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Lecture given by K. Ramachandra on Friday 5 June, 2009, from 12:00 to 1:00.

Little Flowers to Srinivasa Ramanujan (22 December 1887 - 26 April 1920)

In a similar event I have given a lecture, entitled "Little Flowers to G. H. Hardy," [13]. I also gave a lecture entitled "Little flowers to I. M.Vinogradov," [14]. I had the honor of being invited to Vinogradov's 80th birthday in 1971 September, and again on his 90th birthday, in 1981 September. Unfortunately he did not live to be 100.

In this talk, I will give some attractive results and offer them as little flowers to Srinivasa Ramanujan.

Flower 1:

Let a and b be complex constants with $a \neq b$ and $(1-a)(1-b) \neq 0$ and let

$$F(s) = (\zeta(s) - a)(\zeta(s) - b), (s = \sigma + it)$$

Then for all $X \geq X_0(a, b)$, the gaps between X and $2X$ of the ordinates of simple zeroes of $F(s)$ does not exceed $c(a, b) \log \log X$ ($c(a, b)$ depends only on a and b).

Remark. This was proved in a series of papers by R. Balasubramanian, K. Ramachandra, A. Sankaranarayanan, and K. Srinivas.

Reference Notes on the Riemann zeta function - I, II, III, IV, and V. The reference to the final paper V is Hardy-Ramanujan Journal (23) (2000) 2-9. [10][11][4][5][6]

Flower 2:

On some theorems of Littlewood and Selberg - IV, by K. Ramachandra and A. Sankaranarayanan.

Reference Acta Arithmetica Vol 70 (1) (1995) 79-84. [15]

Remark 1. Let $s = \sigma + it$, $s_0 = \frac{1}{2} + it_0$, and let t_0 exceed a large a large positive constant. Then E. C. Titchmarsh proved unconditionally that the region $\sigma \geq \frac{1}{2}$, $|t - t_0| \leq \frac{c_1}{\log\log\log t_0}$ contains at least one zero of $\zeta(s)$.

Remark 2. Assume Riemann Hypothesis. Then A. Selberg proved that the line segment

$$\sigma = \tfrac{1}{2}, |t - t_0| \leq \tfrac{c_2}{\log\log t_0}$$

contains at least one zero of $\zeta(s)$.

Our result is unconditional and runs as follows.
If the region

$$\tfrac{1}{2} \leq \sigma \leq \tfrac{1}{2} + \tfrac{c_3}{\log\log t_0}, |t - t_0| \leq \tfrac{c_3}{\log\log t_0}$$

is zero-free, then the region

$$\sigma \geq \tfrac{1}{2} + \tfrac{c_3}{\log\log t_0}, |t - t_0| \leq c_4 \log\log\log t_0$$

contains at least one zero of $\zeta(s)$.
Here c_1, c_2, c_3 , and c_4 are absolute positive constants.

Flower 3:

Let $Q = Q(x_1, ..., x_k)$ be a positive definite quadratic form with integers as coefficients in $k(\geq 3)$ variables. Consider the function

$$F(s) = \sum (Q(n_1, ..., n_k))^{-s}$$

where the sum is over all $k-$tuples of integers $(n_1, ..., n_k)$ with the exception of $(0, 0, ..., 0)$. Then given any $\delta > 0$ the number of zeros of $F(s)$ in $T \leq t \leq 2T, |\sigma - \frac{1}{2}| \leq \delta$ exceeds $c(\delta)T \log T$ zeros for all $T \geq T_0(\delta)$. Here $T_0(\delta)$ and $c(\delta)$ depend only on δ and Q.

Reference Hardy's Theorem for zeta-functions of quadratic forms (K. Ramachandra and A. Sankaranarayanan) Proc. Indian Acad. Sci (Math. Sci) 106 (3) (1996), 217-226. [9]

Flower 4:

Reference On a problem of Ivić (Hardy-Ramanujan Journal) (23) (2000) 10-19, by K. Ramachandra. [7]

Ivić proved the following. Let $\beta + i\gamma$ run over all the non-trivial zeros of $\zeta(s)$ subject to $\beta > 0, T \leq \gamma \leq 2T$. Let $T \geq T_0$ where T_0 exceeds a large positive constant. Then

$$\sum |\zeta(\tfrac{1}{2} + i\gamma)|^2 \ll_\epsilon T(\log T)^2(\log\log T)^{\frac{3}{2}+\epsilon} \qquad (1)$$

Here $|\zeta(\frac{1}{2} + i\gamma)|$ is replaced by

$$\max|\zeta(s)|$$

taken over all σ with $\frac{1}{2} - \frac{A}{\log T} \leq \sigma \leq 2$ and at the same time all t with

$$|t - \gamma| \leq \frac{B\log\log T}{\log T}$$

At the same time R.H.S. of (1) is replaced by

$$T(\log T)^2 \log\log T$$

Here A and B are arbitrary positive constants. In (1) the constant involved in $\ll_\epsilon$ depends on ϵ , A, and B.

Flower 5:

We have proved

$$\sum_{n \leq x} (d(n))^2 = xP(\log x) + O(x^{\frac{1}{2}}(\log x)^5 \log\log x)$$

where $d(n)$ is as usual and $P(\log x)$ is a polynomial in $\log x$ of degree 3.

Reference (K. Ramachandra and A. Sankaranarayanan) Acta Arithmetica (109) (2003) (No. 4) 349-357. (On an asymptotic formula of Srinivasa Ramanujan) [12]

Flower 6:

Let t be any fixed transcendental number and consider the N numbers

$$2^t, 2^{t^2}, ..., 2^{t^N}$$

The number of algebraic numbers amongst these numbers does not exceed

$$\sqrt{2N}(1+\epsilon)$$

where $\epsilon > 0$ is arbitrary and N exceeds $N_0(\epsilon)$ (depending only on ϵ and t).

Reference (R. Balasubramanian and K. Ramachandra) Transcendental numbers and a lemma in combinatorics, Combinatorics and Applications, Indian Statistical Institute (Calcutta) (1984) 57-59. [3]

The readers would also find it interesting to look at "Some problems of analytic number theory. IV," [1], and its continuation "Some problems of analytic number theory. V," [2], which was presented at An International Conference on Diophantine Equations in honour of Professor T.N. Shorey on his 60th Birthday.

It was the 45th session of the Indian Science Congress, held in Chennai. Professor B. S. Madhava Rao was the president of the mathematics section, and he asked me to present a paper. I sent a paper which relates $\sum_{n=1}^{\infty} \frac{H_n^4}{n^2}$ to a rational multiple of π^4, [8]. When I read this paper in the Indian Science Congress, Professor Madhava Rao encouraged my talk. There were many other results in the paper. All of these results were worked out in a joint paper by me and R. Sitaramachandrarao and published in the Journal of the Madras University, [16]. Myself and my wife had been to the house of Professor B.S. Madhava Rao to pay our regards to Professor and Mrs. Madhava Rao. They gave us a codial welcome and blessed us. Professor B. S. Madhava Rao was very much helpful to me in getting my three or four papers published. I am very much thankful to Professor B.S. Madhava Rao for his help.

I would like to thank Mr. Kishor Bhat for his help in typing this manuscript on Latex.

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