HARDY-RAMANUJAN JOURNAL **34 & 35** (2013), 53-63

K. RAMACHANDRA: REMINISCENCES OF HIS FRIENDS

M. Pavaman Murthy

A few Reminiscences of K.Ramachandra in his early years at TIFR

I joined TIFR in 1958 along with Ramachandra, R. R. Simha and Vasanthi Rao. Ramachandra stood out in this group of four because of his remarkable dedication to mathematics and his knowledge of advanced number theory. He had made up his mind even before joining TIFR to work in number theory. A few months after joining the institute, Ramachandra was studying research papers in number theory when the rest of us in the group were struggling with topics exposed in the "baby seminars". The four of us were good friends. In recent times, whenever I visited TIFR from Chicago, Ramachandra used to come in in to my office and inquire about my family and tell me about his work and his students' work. He always spoke of his students with pride.

Here are some of my reminiscences of Ramachandra in his early years at TIFR.

In the first or second year after joining the institute, one early morning in TIFR hostel in the Old Yacht Club, I still remember witnessing a heated discussion between Ramachandra and Raghavan Narasimhan comparing the contributions of Hilbert and Kummer to number theory. Probably that was the first time that I came to know that there was a famous number theorist called Kummer.

Here is a glimpse of Ramachandra's sense of humor. In our early TIFR years, the institute was located in the Old Yacht Club building adjacent to Gateway of India. Several of the members of the School of Mathematics along with us new comers worked in a large hall with many tables. One day Ramachandra came to my table and showed me a theorem of Siegel and said "See, Siegel proves this theorem in ten pages. I have proved the same in five pages." Indeed Siegel's proof covered ten pages. What Ramachandra had done was to copy verbatim Siegel's proof in his note book in five pages (with very small handwriting)!

Sometime after joining the institute, we four new comers were told that there would not be any oral examinations at the end of our first year, contrary to the usual practice. Instead we were to give some talks on selected topics. Our progress in the first year was to be judged by those talks which were to be held in October 1958 after the summer vacation. I had gone to my hometown Hyderabad, hoping to prepare for my talks. Within a few days after my arrival in Hyderabad I received a letter from Ramachandra who was still at the institute. Ramachandra had written advising me to prepare the talks well as he had heard that KC (Professor K. Chandrasekharan)

was to attend our talks and might ask questions in those talks. It was out of kindness and concern about me that Ramachandra had written that letter. Little did he realize that he had spoiled my vacation!

I will miss Ramachandra.

M. Pavaman Murthy

Michel Waldschmidt

K. Ramachandra: Some reminiscences

I was thrilled when I received a letter from Ramachandra around 1974, who invited me to spend some months at the Tata Institute of Bombay and give a course on transcendental numbers. When I was young it was my dream to visit India, and I did not expect that I would have such an opportunity. I knew very well his paper [1] *Contributions to the theory of transcendental numbers* published in Acta Arithmetica in 1968: this was the main reference of my thesis, submitted in 1972. I was able to pursue his work in several directions, including algebraic groups. My first attempt to prove a new result was motivated by one of his problems which is now called the four exponentials conjecture, which had been proposed independently by S. Lang and Th. Schneider. This has been the problem on which I have spent most of my efforts during all my mathematical life, and it is still open. I believed a couple of times that I had a solution, especially in 1970; it turned out that there was a gap in my argument, but that I could nevertheless get something new: instead of solving the four exponentials conjecture, which is the first open problem proposed by Th. Schneider in his book, I could solve the 8th of these problem, on the transcendence of one at least of the two numbers e^e and e^{e^2}. As a matter of fact, the same solution was found at the same time and independently by W.D. Brownawell. For this result, we shared the Distinguished Award of the Hardy-Ramanujan Society which was attributed to both of us by Ramachandra in 1986.

When I received the invitation of Ramachandra, I decided to accept it and to go to India with my wife. However, shortly afterwards, she became pregnant, so I postponed this visit and came alone, after the birth of my son Alexis in May 1976. I visited TIFR from the end of October to end of December 1976. It was not easy for me to leave my young son for such a long time at his early age. My stay in India has been an unforgettable experience for me. I loved it immediately, even if it took me some time before I could adjust to the food.

My lectures were on *transcendental numbers and group varieties*. Since I was going to deliver lectures on that topic at Collège de France (*cours Peccot*) a few months later, I used this opportunity to polish my presentation. The precise topic was a development of Ramachandra's work with applications to commutative algebraic groups. These notes were going to be published in Astérisque in 1979. I did not

know that Ramachandra had shifted his interest from transcendental number theory to the Riemann zeta function two year earlier (so I had no influence on this shift!); and, most of all, I did not know that he was disliking commutative groups as much as he loved numbers. To mix both was not the best thing to do to please him, but I was innocent. My TIFR course was supposed to be published by the Tata Institute, a research student of Ramachandra was supposed to write it down. I left him the notes (it was not that easy at that time to make xerox copies), but the course was never written down, my notes got lost and I had to reconstruct them from scratch.

This was my first experience of spending some time in a non–French speaking country, and my English was quite poor. To spend two months like this was very efficient from this point of view, and since Ramachandra was among the people with whom I spoke often, I made progress during this stay to understand him better. Later, it happend quite a few times that I was with an English speaking mathematician, from UK or USA, and I served as a translator, repeating with my French accent what Ramachandra said with his Indian accent, and it was helpful for the concerned colleague!

I met Ramachandra again in 1979 in Kingston at Queen's University where we participated in a conference on recent developments in number theory, organized by P. Ribenboim, where I was with my family. I also met him later, in July 1987, again in Canada, during an International number theory conference held at Université Laval organized by Jean–Marie De Koninck and Claude Levesque.

My second trip to India was in 1985. A conference in honor of Bambah was scheduled in Chandigarh. My trip was supported by an agreement between the two Academy of Sciences of India and France; I already had my ticket when the conference was postponed for security reasons. Nevertheless I was allowed by the two academies to maintain the project and I first visited Bombay. In Bombay I was invited by Ramachandra, at his apartment and in the evening at the Tanjore restaurant of the Taj Mahal Hotel. I also went to Madras and visited *Matscience* (which became later *IMSc - Institute of Mathematical Sciences*), and on my way back I visited Delhi.

I came back in 1987 for the centenary of Ramanujan, and I could participate to a conference in Annamalai University (next to Chidambaram in Tamil Nadu), and this was the opportunity for my first trip to Kumbakonam. I visited the home town of Ramanujan three more times later (December of 2007, 2008 and 2009) when I was in the selection committee of the Ramanujan SASTRA Prize.

This visit in 1987 gave me the opportunity to organize my next visit, one year later, with my family. I had been invited by Alf van der Poorten to spend two months (July and August 1987) in Australia. The advantage of my trip to the southern hemisphere was that it was giving us the opportunity to visit India on the way back. This visit to India for my family was threatened at the last minute (at the airport of Sydney, just

before boarding for Bombay) for a question of visa, but fortunately we could make it. With my wife Anne, my son Alexis who was just 12, and my daughter Hélène who was 10, we arrived in Bombay on August 20, 1988, we went to Madras on 22, we visited Pondicherry from 23 to 25, and came back to France on 28. This was a tight schedule, but this has been an unforgettable experience, one of the high points in my life as well as in the life of my children (my daughter Hélène came back on her own to India in 2000 and in 2001). In Bombay we stayed at the Tata Institute. We were invited by Ramachandra, who took us to the Prince of Wales Museum and was our guide. My children were to remember that he had a pink shirt during that visit: this is not common for us that a man would wear a shirt of that color.

After that I was to come back on a regular basis to India, on the average more than once a year, and I met Ramachandra very often. I was there for the two major conferences which were organized for his birthdays, the sixtieth in July 1993 (organized by R. Balasubramanian in Madras) and the seventieth in 2003 (organized by K. Srinivas in Bangalore). I wrote a survey based on Ramachandra's paper [1]. This survey was completed after the first conference and published after the second one [2]. In December 2003, during this conference, I visited Ramachandra's office at NIAS. It was almost empty. Only a picture of G.H. Hardy and a picture of S. Ramanujan were on the wall, to whom he was deeply devoted. He told me that once a year, he had to leave his office which was used by other people for a few days, so he could not keep anything personal there.

I have a specially fond and very moving memory of our meetings in Bangalore in early 2005 (end of January – beginning of February). I was the representative of CIMPA for the school on *Security of computer systems and networks*, organized at the Indian Institute of Science by K. Gopinath. This was only four months after my daughter passed away, and Ramachandra found the right words to speak with me. He also gave me his personal reminiscences. He told me the difficulties he had during his own life. He spoke of his brother and his nephews. He mentioned that his father passed away while he was only 13, that he needed to take care of his family, and that he was fully dedicated to mathematics - this much I already knew! He told me how difficult it has been for him to take certain decisions, like that of moving from one place to another. And, of course, we shared our concerns as fathers who care about their daughters. This is certainly one reason why I had the feeling to be so close to him, and I did my best to meet him as often as possible. I stayed in India two months in December 2009 - January 2010. I was ready to go from Chennai to Bangalore in December, 2009 to visit Ramachandra, but Kishor Bhat, who was taking care of the arrangement, told me that the daughter of Ramachandra had to go to the hospital and he suggested me to postpone my visit, which I did. So this visit took place in January 2010, and this was to be our last meeting. At that time he gave me

some money for P. Philippon, to whom he attributed the Hardy–Ramanujan award. When I told this to Philippon he was grateful and suggested that the money go to an orphanage, which I could do immediately thanks to Prem Prakash in Chennai.

I came back to India for ICM2010 in Hyderabad in August 2010 and for a satellite conference just after in Chennai, I was in transit in Bangalore on the way, but my schedule was too tight and I did not visit him that time - I missed an opportunity.

It has been a great privilege for me to know Ramachandra. I never met anyone else who would be so dedicated to mathematics. I also knew him on a more personal basis. I admire him, he was truly exceptional. I miss him.

Michel Waldschmidt, May 9, 2011

This text is an abstract of a Colloquium talk given at TIFR CAM Bangalore on April 26, 2011, at the invitation of C. S. Aravinda. The author is thankful to C. S. Aravinda for this invitation, to K. Sandeep who took care of the organisation, to Kishor Bhat and K. Srinivas who made possible a visit the same day to Mrs. Ramachandra.

The colloquium talk included also a mathematical discussion concerning Ramachandra's contributions to transcendental number theory. The pdf file of the talk is available on the web site of the author. The main reference is

[1] Ramachandra, K. – *Contributions to the theory of transcendental numbers* (I); Acta Arith., **14** (1968), 65–72; (II), id., 73–88.

A description of this work, which is reprinted in this issue, is given in

[2] Waldschmidt, M. – *On Ramachandra's contributions to transcendental number theory;* Ramanujan Mathematical Society, Lecture Notes Series Number 2, *The Riemann Zeta function and related themes: papers in honour of Professor K. Ramachandra*, Proceedings of International Conference held at National Institute of Advanced Studies, Bangalore 13-15 December, 2003 Ed. R. Balasubramanian, K. Srinivas (2006), 155–179.

K Soundararajan

It is a privilege to have known as great a man and mathematician as Professor Ramachandra. I first met him in 1989 when I was in high school and learning mathematics from Professor Balasubramanian. Balu had mentioned my interests to Ramachandra, and I was astonished to receive shortly afterwards an invitation from Ramachandra to spend a couple of weeks at TIFR. I learnt a great deal from him during those two weeks, especially on the subjects close to his heart – the distribution of prime numbers and the behavior of the zeta function. Equally, I

was struck by his warmth and friendliness and his childlike love and enthusiasm for mathematics. The distance between our years or accomplishments was completely absent in our interactions. To borrow a phrase from "My Fair Lady," Ramachandra treated all flower-girls as duchesses. Over the years, I was fortunate to have many more interactions with him, and I have benefited immensely from his encouragement, advice, and generosity in sharing ideas. And of course, his many beautiful papers in number theory have been a source of inspiration for me and many others.

In 1990, Aleksandar Ivic gave a series of lectures in TIFR on mean-values of the zeta-function. Ramachandra kindly invited me to attend this series, and I spent a very happy month in Bombay learning from these lectures and many conversations with Balu and Ramachandra. I was nearing then the end of my high school years, and was thinking about where to pursue my undergraduate education. I sought advice from Ramachandra on this, and he enthusiastically recommended my going to the University of Michigan to work with Hugh Montgomery. I did so, and my next occasion to meet Ramachandra was on the happy occasion of his sixtieth birthday in 1993 when a celebratory conference was organized at the Institute for Mathematical Sciences, Chennai. I was honored to speak at this conference, and Balu and I wrote a paper *On a conjecture of R. L. Graham* which we dedicated to him.

Ramachandra took great pride and pleasure in the accomplishments of his students – and I count myself as a student/grandstudent of his – and a nice result by one of his students gave him even more joy than his own great theorems. Through the 70's and 80's Ramachandra wrote a series of path-breaking papers on the zeta function, making great progress on understanding the moments and extreme values of the zeta function. Shortly after his sixtieth birthday conference, I was reading one of these beautiful papers (*J. London Math. Soc.*, 1975) on the fourth moment of the zeta-function, and this directly inspired me to work out new lower bounds for moments of the zeta-function. I sent the paper and related work to Ramachandra who was absolutely delighted. This reaction from one of the pioneers in the field was a source of great encouragement to me. A little later, Ramachandra himself wrote another paper on fractional moments of the zeta-function, and at the risk of seeming immodest I am very proud to say that he dedicated this paper to my 23rd birthday! No one except Ramachandra would think of dedicating a paper to someone's 23rd birthday, and to him this would have seemed perfectly natural!

After Ramachandra retired from TIFR and moved to Bangalore I had a couple of occasions to meet him. In 1997 he invited me to NIAS to give a couple of lectures and I spent a happy week with him. I next saw him at the 2003 meeting of the American Math Society in Bangalore. As always he was full of ideas and spoke excitedly of the many problems he was thinking about. The AMS meeting followed a conference in

honor of Ramachandra's 70th birthday, which unfortunately I missed, but Granville and I wrote a paper in his honor on one of his favorite topics (extreme values of the zeta function). It did not occur to me that this would be the last time I saw him. We of course kept in touch over the next several years: I would always receive new year cards, and the next issue of the Hardy-Ramanujan Journal, and he always forgave my tardy responses. I thought I would see him at the 2010 ICM, or at Balu's 60th birthday conference, but that was not to be. I miss him greatly.

Prabhakar Vaidya

I, like the rest of us at NIAS, have not had sufficient time to reflect on the magnitude of the great personal loss that I have suffered. I am going to mention one immediate thing that springs to my mind.

I will miss visiting his office. His office was physically quite close to mine. Yet, the few steps I often took to visit his office were nothing short of magical. Once you entered his door, I was transformed into a magical world. There were these two towering portraits of Hardy and Ramanujan. I would look at them in reverence and then my eyes would wander to his scribbling on the whiteboard. He would notice where they had rested and his face would be lit. "This one is by Soundararajan. He has improved the result of Montgomery, which was in turn an improvement of the famous result of Vinogradov...."

(This was his world. Numbers, theorems, Hardy, Ramanujan. Michigan was mostly Montgomery and when he announced at our faculty meeting that Vinogradov had passed away, he could not stop his tears.)

He would notice my confusion as I try to grasp this new result on the board and say in a kind voice, "oh don't worry about these constants, they don't matter...". I keep asking myself, "log log log of x?.." How on earth does anyone think of this? He would start explaining and I would look at his face and say to myself, "Can God have a kinder face?"

No matter how many times I visited his office, the result was the same. I was a much purer version of me than the person who entered. I was back in my childhood, worshipping Ramanujan, dreaming that one day I will prove Fermat's last theorem, I would be lost in my books, in numbers, in dreams....

That boy is now mostly gone. And yet, in his room, he was back. The innocence, the purity Professor Ramachandra, radiated it to us. The glow would last for while, even after I used to leave his room. I knew that this was a very special Darshana. Yes, my rest of the day would sail quite smoothly now.

Was I really this lucky to have known this extraordinary genius, this man whose devotion to Ramanujan was greater than Hanuman's for Rama, this utterly, extraordinarily humble man, this living saint, who chose to say hello to me?

Matti Jutila

Remembering K. Ramachandra

I met professor Ramachandra first time in September 1971 in Moscow, where we participated a number theory conference in honor of the 80th birthday of academician I. M. Vinogradov; by the way, we met again in 1981 at a similar occasion celebrating the 90th birthday of Vinogradov. We had common interests in number theory, in particular prime numbers and Dirichlet series, so our scientific contacts and personal friendship got started immediately and lasted forty years until Ramachandra's death. His talk [7] in Moscow turned out to be quite influential for me, and therefore I would like to dwell for a moment on this topic and related questions. Ramachandra considered the following problem: given k consecutive numbers $n+1, \cdots, n+k$, what is the biggest prime factor, say P, occurring in these numbers ? Erdös had proved in an elementary way that $P \gg k \log k$, but improving this bound turned out to be a highly nontrivial problem, as commented by Erdös himself. As a reformulation, one may ask about gaps between numbers with a large prime factor. Extreme cases of such numbers are the prime numbers p_n themselves, and by the classical theory of Ingham we have a relation between zero-density estimates for Riemann's zeta-function $\zeta(s)$ and the gaps between primes. Let $N(\alpha, T)$ denote the number of zeros of $\zeta(s)$ in the domain Re $s \geq \alpha$, $|\text{Im } s| \leq T$. The *density hypothesis*

$$N(\alpha, T) \ll T^{2(1-\alpha)+\varepsilon} \tag{1}$$

for $1/2 \leq \alpha \leq 1$ and $T \geq 1$ implies the bound (see [1], eq. (12.83))

$$p_{n+1} - p_n \ll p_n^{1/2+\varepsilon}. \tag{2}$$

However, the density hypothesis is still unproved; I proved it for $\alpha \geq 11/14$, and this bound has been improved somewhat by J. Bourgain. Now, as an enlargement of the sequence of primes, consider the sequence of $r_n^{(\beta)}$ of the numbers r having a prime factor $p \geq r^\beta$ for given $\beta \in (0, 1)$. As an analogue of (2), one may now pose the following problem: to find β, as large as possible, such that (2) holds if p_n is replaced by $r_n^{(\beta)}$. Ramachandra had shown this for $\beta = 5/8$, and in [3] I obtained $\beta = 2/3$. My argument was based on a *weighted density theorem*, that is a density theorem where the zeros ρ are weighted by $|f(\rho)|$ with $f(s)$ a given Dirichlet polynomial. This approach fails to give any result for gaps of length not exceeding the square root of the numbers in question. On the other hand, Ramachandra's method, which was based on Selberg's sieve and van der Corput's method, was applicable more generally. The same was true for my paper [4], where I used Vinogradov's method for exponential sums over primes. However, for extremely large n compared with k, another powerful tool was needed, namely Baker's method from the transcendental number theory, and this very delicate case was treated by Ramachandra and Shorey.

As a combination of all cases, the final result was

$$P \gg k \log k \log\log k / \log\log\log k,$$

a sharpening of the above mentioned result of Erdös. Interestingly, its proof required combining methods of essentially different nature. Another situation like this will be discussed next, namely estimating gaps between primes.

Let θ be a number such that for any positive ε we have

$$p_{n+1} - p_n \ll p_n^{\theta+\varepsilon}.$$

The value $\theta = 7/12$ (see [1], eq. (12.68)) due to M. N. Huxley is presently the best result obtained by complex analytic methods using zero-density estimates for the zeta-function. Henryk Iwaniec and myself were visiting the Institute Mittag-Leffler in Stockholm in 1977-8, and Iwaniec made an attempt to apply his version of the linear sieve to gaps between primes. It turned out that a certain version of the weighted density theorem was helpful in this context, and in [2] we showed that $\theta = 13/23$ is admissible. Note that $7/12 = 0.5833...$, while $13/23 = 0.5652...$; the last mentioned value has been improved afterwards by more refined techniques. It is fair to say that Ramachandra was indirectly involved in this development, for his problem on numbers with a large prime factor motivated me to consider weighted density theorems, and such a device found unexpectedly an application to the gap problem for primes.

One of Ramachandra's favorite topics was estimating moments of the zeta-function. A typical case is

$$I_k(T) = \int_1^T |\zeta(1/2 + it)|^{2k}\, dt,$$

and one may also consider moments over short intervals and over different lines. The cases $k = 1, 2$ are classical and asymptotic formulae for $I_k(T)$ are known, whereas the other cases are more problematic. D. R. Heath-Brown proved that on Riemann's Hypothesis

$$T(\log T)^{k^2} \ll I_k(T) \ll T(\log T)^{k^2} \tag{3}$$

for $0 \le k \le 2$. Moreover, the lower bound holds for all positive k. The implied constants here depend on k. As to unconditional results, the lower bound holds if k is rational, and the same is true for the upper bound if $k = 1/n$, where n is a natural number. My contribution [5] to the topic was that if $k = 1/n$, then the implied constants in (3) can be taken to be absolute, thus independent of n. As the choice of n is now flexible, one may deduce results on the value distribution of the zeta-function. Again the work of Ramachandra was of pioneering importance, and his monograph [8] gives an account of moment problems together with other related topics.

In the context of the fourth moment of the zeta-function, Ramachandra's *reflection principle* should be mentioned (see [1], Sec. 4.4). It is a flexible variant of the approximate functional equation for $\zeta^2(s)$, and since its proof requires only the functional equation, it can be immediately generalized to other Dirichlet series having a functional equation of the Riemann type. This device is now a standard tool in analytic number theory.

In 1985, Ramachandra kindly arranged for me a position as a visiting professor at the Tata Institute of Fundamental Research, a fortunate opportunity indeed to work at this highly reputed scientific center. That time I was developing a method, combining ideas of the methods of Hardy-Littlewood and van der Corput, in the theory of exponential sums involving the divisor function or Fourier coefficients of cusp forms. My lectures [6] on this topic appeared in the Tata Lecture Notes series, and I felt it as a great honor to have a book published in a series including authors such as C. L. Siegel and many other famous names. Curiously, Bombieri and Iwaniec were working at the same time on an analogous method for zeta-sums, that is for segments of the zeta-series.

Ramachandra organized visits to the Tata Institute for several other number theorists as well, so my friends Y. Motohashi, A. Ivić and M. N. Huxley were enjoying his hospitality. Moreover, Motohashi and Ivić also published their lectures in the prestigious Tata series.

Professor Ramachandra carried on his mathematical activity with full force even after the retirement. Indeed, last year I was still in contact with him when he informed me that my paper on the estimation of the Mellin transform of Hardy's function had been accepted for publication in the Hardy-Ramanujan Journal. Therefore the sad message that he had passed away came so unexpectedly for me, and my sorrow was mixed with the relief that he was fortunate to retain and continue his devotion to mathematics till the very end of his life.

References

[1] A. Ivić, *The Riemann zeta-function*, (second. ed., Dover, Mineola, New York, 2003), John Wiley & Sons, Inc., New York, 1985.
[2] H. Iwaniec and M. Jutila, *Primes in short intervals*, Arkiv Mat. **17** (1979), 167–176.
[3] M. Jutila, *On numbers with a large prime factor*, J. Indian Math. Soc. **37** (1973), 43–53.
[4] M. Jutila, *On numbers with a large prime factor II*, J. Indian Math. Soc. **38** (1974), 125–130.
[5] M. Jutila, *On the value distribution of the zeta-function on the critical line*, Bull. London Math. Soc. **15** (1983), 513–518.
[6] M. Jutila, *Lectures on a method in the theory of exponential sums*, Lecture Notes in Mathematics and Physics vol. 80, Tata Institute of Fundamental Research, Springer Verlag, Berlin - New York, 1987.
[7] K. Ramachandra, *Largest prime factor of the product of k-consecutive integers*, Trudy Mat. Inst. Steklov **132** (1973), 77–81.

[8] K. Ramachandra, *On the Mean-Value and Omega-Theorems for the Riemann Zeta-Function*, Lecture notes in Mathematics and Physics vol. 85, Tata Institute of Fundamental Research, Springer-Verlag, Berlin - New York, 1995.

Department of Mathematics
University of Turku
FI-20014 Turku, Finland

E-mail address: jutila@utu.fi