ON THE CLASS NUMBER FORMULA OF CERTAIN REAL QUADRATIC FIELDS

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To Professor Tsuyoshi Uehara, with great respect and friendship

Abstract. In this note we give an alternate expression of class number formula for real quadratic fields with discriminant \( d \equiv 5 \mod 8 \).

1. Introduction

Let \( p \) be an odd prime with \( p \equiv 1 \mod (4) \) and \( \varepsilon \) be the fundamental unit of \( \mathbb{Q}(\sqrt{p}) \).

It is well known that the class number \( h \) in this case is given by

\[
h = \frac{\sqrt{p}}{2 \log \varepsilon} L(p)
\]

where \( L(p) = \sum_{k=1}^{\infty} \left( \frac{p}{k} \right) k^{-1} \). From here one can deduce classical Dirichlet’s class number formula

\[
\varepsilon^{2h} = \frac{\prod_n (1 - \theta^n)}{\prod_v (1 - \theta^v)}
\]

where \( \theta \) is the primitive \( p \)-th root of unity; \( v \) and \( n \) run over the quadratic residues and non-residues of \( p \) respectively in the interval \( (0, p) \). This formula exhibit class number \( h \) of the corresponding real quadratic field in a “transcendental” manner. A somewhat simpler formula was proved by P. Chowla [3] in the case of prime discriminant of the form \( p \equiv 5 \mod 8 \). She took the square root of the above classical formula to derive her formula.

We derive an alternate class number formula for real quadratic fields with discriminant \( d \equiv 5 \mod 8 \). We appeal to another form of the class number formula for \( \varepsilon^h \) and transform it using another result of Dirichlet for 1/4-th character sum, in such a way that the obtained result has the effect of taking the square root of the class number formula for \( \varepsilon^{2h} \) as was done by P. Chowla. Our result can elucidate other generalisations of Mitsuhiro, Nakahara and Uehara for \( \varepsilon^{2h} \) for general real quadratic fields.

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2. Square root of the class number formula

Let $d$ be a positive discriminant. We denote by $h$ the class number of the real quadratic field $\mathbb{Q}(\sqrt{d})$ and by $\varepsilon > 1$ its fundamental unit. We consider the following version ([2, Theorem 3, p. 246]) of the classical class number formula of Dirichlet

$$
\varepsilon^h = \frac{\prod_{0<n<d/2} \sin \frac{\pi n}{d}}{\prod_{0<v<d/2} \sin \frac{\pi v}{d}},
$$

where $v$ and $n$ will always denote integers such that the Kronecker symbols $\left( \frac{v}{d} \right) = 1$ and $\left( \frac{n}{d} \right) = -1$ respectively.

In 1966, P. Chowla [3] showed that for $d = p \equiv 5 \mod 8$, one has:

$$
\varepsilon^h = (-1)^m \prod_{0<n<p/2} \left[ 2 \cos \frac{2\pi}{p} \left\{ \frac{n}{2} \right\} \right],
$$

where $m$ denotes the number of odd $n$’s with $n \leq p/2$ and for any odd positive integer $d$, the symbol $\left\{ n/2 \right\}$ denotes the solution $k$ of congruence equation $2X \equiv n \mod d$ with $0 < k < d$. In the same paper she also shows that $m$ can be computed via the identity:

$$
m = \frac{1}{2} \left[ \frac{p - 1}{4} + H \right],
$$

where $H$ denotes the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$.

The goal of this note is to extend Chowla’s formula to the case when $d$ is not necessarily prime. More precisely, we will prove the following:

**Theorem 2.1.** Let $d \equiv 5 \mod 8$ be a positive discriminant and let $h$ and $\varepsilon > 1$ be the class number and the fundamental unit of $\mathbb{Q}(\sqrt{d})$, respectively. Furthermore let $H$ denote the class number of $\mathbb{Q}(\sqrt{-d})$. Then

$$
\varepsilon^h = (-1)^m \prod_{v < \frac{d}{2}} 2 \cos \frac{2\pi}{d} \left\{ \frac{v}{2} \right\}, \quad \varepsilon^{-h} = (-1)^l \prod_{n < \frac{d}{2}} 2 \cos \frac{2\pi}{d} \left\{ \frac{n}{2} \right\}
$$

(3)

$$
m = \frac{1}{2} \left( \frac{\varphi(d)}{2} - \varphi(d, \frac{d}{4}) + \frac{1}{2} H \right), \quad l = \frac{1}{2} \left( \frac{\varphi(d)}{2} - \varphi(d, \frac{d}{4}) - \frac{1}{2} H \right)
$$

where $\varphi(a,b)$ denote the number of rational integers $k$ such that $0 < k < b$ and $(k,a) = 1$.

Let $T$, $U$, $T_1$ and $U_1$ be the integers defined by the identities

$$
\frac{T + U\sqrt{d}}{2} = \varepsilon^h, \quad \frac{T_1 + U_1\sqrt{d}}{2} = \varepsilon^h.
$$
Then,

\[ T_1 = \varepsilon^h + N(\varepsilon)^h \varepsilon^{-h} = (\varepsilon) \frac{\phi(d, d, 4)}{4} + 2^r - 2 \quad \text{if any prime } p \mid d \text{ are } p \equiv 3 \mod 4, \]

where \( r \) is the number of prime factors of \( d \).

Since \( d \equiv 1 \mod 4 \), the function \( \varphi(d, \frac{d}{4}) \) can be computed as in [6]:

\[ \varphi(d, \frac{d}{4}) = \begin{cases} \frac{\varphi(d)}{4} + 2^r - 2 & \text{if any prime } p \mid d \text{ are } p \equiv 3 \mod 4, \\ \frac{\varphi(d)}{4} & \text{otherwise}, \end{cases} \]

Another classical formulation of the Dirichlet class number formula states that,

\[ \varepsilon^{2h} = \frac{F_-(\theta)}{F_+(\theta)}, \]

where \( \theta = e^{\frac{2\pi i}{d}} \) denotes the \( d \)-th root of 1 with lowest argument and

\[ F_\pm(X) = \prod_{\begin{subarray}{c} a < d \\ (\frac{a}{d}) = \pm 1 \end{subarray}} (X^a - 1). \]

In [6], Mitsuhiro, Nakahara and Uehara proved a result which allows to compute \( T \) for a general discriminant \( d \neq 12 \). If \( \Phi_d(x) \) denotes the \( d \)-cyclotomic polynomial, then their result states that \( T \) is the least positive residue (mod \( \Phi_d(2) \)) of the quantity:

\[ \frac{F_-(2)}{F_+(2)} + \frac{F_+(2)}{F_-(2)}. \]

A consequence of Theorem 2.1 is:

**Corollary 2.2.** Suppose \( d \equiv 5 \mod 8 \). Then \( T_1 \) is the least positive residue mod \( \Phi_d(2) \) of

\[ (-1)^m \left( \prod_{v<\frac{d}{2}} (2^v + 2^{-v}) + \prod_{v<\frac{d}{2}} (2^{2v} + 2^{-2v}) \right). \]
Remark 1. These results may have relation with the class number relation for the biquadratic field \((5) \mathbb{Q}(i, \sqrt{d})\) for a square-free integer \(d \equiv 1 \mod 4\), which is a Galois extension of \(\mathbb{Q}\) and its Galois group is \(V_4\) with subfields \(\mathbb{Q}(\sqrt{d})\), \(\mathbb{Q}_4 = \mathbb{Q}(\sqrt{-4})\) and \(\mathbb{Q}(\sqrt{-d})\).

3. Proofs

Proof of Theorem 2.1. The Kronecker symbol is even when \(d > 0\). Hence (1) follows from (6) by rewriting it as

\[ \varepsilon^{2h} = \frac{\prod_{0<n<d} \sin \frac{\pi n}{d}}{\prod_{0<v<d/2} \sin \frac{\pi v}{d}} = \left( \frac{\prod_{0<n<d/2} \sin \frac{\pi n}{d}}{\prod_{0<v<d/2} \sin \frac{\pi v}{d}} \right)^2. \]

As \(\left(\frac{d}{2}\right) = -1\), the non-residues are given in the form \(2v\) ([3, p.54]). Hence (1) may be written as

\[ \varepsilon^h = \prod \frac{\sin \frac{2\pi v}{d}}{\sin \frac{\pi v}{d}} = \prod_{0<v<d/2} 2 \cos \frac{\pi v}{d}. \]

Noting that \(\left\{ \frac{v}{2} \right\}\) also means the least positive residue mod \(d\) of \(2^{-1}v\), where \(2^{-1}\) is the inverse of \(2\) mod \(d\):

\[ \left\{ \frac{v}{2} \right\} = 2^{-1}v - d \left\lfloor \frac{2^{-1}v}{d} \right\rfloor, \]

\([x]\) denotes the integral part of \(x\). Thus we can express \(\left\{ \frac{v}{2} \right\}\) more concretely as

\[ \left\{ \frac{v}{2} \right\} = \begin{cases} \frac{v}{2} & v \text{ even,} \\ \frac{v+d}{2} & v \text{ odd.} \end{cases} \]

Hence, if we replace \(v\) in (8) by \(\left\{ \frac{v}{2} \right\}\), then those terms with odd \(v\) are negative and there are \(m\) of them, where \(m\) indicates the number of odd residues in \((0, \frac{d}{2})\). Hence it follows that

\[ \varepsilon^h = (-1)^m \prod_{0<v<d/2} 2 \cos 2\pi \left\{ \frac{v}{2} \right\}. \]

The remaining part is the same as in [3] and we prove as she has done by elementary argument that,

\[ m = \frac{1}{2} \left\{ \varphi(d) - \varphi(d, \frac{d}{4}) + \frac{1}{4} S_{1/4} \right\}. \]

Where \(2S_{1/4} = H\), which is a form of Dirichlet class number formula for \(\mathbb{Q}(\sqrt{-d})\). Moving in a similar fashion we can prove the case of \(\varepsilon^{-h}\) and \(l\).

We now have the proof modulo the fact that \(N(\varepsilon)^h = 1\) or \(-1\) and we prove it in the course of the proof of Corollary 2.2.
Proof. (Proof of Corollary 2.2) Our proof is similar to Chowlas’ [4, Proof of Theorem 1] by appealing to the Gauss sum

\[
\sqrt{d} = \sum_{a<d} \chi_d(a)\theta^a = \sum_{\nu<d} \theta^{\nu} - \sum_{n<d} \theta^n
\]

This follows by taking the square root of the formula

\[
(-1)^{d+1}d = \left(\sum_{a<d} \chi_d(a)\theta^a\right)^2.
\]

Substituting (11) in (2), we deduce that,

\[
\frac{1}{2} \left(T_1 + U_1 \left(\sum_{\nu<d} \theta^{\nu} - \sum_{n<d} \theta^n\right)\right) = \varepsilon^h = (-1)^m \prod_{\nu<\frac{d}{2}} \left(\theta^{\left\lceil \frac{\nu}{2} \right\rceil} + \theta^{-\left\lceil \frac{\nu}{2} \right\rceil}\right).
\]

Changing $\theta$ in (11) by $\theta^4$ and $\theta^2$, respectively, thereby noting that $2 \left\lceil \frac{\nu}{2} \right\rceil \equiv \nu \mod d$ and $\chi_d(2) = -1$, we obtain two expressions for $d$. Substituting this in (12), we deduce that

\[
\varepsilon^h = (-1)^m \prod_{\nu<\frac{d}{2}} (\theta^{2\nu} + \theta^{-2\nu})
\]

and

\[
\varepsilon'^h = (-1)^m \prod_{\nu<\frac{d}{2}} (\theta^{\nu} + \theta^{-\nu}),
\]

where $\varepsilon'$ is the conjugate of $\varepsilon$. Now summing (13) and (14) we have,

\[
T_1 = \varepsilon^h + \varepsilon'^h,
\]

which is (3).

The number of $\nu$ such that $\theta^{\nu} + \theta^{-\nu} < 0$ and $0 < \nu < d/2$ is $l$, and $\varepsilon' = N(\varepsilon)\varepsilon^{-1}$.

Thus $N(\varepsilon)^h = (\varepsilon)^{m+l}$, and this completes the proof of Theorem 2.1.

It remains to prove that $T_1$ is the least positive residue mod $\Phi_d(2)$. The right-hand side is estimated by

\[
T_1 < 2^{\varepsilon(d)} + 1 < 2^{\varepsilon(d)-2}
\]

thus it is less than $\Phi_d(2)$ (cf. [6, p.101]). Finally, the polynomial

\[
P(X) = X^{d^2} \left(T_1 - (-1)^m \left(\prod_{\nu<\frac{d}{2}} (X^{\nu} + X^{-\nu}) + \prod_{\nu<\frac{d}{2}} (X^{2\nu} + X^{-2\nu})\right)\right)
\]

is divisible by $\Phi_d(X)$ because it vanishes at $X = \theta$ and $\Phi_d(X)$ is the minimal polynomial for $\theta$. \qed
In [6], Mitsuhiro, Nakahara and Uehara considered a generalization of another result of Chowlas for $\varepsilon^{2h}$. Let $T = T_2, U = U_2$ be defined by

$$\frac{T + U\sqrt{d}}{2} = \varepsilon^{2h}. $$

Then their theorem states that,

$$T = T_2 = \varepsilon^{2h} + \varepsilon^{-2h}$$

is the least positive residue mod $\Phi_d(2)$ of $F^-(2) + F^+(2)$. Their proof is similar to that of Chowlas’ except for one point in which they need to prove a key lemma which shows that $|\theta^{\pm n}| < \sqrt{2}$ for non-residue $n$ in the first $1/4$-th interval. Thus establishing that the above value is less than $\Phi_d(2)$.

We conclude by showing that with Dirichlet’s formula for the $1/4$-th sum one can give a simpler proof of their Lemma [6, Lemma].

**Proposition 3.1.** For any discriminant $d$ except for $5, 8, 12, 24$, there exists a non-residue mod $d$ in the first $1/4$-th interval.

**Proof.** Assume that $\chi_d(a) = 1$ for all $a$ in $0 << \frac{1}{4}d$. Then $\chi_d(a) = -1$ in $\frac{1}{4}d << \frac{3}{4}d$ and $\chi_d(a) = 1$ in $\frac{3}{4}d << d$. We distinguish cases following [6].

- $d \equiv 1 \mod 4$ with $d > 5$. $\chi_d(2) = 1$ and $0 < 2 < \frac{d}{4}$. Hence there is an $\alpha$ such that $\frac{d}{2} < 2^\alpha < \frac{d}{4}$ for which $\chi_d(2^\alpha) = 1$, a contradiction.
- $d = 4f$, $f \equiv 3 \mod 4$, $f > 3$. $f^2 \equiv 1 \mod 4$ and $\chi_d(f + 2) = -1$, so that $(f + 2)^3 \equiv 3f + 8 \mod d$. For $f = 7$, $3f + 8 \equiv 1 \mod d$ while for $f > 7$, $\frac{3}{4}d < 3f + 8 < d$ and $\chi_d(3f + 8) = -1$, a contradiction.
- $d = 8f$. There are two cases. If $f \equiv 1 \mod 4$, then $\chi_d(3f - 2) = -1$ and $(3f - 2)^3 \equiv f - 8 \mod d$, so that $\chi_d(f - 8) = -1$. If $f \equiv 3 \mod 4$, then $3f + 2$ works.

$$\square$$

**Remark 2.** Noting that $T_1^2 = T - 2$ (which follows from the Pell equation $T_1^2 - U_1^2d = 4$), we see that to treat the general case of Mitsuhiro, Nakahara and Uehara, we can work with (5) as in Chowlas’ and appeal to their Lemma.

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