stands. Here we have to continue the function $(\alpha/\beta)^s$ analytically to the region (2.11) by the exponential law (2.15).

On the other hand, the hypergeometric functions in the right hand side of (2.13) is analytically continued into the region (2.11) by the same form, since the argument $\frac{\alpha}{\beta}$ does not cross the segment $[1, \infty)$.

As a result, the true form of analytic continuation of $B(\nu, 2s-\nu)F(s, \nu; 1+$

$s-\nu; 1-\frac{\alpha}{\beta}$) ($\alpha$ and β are interchanged) into the region (2.11) is given by

(2.16)
$$\begin{aligned} &B(\nu, 2s-\nu)F\left(s, \nu; 1+s-\nu; 1-\frac{\alpha}{\beta}\right) \\ &= \alpha^{-s}\beta^{s}\frac{\Gamma(2s-\nu)\Gamma(\nu-s)}{\Gamma(s)}F\left(s, 2s-\nu; 1+s-\nu; \frac{\beta}{\alpha}\right) \\ &+ \alpha^{-\nu}\beta^{\nu}\frac{\Gamma(\nu)\Gamma(s-\nu)}{\Gamma(s)}F\left(s, \nu; 1-s+\nu; \frac{\beta}{\alpha}\right). \end{aligned}$$

Substituting (2.16) into (2.9), we also get

(2.17)
$$\begin{aligned} &(x+\alpha)^{-s}(x+\beta)^{-s} \\ &= \frac{\alpha^{-2s}}{2\pi i}\int_{(c)} \alpha^{\nu}\frac{\Gamma(2s-\nu)\Gamma(\nu-s)}{\Gamma(s)}F\left(s, 2s-\nu; 1+s-\nu; \frac{\beta}{\alpha}\right)x^{-\nu}d\nu \\ &+ \frac{\alpha^{-s}\beta^{-s}}{2\pi i}\int_{(c)} \beta^{\nu}\frac{\Gamma(\nu)\Gamma(s-\nu)}{\Gamma(s)}F\left(s, \nu; 1-s+\nu; \frac{\beta}{\alpha}\right)x^{-\nu}d\nu \end{aligned}$$

for $0 < c < 2\sigma$.

Remark. In (2.9) and (2.10), we assumed that

$$0 < \Re\nu < 2\Re s.$$

The first term in the right hand side of (2.13) have poles at

$$\nu = s-n \quad n \in \mathbb{N},$$

while as is easily checked by [4, p.243 (9.2.14)], the second term has poles at the same points with the residue of opposite sign, which is compatible with the holomorphy of the left hand side.

3 Treatment of $J_1(s)$ and $J_2(s)$

Let $1 < \sigma := \Re s < 2$. To treat $J_1(s)$ we apply (2.14) with $\alpha = \omega n$ and $\beta = \bar{\omega}n$ (in which $\alpha^s = \omega^s n^s, \beta^s = \bar{\omega}^s n^s$) and sum over m and n. Then we obtain

(3.1)
$$\begin{aligned} J_1(s) &= \frac{\bar{\omega}^{-2s}}{2\pi i}\int_{(c)} \bar{\omega}^{\nu}\frac{\Gamma(2s-\nu)\Gamma(\nu-s)}{\Gamma(s)}F\left(s, 2s-\nu; 1+s-\nu; \frac{\omega}{\bar{\omega}}\right)\zeta(2s-\nu)\zeta(\nu)d\nu \\ &+ \frac{\omega^{-s}\bar{\omega}^{-s}}{2\pi i}\int_{(c)} \omega^{\nu}\frac{\Gamma(\nu)\Gamma(s-\nu)}{\Gamma(s)}F\left(s, \nu; 1-s+\nu; \frac{\omega}{\bar{\omega}}\right)\zeta(2s-\nu)\zeta(\nu)d\nu, \end{aligned}$$

where

$$1 < c < 2\sigma - 1. \tag{3.2}$$

On the other hand, we apply (2.17) to J_2 with $\alpha = -\bar{\omega}n = e^{\pi i}\bar{\omega}n$ and $\beta = -\omega n = e^{-\pi i}\omega n$ to deduce

(3.3)

$$\begin{aligned}
J_2(s) &= \sum_{n=1}^{\infty} I(s; -\bar{\omega}n, -\omega n)\\
&= \frac{(e^{\pi i}\bar{\omega})^{-2s}}{2\pi i}\int_{(c)} (e^{\pi i}\bar{\omega})^{\nu}\frac{\Gamma(2s-\nu)\Gamma(\nu-s)}{\Gamma(s)}F\left(s, 2s-\nu; 1+s-\nu; \frac{\omega}{\bar{\omega}}\right)\\
&\qquad\qquad \times \zeta(2s-\nu)\zeta(\nu)d\nu\\
&+ \frac{(e^{\pi i}\bar{\omega})^{-s}(e^{-\pi i}\omega)^{-s}}{2\pi i}\int_{(c)} (e^{-\pi i}\omega)^{\nu}\frac{\Gamma(\nu)\Gamma(s-\nu)}{\Gamma(s)}F\left(s, \nu; 1-s+\nu; \frac{\omega}{\bar{\omega}}\right)\\
&\qquad\qquad \times \zeta(2s-\nu)\zeta(\nu)d\nu.
\end{aligned}$$

We now shift the line of integration to $\Re\nu = -\varepsilon$, $\varepsilon > 0$ encountering the poles at $\nu = 1$ (from $\zeta(\nu)$) and $\nu = 0$ (from $\Gamma(\nu)$), where, as remarked at the end of §2, the point $\nu = s - 1$ is not a pole of the integrands of $J_1(s)$ and $J_2(s)$.

To find the residues we use the formulas:

$$F(a, b; a; z) = (1-z)^{-b} \tag{3.4}$$

$$F(a, 0; c; z) = 1. \tag{3.5}$$

For J_1, we have

$$\begin{aligned}
\operatorname*{Res}_{\nu=1} &= \frac{1}{(\bar{\omega}-\omega)^{2s-1}}\frac{\Gamma(2s-1)\Gamma(1-s)}{\Gamma(s)}\zeta(2s-1)\\
&\quad + \bar{\omega}^{-s}\omega^{1-s}\frac{\Gamma(s-1)}{\Gamma(s)}F(s, 1; 2-s; \frac{\omega}{\bar{\omega}})\zeta(2s-1)
\end{aligned}$$

and

$$\operatorname*{Res}_{\nu=0} = -\frac{1}{2}(\omega\bar{\omega})^{-s}\zeta(2s),$$

while for J_2,

$$\operatorname*{Res}_{\nu=1} = -e^{-2\pi is}\frac{1}{(\bar\omega-\omega)^{2s-1}}\frac{\Gamma(2s-1)\Gamma(1-s)}{\Gamma(s)}\zeta(2s-1)$$
$$-\bar\omega^{-s}\omega^{1-s}\frac{\Gamma(s-1)}{\Gamma(s)}F(s,1;2-s;\frac{\omega}{\bar\omega})\zeta(2s-1)$$

and

$$\operatorname*{Res}_{\nu=0} = -\frac{1}{2}(\omega\bar\omega)^{-s}\zeta(2s).$$

Hence

(3.6)

$$\begin{aligned}
J_1(s) = {} & \frac{1}{(\bar\omega-\omega)^{2s-1}}\frac{\Gamma(2s-1)\Gamma(1-s)}{\Gamma(s)}\zeta(2s-1) \\
& + \bar\omega^{-s}\omega^{1-s}\frac{\Gamma(s-1)}{\Gamma(s)}F\left(s,1;2-s;\frac{\omega}{\bar\omega}\right)\zeta(2s-1) \\
& - \frac{1}{2}(\omega\bar\omega)^{-s}\zeta(2s) \\
& + \frac{\bar\omega^{-2s}}{2\pi i}\int_{(-\varepsilon)}\omega^{-\nu}\frac{\Gamma(2s-\nu)\Gamma(\nu-s)}{\Gamma(s)}F\left(s,2s-\nu;1+s-\nu;\frac{\omega}{\bar\omega}\right) \\
& \qquad \times\zeta(2s-\nu)\zeta(\nu)d\nu \\
& + \frac{(\omega\bar\omega)^{-s}}{2\pi i}\int_{(-\varepsilon)}\omega^{\nu}\frac{\Gamma(\nu)\Gamma(s-\nu)}{\Gamma(s)}F(s,\nu;1-s+\nu;\frac{\omega}{\bar\omega})\zeta(2s-\nu)\zeta(\nu)d\nu
\end{aligned}$$

and

(3.7)

$$\begin{aligned}
J_2(s) = {} & -\frac{e^{-2\pi is}}{(\bar\omega-\omega)^{2s-1}}\frac{\Gamma(2s-1)\Gamma(1-s)}{\Gamma(s)}\zeta(2s-1) \\
& - \bar\omega^{-s}\omega^{1-s}\frac{\Gamma(s-1)}{\Gamma(s)}F(s,1;2-s;\frac{\omega}{\bar\omega})\zeta(2s-1) \\
& - \frac{1}{2}(\omega\bar\omega)^{-s}\zeta(2s) \\
& + \frac{(e^{\pi i}\bar\omega)^{-2s}}{2\pi i}\int_{(-\varepsilon)}(e^{\pi i}\omega)^{\nu}\frac{\Gamma(2s-\nu)\Gamma(\nu-s)}{\Gamma(s)}F(s,2s-\nu;1+s-\nu;\frac{\omega}{\bar\omega}) \\
& \qquad \times\zeta(2s-\nu)\zeta(\nu)d\nu \\
& + \frac{\omega\bar\omega)^{-s}}{2\pi i}\int_{(-\varepsilon)}(e^{-\pi i}\omega)^{\nu}\frac{\Gamma(\nu)\Gamma(s-\nu)}{\Gamma(s)}F(s,\nu;1-s+\nu;\frac{\omega}{\bar\omega})\zeta(2s-\nu)\zeta(\nu)d\nu.
\end{aligned}$$

We calculate $J_1(s) + J_2(s)$ by summing the corresponding terms separately. The second terms cancel each other and the sum of the third term is

$$-(\omega\bar{\omega})^{-s}\zeta(2s) = -\left(\frac{a}{c}\right)^s \zeta(2s). \tag{3.8}$$

If factor out in the sum of terms, we have the remaining factor $1 - e^{-2\pi i s} = e^{-\pi i s}2i\sin\pi s$, which is $2\pi i e^{-\pi i s}\frac{1}{\Gamma(1-s)\Gamma(s)}$ by the reciprocity relation. Therefore the sum of first terms is

$$\frac{2\pi i e^{-\pi i s}}{(\bar{\omega}-\omega)^{2s-1}}\frac{\Gamma(2s-1)}{\Gamma(s)^2}\zeta(2s-1),$$

which becomes by (1.3) and (1.4) and by the duplication formula for the gamma function

$$\frac{2^{2s-1}a^{2s-1}\sqrt{\pi}}{\Delta^{s-\frac{1}{2}}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1). \tag{3.9}$$

We denote the sum of the fourth terms by $T_1(s)$ and that of the fifth terms by $T_2(s)$, respectively. Substituting the expression for $J_1(s) + J_2(s)$ obtained by adding (3.8), (3.9) etc., we deduce that

$$S = -\frac{2}{c^s}\zeta(2s) + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Delta^{s-\frac{1}{2}}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1) + \frac{2}{a^s}\{T_1(s) + T_2(s)\}, \tag{3.10}$$

where

$$\begin{aligned} T_1(s) = \frac{\bar{\omega}^{-2s}}{2\pi i}\int_{(-\varepsilon)} &\left(1 + e^{\pi i(\nu-2s)}\right)\bar{\omega}^{\nu}\frac{\Gamma(2s-\nu)\Gamma(\nu-s)}{\Gamma(s)} \\ &\times F\left(s, 2s-\nu; 1+s-\nu; \frac{\omega}{\bar{\omega}}\right)\zeta(2s-\nu)\zeta(\nu)d\nu \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} T_2(s) = \frac{\bar{\omega}^{-s}\omega^{-s}}{2\pi i}\int_{(-\varepsilon)} &\left(1 + e^{-\pi i\nu}\right)\omega^{\nu}\frac{\Gamma(\nu)\Gamma(s-\nu)}{\Gamma(s)} \\ &\times F\left(s, \nu; 1-s+\nu; \frac{\omega}{\bar{\omega}}\right)\zeta(2s-\nu)\zeta(\nu)d\nu. \end{aligned} \tag{3.12}$$

Hence we deduce the integral formula:

Theorem 3.1. *We have for* $\sigma > 0$

$$\begin{aligned} \zeta_Q(s) = &\frac{2}{a^s}\zeta(2s) + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Delta^{s-\frac{1}{2}}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1) \\ &+ \frac{2}{a^s}\{T_1(s) + T_2(s)\}, \end{aligned} \tag{3.13}$$

where $T_1(s)$ *and* $T_2(s)$ *are given by* (3.11) *and* (3.12) *respectively.*

4 The Kronecker limit formula

It is easy to see that

$$\begin{aligned}(4.1)\qquad &\frac{2^{2s}a^{s-1}\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)\Delta^{s-\frac{1}{2}}}\zeta(2s-1)\\ &=\frac{2\pi}{\sqrt{\Delta}}\cdot\frac{1}{s-1}+\frac{2\pi}{\sqrt{\Delta}}\left(2\gamma+\log\frac{a}{\Delta}\right)+O(s-1),\end{aligned}$$

where γ indicates Euler's constant (cf. also [6, (36), p.96]). It therefore suffices to find the limit of $T_1(s)$ and $T_2(s)$ as $s \to 1$.

First we find that

$$\begin{aligned}\lim_{s\to 1} T_1(s) = T_1(1) = \frac{\bar{\omega}^{-2}}{2\pi i}\int_{(-\varepsilon)} &(1+e^{\pi i\nu})\bar{\omega}^{\nu}\Gamma(2-\nu)\Gamma(\nu-1)\\ &\times F\left(1, 2-\nu; 2-\nu; \frac{\omega}{\bar{\omega}}\right)\zeta(2-\nu)\zeta(\nu)d\nu.\end{aligned}$$

By using (3.4), the functional equation of the Riemann zeta-function

$$\zeta(\nu) = 2^{\nu}\pi^{\nu-1}\sin\frac{|pi\nu}{2}\Gamma(1-\nu)\zeta(1-\nu)$$

and changing the variable $\mu = 1-\nu$, we have

$$T_1(1) = -\frac{2\pi i}{\bar{\omega}-\omega}\frac{1}{2\pi i}\int_{(1+\varepsilon)}(2\pi i\bar{\omega})^{-\mu}\Gamma(\mu)\zeta(1+\mu)\zeta(\mu)d\mu.$$

We note that

$$\zeta(1+\mu)\zeta(\mu) = \sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n^{1+\mu}},$$

where $\sigma_c(n) = \sum_{d|n} d^c$, hence we obtain

$$\begin{aligned}(4.2)\qquad T_1(1) &= -\frac{2\pi i}{\bar{\omega}-\omega}\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}\frac{1}{2\pi i}\int_{(1+\varepsilon)}\Gamma(\mu)(2\pi i n\bar{\omega})^{-\mu}d\mu\\ &= -\frac{2\pi i}{\bar{\omega}-\omega}\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}e^{-2\pi i n\bar{\omega}}.\end{aligned}$$

Similarly we have

$$(4.3)\qquad T_1(1) = -\frac{2\pi i}{\bar{\omega}-\omega}\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}e^{2\pi i n\omega}.$$

Substituting (4.1), (4.2) and (4.3) into (3.13), we conclude that

$$(4.4)\quad \zeta_Q(s) = \frac{2\pi}{\sqrt{\Delta}} \cdot \frac{1}{s-1} + \frac{2\pi}{\sqrt{\Delta}}\left(2\gamma + \log\frac{a}{\Delta}\right) + \frac{2\zeta(2)}{a} + \frac{4\pi}{\sqrt{\Delta}}\left\{\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}e^{-2\pi in\bar{\omega}} + \sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}e^{2\pi in\omega}\right\} + O(s-1).$$

The remaining part is standard routine and can be summarized as follows.

Integrating the Lambert series

$$\sum_{n=1}^{\infty}\sigma(n)e^{-ns} = \sum_{n=1}^{\infty}\frac{n}{e^{ns}-1}, \quad \Re s > 0,$$

we have

$$(4.5)\qquad \sum_{n=1}^{\infty}\frac{\sigma(n)}{n}e^{-ns} = -\sum_{n=1}^{\infty}\log(1-e^{-ns}).$$

On the other hand, the Dedekind eta-function is defined by

$$\eta(\tau) = e^{\frac{2\pi i\tau}{24}}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}), \quad \Im\tau > 0,$$

whence

$$(4.6)\qquad \log\eta(\tau) = \frac{\pi i\tau}{12} + \sum_{n=1}^{\infty}\log(1-e^{2\pi in\tau}), \quad \Im\tau > 0.$$

Combining (4.5) and (4.7), we have

$$(4.7)\qquad \sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}e^{2\pi in\tau} = \frac{\pi i\tau}{12} - \log\eta(\tau),$$

so that

$$(4.8)\qquad \sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}e^{-2\pi in\bar{\omega}} + \sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}e^{2\pi in\omega} = -\log\eta(\omega)\eta(-\bar{\omega}) - \frac{\pi\sqrt{\Delta}}{12a}.$$

Subatituting (4.8) in (4.4), we obtain

Corollary 4.1 (The Kronecker limit formula).

$$\lim_{s\to 1}\left\{\zeta_Q(s)-\frac{2\pi}{\sqrt{\Delta}}\frac{1}{s-1}\right\}=\frac{4\pi\gamma}{\sqrt{\Delta}}+\frac{2\pi}{\sqrt{\Delta}}\log\frac{a}{\Delta}-\frac{4\pi}{\sqrt{\Delta}}\log\eta(\omega)\eta(-\bar{\omega}).$$

Remark. The following equivalent form of (4.4)

$$\begin{aligned}\zeta_Q(s)=&\frac{\frac{2\pi}{\sqrt{\Delta}}}{s-1}+\frac{2\pi}{\sqrt{\Delta}}\left(2\gamma+\log\frac{a}{\Delta}\right)+\frac{\pi^2}{3a}\\&+\frac{8\pi}{\sqrt{\Delta}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\cos\frac{\pi nb}{a}e^{-\frac{n\pi\sqrt{\Delta}}{a}}+O(s-1),\end{aligned}$$

is Formula (39) of [6, p.97].

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