Hardy-Ramanujan Journal 48 (2025) Part B, 3-20 submitted 25/02/2025, accepted 25/06/2025, revised 10/12/2025

Continued Fractions in Entries 34 and 25 of Chapter 12 in Ramanujan's Second Notebook

Bruce C. Berndt, Victoria J. Reuter

Abstract. Chapter 12 in Ramanujan's second notebook is devoted to continued fractions, many of which are evaluated in terms of Gamma functions. Using the theory of hypergeometric functions, we give new proofs of two of the most attractive continued fractions, and also two new companions.

Keywords. continued fractions, hypergeometric functions, Ramanujan's notebooks.

2010 Mathematics Subject Classification. Primary, 11A55; Secondary, 33C20, 33C90.

1. Introduction

Chapter 12 in Ramanujan's second notebook [Ramanujan] contains a plethora of continued fractions, many involving the gamma function. The continued fractions for quotients of gamma functions are particularly beautiful. With multiple parameters, the continued fractions exhibit especially attractive symmetries. In his book [Ber89, Chapter 12], the first author gave an account of all of these continued fractions. For those continued fractions that had previously been proved in the literature, he deferred to these sources. Ramanujan did not provide clues for any of his proofs. However, preceding Chapter 12 in the second notebook are two chapters on hypergeometric series. It therefore is possible that Ramanujan employed hypergeometric series in at least some of his proofs.

In this paper, we employ hypergeometric series to give a new proof of one of Ramanujan's signature continued fractions, Entry 34 in his second notebook. Entry 34 has a companion, which is not in Ramanujan's notebooks. We give two proofs. First, the companion is, in fact, a corollary of Entry 34; second, using hypergeometric series, we provide a proof *ad hoc*.

Entry 25 in Ramanujan's second notebook [Ramanujan], [Ber89, p. 140] is also attractive. It has a companion, which is not given in the notebooks, and which we prove. As a corollary, we obtain Entry 25. There are two proofs of Entry 25 in [Ber89, pp. 41–45]. One involves hypergeometric functions, and the other is considerably deeper. Our proof involves hypergeometric functions, but is different from that of the first author [Ber89].

We also utilize hypergeometric functions to establish Lemma 2.4, possibly a new continued fraction for a quotient of two ${}_3F_2$ hypergeometric functions.

2. Preliminary Results

Before proving Entry 34, we begin with some preliminary identities and recurrences. The first of these is Dixon's Identity, which is found in Ramanujan's notebooks [Ramanujan, Chapter 10, Entry 7], [Ber89, p. 15], [AAR99, p. 72].

Theorem 2.1. (Dixon's Identity) *If* $\Re(\frac{1}{2}a - b - c + 1) > 0$, *then*

$$ {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b+1,\, a-c+1 \end{matrix}; 1\right) = \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c+1)}. \tag{2.1} $$

We thank episciences.org for providing open access hosting of the electronic journal *Hardy-Ramanujan Journal*

The second is Kummer's Theorem [AAR99, Corollary 3.1.2, p. 126],

$$ {}_2F_1\left(\begin{matrix} a,\, b \\ a-b+1 \end{matrix};-1\right) = \frac{\Gamma(a-b+1)\Gamma(\frac{1}{2}a+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(a+1)}. \tag{2.2} $$

Thirdly, the Beta function is given by

$$ \mathrm{B}(\alpha,\beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \Re(\alpha)>0,\ \Re(\beta)>0. \tag{2.3} $$

Several recurrence formulas for hypergeometric functions are used in the sequel. The second recurrence below is used in the proof of Lemma 2.4, which is the main continued fraction to be employed to prove Entry 34.

Proposition 2.2. *If $\Re(d+e-a-b-c)>0$, then*

$$ \begin{aligned} \frac{ab(e-c)}{e}{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix};1\right) &= (d-a)(d-b){}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d+1,\, e \end{matrix};1\right) \\ &\quad + d(a+b-d){}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix};1\right). \end{aligned} \tag{2.4} $$

Proof. We begin with an identity from [AAR99, Equation (2.5.10), page 97], namely,

$$ \begin{aligned} ab(1-x){}_2F_1\left(\begin{matrix} a+1,\, b+1 \\ d+1 \end{matrix};x\right) &= (d-a)(d-b){}_2F_1\left(\begin{matrix} a,\, b \\ d+1 \end{matrix};x\right) \\ &\quad + d(a+b-d){}_2F_1\left(\begin{matrix} a,\, b \\ d \end{matrix};x\right). \end{aligned} $$

Assume that $\Re(c)>0$ and $\Re(e-c)>0$, multiply both sides by $x^{c-1}(1-x)^{e-c-1}$, and integrate with respect to x from 0 to 1. We thereby obtain

$$ \begin{aligned} &ab\int_0^1 x^{c-1}(1-x)^{e-c}{}_2F_1\left(\begin{matrix} a+1,\, b+1 \\ d+1 \end{matrix};x\right)dx \\ &= (d-a)(d-b)\int_0^1 x^{c-1}(1-x)^{e-c-1}{}_2F_1\left(\begin{matrix} a,\, b \\ d+1 \end{matrix};x\right)dx \\ &\quad + d(a+b-d)\int_0^1 x^{c-1}(1-x)^{e-c-1}{}_2F_1\left(\begin{matrix} a,\, b \\ d \end{matrix};x\right)dx. \end{aligned} \tag{2.5} $$

We integrate each of these term-by-term. Thus, with the use of (2.3) below, we deduce that

$$ \begin{aligned} &ab\int_0^1 x^{c-1}(1-x)^{e-c}{}_2F_1\left(\begin{matrix} a+1,\, b+1 \\ d+1 \end{matrix};x\right)dx \\ &= ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(d+1)_n n!}\int_0^1 x^{n+c-1}(1-x)^{e-c}\,dx \\ &= ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(d+1)_n n!}\mathrm{B}(n+c,e-c+1) \\ &= ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(d+1)_n n!}\frac{\Gamma(n+c)\Gamma(e-c+1)}{\Gamma(n+c+e-c+1)} \\ &= \frac{ab\Gamma(e-c+1)\Gamma(c)}{\Gamma(e+1)}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(d+1)_n n!}\frac{\Gamma(n+c)}{\Gamma(c)}\frac{\Gamma(e+1)}{\Gamma(n+e+1)} \end{aligned} $$

$$= \frac{ab\Gamma(e-c+1)\Gamma(c)}{\Gamma(e+1)} \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n(c)_n}{(d+1)_n n!(e+1)_n}$$
$$= \frac{ab(e-c)\Gamma(e-c)\Gamma(c)}{e\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right). \tag{2.6}$$

Similarly, by using the same argument as above, we find that

$$(d-a)(d-b)\int_0^1 x^{c-1}(1-x)^{e-c-1} {}_2F_1\left(\begin{matrix} a,\, b \\ d+1 \end{matrix}; x\right) dx$$
$$= \frac{(d-a)(d-b)\Gamma(e-c)\Gamma(c)}{\Gamma(e)} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d+1,\, e \end{matrix}; 1\right) \tag{2.7}$$

and

$$d(a+b-d)\int_0^1 x^{c-1}(1-x)^{e-c-1} {}_2F_1\left(\begin{matrix} a,\, b \\ d \end{matrix}; x\right) dx$$
$$= \frac{d(a+b-d)\Gamma(e-c)\Gamma(c)}{\Gamma(e)} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right). \tag{2.8}$$

Putting (2.6)–(2.8) in (2.5), we arrive at

$$\frac{ab(e-c)\Gamma(e-c)\Gamma(c)}{e\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right)$$
$$= \frac{(d-a)(d-b)\Gamma(e-c)\Gamma(c)}{\Gamma(e)} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d+1,\, e \end{matrix}; 1\right)$$
$$+ \frac{d(a+b-d)\Gamma(e-c)\Gamma(c)}{\Gamma(e)} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right). \tag{2.9}$$

Dividing both sides of (2.9) by $\Gamma(e-c)\Gamma(c)/\Gamma(e)$ yields (2.4). We use analytic continuation to remove the restrictions $\Re(c) > 0$ and $\Re(e-c) > 0$ and obtain Proposition 2.2.

Proposition 2.3. *If $\Re(d+e-a-b-c) > 0$, then*

$$ {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right) = \frac{d-b}{d} {}_3F_2\left(\begin{matrix} a+1,\, b,\, c \\ d+1,\, e \end{matrix}; 1\right) + \frac{b(e-c)}{de} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right). \tag{2.10}$$

Proof. We begin with an identity from K. G. Ramanathan's paper [Ram87, Equation (9)],

$$ {}_2F_1\left(\begin{matrix} a,\, b \\ d \end{matrix}; x\right) = \frac{d-b}{d} {}_2F_1\left(\begin{matrix} a+1,\, b \\ d+1 \end{matrix}; x\right) + \frac{b(1-x)}{d} {}_2F_1\left(\begin{matrix} a+1,\, b+1 \\ d+1 \end{matrix}; x\right).$$

As in the proof of the previous proposition, we assume $\Re(c) > 0$ and $\Re(e-c) > 0$, multiply both sides by $x^{c-1}(1-x)^{e-c-1}$, and then integrate with respect to x from 0 to 1. This yields

$$\int_0^1 x^{c-1}(1-x)^{e-c-1} {}_2F_1\left(\begin{matrix} a,\, b \\ d \end{matrix}; x\right) dx$$
$$= \frac{d-b}{d}\int_0^1 x^{c-1}(1-x)^{e-c-1} {}_2F_1\left(\begin{matrix} a+1,\, b \\ d+1 \end{matrix}; x\right) dx$$
$$+ \frac{b}{d}\int_0^1 x^{c-1}(1-x)^{e-c} {}_2F_1\left(\begin{matrix} a+1,\, b+1 \\ d+1 \end{matrix}; x\right) dx. \tag{2.11}$$

We evaluate each of these integrals term-by-term, again making use of the Beta function (2.3) in our calculations. Since the details are identical to those in the proof of Proposition 2.2, we forego them. Hence, we find that

$$\int_0^1 x^{c-1}(1-x)^{e-c-1} {}_2F_1\left(\begin{matrix} a,\, b \\ d \end{matrix}; x\right) dx = \frac{\Gamma(e-c)\Gamma(c)}{\Gamma(e)} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right), \tag{2.12}$$

$$\frac{d-b}{d}\int_0^1 x^{c-1}(1-x)^{e-c-1} {}_2F_1\left(\begin{matrix} a+1,\, b \\ d+1 \end{matrix}; x\right) dx = \frac{(d-b)\Gamma(e-c)\Gamma(c)}{d\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b,\, c \\ d+1,\, e \end{matrix}; 1\right), \tag{2.13}$$

and

$$\begin{aligned} &\frac{b}{d}\int_0^1 x^{c-1}(1-x)^{e-c} {}_2F_1\left(\begin{matrix} a+1,\, b+1 \\ d+1 \end{matrix}; x\right) dx \\ &\quad = \frac{b(e-c)\Gamma(e-c)\Gamma(c)}{de\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right). \end{aligned} \tag{2.14}$$

Hence, by (2.12)–(2.14), we find that (2.11) is transformed into

$$\begin{aligned} \frac{\Gamma(e-c)\Gamma(c)}{\Gamma(e)} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right) &= \frac{(d-b)\Gamma(e-c)\Gamma(c)}{d\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b,\, c \\ d+1,\, e \end{matrix}; 1\right) \\ &\quad + \frac{b(e-c)\Gamma(e-c)\Gamma(c)}{de\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right). \end{aligned} \tag{2.15}$$

Proposition 2.3 is obtained by dividing both sides of (2.15) by $\Gamma(e-c)\Gamma(c)/\Gamma(e)$ and using analytic continuation to remove the conditions $\Re(c) > 0$ and $\Re(e-c) > 0$.

Lemma 2.4. *If $\Re(d+e-a-b-c) > 0$, $d \neq c$, and $e-b \neq 1$, then*

$$\begin{aligned} d\frac{{}_3F_2\left(\begin{smallmatrix} a,b,c \\ d,e \end{smallmatrix}; 1\right)}{{}_3F_2\left(\begin{smallmatrix} a,b+1,c \\ d+1,e \end{smallmatrix}; 1\right)} &= (d-c) + \frac{c(e-a)}{e-b-1+} \frac{(b+1)(d+1-a)}{d-c} + \frac{(c+1)(e+1-a)}{e-b-1} \\ &\quad + \frac{(b+2)(d+2-a)}{d-c} + \frac{(c+2)(e+2-a)}{e-b-1} + \cdots \\ &= (d-c) + \frac{c(e-a)}{e-b-1+} \operatorname*{\mathbf{K}}_{k=1}^{\infty} \frac{p_k}{q_k}, \end{aligned} \tag{2.16}$$

where

$$\begin{cases} p_{2k-1} &= (b+k)(d+k-a), \\ q_{2k-1} &= d-c, \\ p_{2k} &= (c+k)(e+k-a), \\ q_{2k} &= e-b-1. \end{cases} \tag{2.17}$$

Proof. Simultaneously replace a by b, b by c, and c by a in Proposition 2.3 to obtain

$${}_3F_2\left(\begin{matrix} b,\, c,\, a \\ d,\, e \end{matrix}; 1\right) = \frac{d-c}{d} {}_3F_2\left(\begin{matrix} b+1,\, c,\, a \\ d+1,\, e \end{matrix}; 1\right) + \frac{c(e-a)}{de} {}_3F_2\left(\begin{matrix} b+1,\, c+1,\, a \\ d+1,\, e+1 \end{matrix}; 1\right). \tag{2.18}$$

We rearrange (2.18) into the form

$$d\frac{{}_3F_2\left(\begin{smallmatrix} a,b,c \\ d,e \end{smallmatrix}; 1\right)}{{}_3F_2\left(\begin{smallmatrix} a,b+1,c \\ d+1,e \end{smallmatrix}; 1\right)} = (d-c) + \frac{c(e-a)}{e} \frac{{}_3F_2\left(\begin{smallmatrix} a,b+1,c+1 \\ d+1,e+1 \end{smallmatrix}; 1\right)}{{}_3F_2\left(\begin{smallmatrix} a,b+1,c \\ d+1,e \end{smallmatrix}; 1\right)},$$

and iterate this to generate

$$\begin{aligned}
d\frac{{}_3F_2\left({a,b,c \atop d,e};1\right)}{{}_3F_2\left({a,b+1,c \atop d+1,e};1\right)} &= (d-c)+\frac{c(e-a)}{1}\frac{1}{e\frac{{}_3F_2\left({a,b+1,c \atop d+1,e};1\right)}{{}_3F_2\left({a,b+1,c+1 \atop d+1,e+1};1\right)}} \\
&= (d-c)+\frac{c(e-a)}{1}\frac{1}{e\frac{{}_3F_2\left({a,c,b+1 \atop e,d+1};1\right)}{{}_3F_2\left({a,c+1,b+1 \atop e+1,d+1};1\right)}} \\
&= (d-c)+\frac{c(e-a)}{1}\frac{1}{(e-(b+1))+\frac{(b+1)((d+1)-a)}{d+1}\frac{{}_3F_2\left({a,c+1,b+2 \atop e+1,d+2};1\right)}{{}_3F_2\left({a,c+1,b+1 \atop e+1,d+1};1\right)}} \\
&= (d-c)+\frac{c(e-a)}{e-b-1+}\frac{(b+1)(d+1-a)}{(d+1)\frac{{}_3F_2\left({a,b+1,c+1 \atop d+1,e+1};1\right)}{{}_3F_2\left({a,b+2,c+1 \atop d+2,e+1};1\right)}} \\
&= (d-c)+\frac{c(e-a)}{e-b-1+}\frac{(b+1)(d+1-a)}{d-c}+\frac{(c+1)(e+1-a)}{(e+1)\frac{{}_3F_2\left({a,c+1,b+2 \atop e+1,d+2};1\right)}{{}_3F_2\left({a,c+2,b+2 \atop e+2,d+2};1\right)}} \\
&= (d-c)+\frac{c(e-a)}{e-b-1+}\frac{(b+1)(d+1-a)}{d-c}+\frac{(c+1)(e+1-a)}{e-b-1}+\cdots.
\end{aligned}$$

We finally obtain the continued fraction

$$d\frac{{}_3F_2\left({a,b,c \atop d,e};1\right)}{{}_3F_2\left({a,b+1,c \atop d+1,e};1\right)} = (d-c)+\frac{c(e-a)}{e-b-1+}\mathop{\mathbf{K}}_{k=1}^{\infty}\frac{p_k}{q_k},$$

where p_k and q_k are given by (2.17).

The hypothesis $\Re(d+e-a-b-c)>0$ is needed to ensure the convergence of the hypergeometric series on the left-hand side of (2.16). In view of the continued fraction on the right-hand side of (2.16), the remaining two conditions are necessary. To show that the hypotheses of Lemma 2.4 are sufficient, we can use an argument like those cited in the applications of Lemma 2.4 that follow.

For a simple example, set $e=a$ in (2.4). Trivially, the right side terminates and equals $d-c$, while the left side becomes $d\cdot{}_2F_1(b,c;d;1)/{}_2F_1(b+1,c;d+1;1)$. If we apply Gauss's summation formula to each of these two hypergeometric functions, after simplification, we obtain $d-c$.

We are now ready to state and prove Entry 34 of Ramanujan's second notebook.

3. Entry 34 and Its Proof

Theorem 3.1. (Entry 34) *Suppose that n is an odd integer or m is an even integer, or that $\Re(x)>0$, where m and n are arbitrary complex numbers. Define*

$$P=\frac{\Gamma(\frac14(x+m+n+1))\Gamma(\frac14(x+m-n+1))\Gamma(\frac14(x-m+n+3))\Gamma(\frac14(x-m-n+3))}{\Gamma(\frac14(x-m+n+1))\Gamma(\frac14(x-m-n+1))\Gamma(\frac14(x+m+n+3))\Gamma(\frac14(x+m-n+3))}. \tag{3.19}$$

Then,

$$\frac{1-P}{1+P}=\frac{m}{x+}\frac{1^2-n^2}{x}+\frac{2^2-m^2}{x}+\frac{3^2-n^2}{x}+\frac{4^2-m^2}{x}+\cdots. \tag{3.20}$$

Proof. First, we solve (2.4) for ${}_3F_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};1\right)$. This yields

$$
{}_3F_2\left(\begin{matrix}a,\,b,\,c\\d,\,e\end{matrix};1\right)=\frac{ab(e-c)}{de(a+b-d)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right)
-\frac{(d-a)(d-b)}{d(a+b-d)}{}_3F_2\left(\begin{matrix}a,\,b,\,c\\d+1,\,e\end{matrix};1\right), \tag{3.21}
$$

which can be combined with (2.10) to deduce that

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix}a,\,b,\,c\\d,\,e\end{matrix};1\right)\\
&\quad=\frac{d-b}{d}{}_3F_2\left(\begin{matrix}a+1,\,b,\,c\\d+1,\,e\end{matrix};1\right)+\frac{b(e-c)}{de}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right)\\
&\quad=\frac{ab(e-c)}{de(a+b-d)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right)-\frac{(d-a)(d-b)}{d(a+b-d)}{}_3F_2\left(\begin{matrix}a,\,b,\,c\\d+1,\,e\end{matrix};1\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{d-b}{d}{}_3F_2\left(\begin{matrix}a+1,\,b,\,c\\d+1,\,e\end{matrix};1\right)&+\frac{(d-a)(d-b)}{d(a+b-d)}{}_3F_2\left(\begin{matrix}a,\,b,\,c\\d+1,\,e\end{matrix};1\right)\\
&=\left\{\frac{ab(e-c)}{de(a+b-d)}-\frac{b(e-c)}{de}\right\}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right)\\
&=\frac{b(e-c)(d-b)}{de(a+b-d)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right).
\end{aligned} \tag{3.22}
$$

Divide (3.22) by $(d-b)/d$, exchange a and b, and isolate ${}_3F_2\left(\begin{matrix}a,b+1,c\\d+1,e\end{matrix};1\right)$ to see that

$$
{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\d+1,\,e\end{matrix};1\right)=\frac{a(e-c)}{e(a+b-d)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right)
-\frac{d-b}{a+b-d}{}_3F_2\left(\begin{matrix}a,\,b,\,c\\d+1,\,e\end{matrix};1\right). \tag{3.23}
$$

We now let $d=a-b$ and $e=a-c+1$ in (3.23) to obtain

$$
{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\a-b+1,\,a-c+1\end{matrix};1\right)=\frac{a(a-2c+1)}{(a-c+1)(2b)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c\\a-b+1,\,a-c+2\end{matrix};1\right)
-\frac{a-2b}{2b}{}_3F_2\left(\begin{matrix}a,\,b,\,c\\a-b+1,\,a-c+1\end{matrix};1\right). \tag{3.24}
$$

With two applications of Dixon's Identity (2.1) on the right-hand side of (3.24), we find that

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\a-b+1,\,a-c+1\end{matrix};1\right)\\
&=\frac{a(a-2c+1)\Gamma(a-b+1)\Gamma(a-c+2)\Gamma(\frac12(a+1)+1)\Gamma(\frac12(a+1)-b-c)}{(a-c+1)(2b)\Gamma(\frac12(a+1)-b)\Gamma(\frac12(a+1)-c+1)\Gamma(a+2)\Gamma(a-b-c+1)}\\
&\quad-\frac{(a-2b)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac12 a+1)\Gamma(\frac12 a-b-c+1)}{2b\Gamma(\frac12 a-b+1)\Gamma(\frac12 a-c+1)\Gamma(a+1)\Gamma(a-b-c+1)}\\
&=\frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac12(a+1))\Gamma(\frac12(a+1)-b-c)}{2b\Gamma(\frac12(a+1)-b)\Gamma(\frac12(a+1)-c)\Gamma(a)\Gamma(a-b-c+1)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{2b\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a)\Gamma(a-b-c+1)}\\
&=\frac{\Gamma(a-b+1)\Gamma(a-c+1)}{2b\Gamma(a)\Gamma(a-b-c+1)}\left\{\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)}-\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)}\right\}\\
&=\frac{\Gamma(a-b+1)\Gamma(a-c+1)}{2b\Gamma(a)\Gamma(a-b-c+1)}\left\{S-T\right\}, \qquad (3.25)
\end{aligned}
$$

where

$$
\begin{cases}
S = \dfrac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)},\\[2ex]
T = \dfrac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)}.
\end{cases} \qquad (3.26)
$$

Similarly, under the substitutions $d=a-b$ and $e=a-c+1$, (3.21) becomes

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix}a,\, b,\, c\\ a-b,\, a-c+1\end{matrix};1\right)\\
&\quad=\frac{ab(a-2c+1)}{(a-b)(a-c+1)(2b)}{}_3F_2\left(\begin{matrix}a+1,\, b+1,\, c\\ a-b+1,\, a-c+2\end{matrix};1\right)\\
&\qquad-\frac{(-b)(a-2b)}{(a-b)(2b)}{}_3F_2\left(\begin{matrix}a,\, b,\, c\\ a-b+1,\, a-c+1\end{matrix};1\right)\\
&\quad=\frac{a(a-2c+1)\Gamma(a-b+1)\Gamma(a-c+2)\Gamma(\frac{1}{2}(a+1)+1)\Gamma(\frac{1}{2}(a+1)-b-c)}{2(a-b)(a-c+1)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c+1)\Gamma(a+2)\Gamma(a-b-c+1)}\\
&\qquad+\frac{(a-2b)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c+1)}{2(a-b)\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c+1)}\\
&\quad=\frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{2(a-b)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c+1)}\\
&\qquad+\frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{2(a-b)\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a)\Gamma(a-b-c+1)}\\
&\quad=\frac{\Gamma(a-b+1)\Gamma(a-c+1)}{2(a-b)\Gamma(a)\Gamma(a-b-c+1)}\times\\
&\qquad\quad\left\{\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)}+\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)}\right\}\\
&\quad=\frac{\Gamma(a-b+1)\Gamma(a-c+1)}{2(a-b)\Gamma(a)\Gamma(a-b-c+1)}\left\{S+T\right\}, \qquad (3.27)
\end{aligned}
$$

by (3.26).

Finally, using (3.25) and (3.27), we find that

$$
\begin{aligned}
(a-b)\frac{{}_3F_2\left(\begin{matrix}a,\, b,\, c\\ a-b,\, a-c+1\end{matrix};1\right)}{{}_3F_2\left(\begin{matrix}a,\, b+1,\, c\\ a-b+1,\, a-c+1\end{matrix};1\right)}
&=\frac{(a-b)\dfrac{\Gamma(a-b+1)\Gamma(a-c+1)}{2(a-b)\Gamma(a)\Gamma(a-b-c+1)}\left\{S+T\right\}}{\dfrac{\Gamma(a-b+1)\Gamma(a-c+1)}{2b\Gamma(a)\Gamma(a-b-c+1)}\left\{S-T\right\}}\\
&=\frac{b(S+T)}{S-T}\\
&=b\frac{1+R}{1-R}, \qquad (3.28)
\end{aligned}
$$

where
$$R=\frac{T}{S}=\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}.$$
Note that if $a=\frac{1}{2}x+b+c$, $b=\frac{1}{2}m$, and $c=\frac{1}{2}(n+1)$, then
$$\begin{aligned}R&=\frac{\Gamma(\frac{1}{4}(x+m+n+1))\Gamma(\frac{1}{4}(x+m-n+1))\Gamma(\frac{1}{4}(x-m+n+3))\Gamma(\frac{1}{4}(x-m-n+3))}{\Gamma(\frac{1}{4}(x-m+n+1))\Gamma(\frac{1}{4}(x-m-n+1))\Gamma(\frac{1}{4}(x+m+n+3))\Gamma(\frac{1}{4}(x+m-n+3))}\\&=P,\end{aligned}$$

where we have recalled the definition of P from (3.19).

Letting $d=a-b$ and $e=a-c+1$ in (2.16), we find that
$$(a-b)\frac{{}_3F_2\left({a,b,c \atop a-b,a-c+1};1\right)}{{}_3F_2\left({a,b+1,c \atop a-b+1,a-c+1};1\right)}=(a-b-c)+\frac{c(1-c)}{a-b-c+}\mathop{\mathbf{K}}_{k=1}^{\infty}\frac{r_k}{s_k}, \tag{3.29}$$

where
$$\begin{aligned}r_{2k-1}&=(k+b)(k-b),\\s_{2k-1}&=a-b-c,\\r_{2k}&=(k+c)(k+1-c),\\s_{2k}&=a-b-c.\end{aligned}$$

Again, letting $a=\frac{1}{2}x+b+c$, $b=\frac{1}{2}m$, $c=\frac{1}{2}(n+1)$, we see that
$$\begin{aligned}s_{2k-1}=s_{2k}&=\frac{1}{2}x,\\r_{2k-1}&=\left(k+\frac{1}{2}m\right)\left(k-\frac{1}{2}m\right)=\frac{1}{4}\left((2k)^2-m^2\right),\\r_{2k}&=\left(k+\frac{1}{2}(n+1)\right)\left(k+1-\frac{1}{2}(n+1)\right)=\frac{1}{4}\left((2k+1)^2-n^2\right),\end{aligned}$$

so the continued fraction in (3.29) can be written as
$$\begin{aligned}&(a-b-c)+\frac{c(1-c)}{a-b-c+}\mathop{\mathbf{K}}_{k=1}^{\infty}\frac{r_k}{s_k}\\&=\frac{x}{2}+\frac{\frac{1}{4}(1^2-n^2)}{x/2}+\frac{\frac{1}{4}(2^2-m^2)}{x/2}+\frac{\frac{1}{4}(3^2-n^2)}{x/2}+\frac{\frac{1}{4}(4^2-m^2)}{x/2}+\cdots\\&=\frac{x}{2}+\frac{\frac{1}{2}(1^2-n^2)}{x}+\frac{2^2-m^2}{x}+\frac{3^2-n^2}{x}+\frac{4^2-m^2}{x}+\cdots.\end{aligned} \tag{3.30}$$

Combining (3.28), (3.29), and (3.30), we find that
$$\frac{m}{2}\frac{1+P}{1-P}=\frac{x}{2}+\frac{\frac{1}{2}(1^2-n^2)}{x}+\frac{2^2-m^2}{x}+\frac{3^2-n^2}{x}+\frac{4^2-m^2}{x}+\cdots. \tag{3.31}$$

Multiplying both sides of (3.31) by 2, taking reciprocals of both sides, and then multiplying by m, we obtain
$$\frac{1-P}{1+P}=\frac{m}{x+}\frac{1^2-n^2}{x}+\frac{2^2-m^2}{x}+\frac{3^2-n^2}{x}+\frac{4^2-m^2}{x}+\cdots,$$
which is the identity (3.20) in Theorem 3.1.

The region of convergence for Entry 34 is given on page 432 in Lisa Jacobsen's paper [Jac89]. This then completes the proof.

4. Identities related to Entries 34 and 25

In proving Entry 34, we used an initial substitution $d = a - b$, $e = a - c + 1$ in the main continued fraction (2.16). Starting with the substitution $d = a - b - 1$, $e = a - c$ yields a similar identity, given below as Theorem 4.1. Theorem 4.1 has a corollary that is interesting in that it appears to be a companion to Entry 25 in Ramanujan's second notebook. This corollary, Theorem 5.1, is discussed and proved in the next section.

Theorem 4.1. *Suppose that $m \equiv 0 (\mathrm{mod}\ 4)$ or $n \equiv 2 (\mathrm{mod}\ 4)$, or that $\Re(x) > 0$, with m and n arbitrary complex numbers. Define*

$$P = \frac{\Gamma(\frac{1}{8}(x+m+n+2))\Gamma(\frac{1}{8}(x+m-n+2))\Gamma(\frac{1}{8}(x-m+n+6))\Gamma(\frac{1}{8}(x-m-n+6))}{\Gamma(\frac{1}{8}(x-m+n+2))\Gamma(\frac{1}{8}(x-m-n+2))\Gamma(\frac{1}{8}(x+m+n+6))\Gamma(\frac{1}{8}(x+m-n+6))}. \tag{4.32}$$

Then

$$\frac{1-P}{1+P} = \frac{m}{x+}\ \frac{2^2-n^2}{x}\ +\ \frac{4^2-m^2}{x}\ +\ \frac{6^2-n^2}{x}\ +\ \frac{8^2-m^2}{x}\ +\cdots.$$

We break up the proof into two parts. First, we prove this identity without regard for its region of convergence. Second, we discuss the convergence of the continued fraction at the end of our proof.

Proof. As in the proof of Entry 34, we use the relations (2.4) and (2.10) and the continued fraction (2.16). In (2.4), we exchange b and c and also d and e, and then replace b by $b+1$ and d by $d+1$. After a rearrangement, we find that

$$\begin{aligned} {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) &= \frac{ac(d-b)}{(d+1)e(a+c-e)}{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ d+2,\, e+1 \end{matrix}; 1\right) \\ &\quad - \frac{(e-a)(e-c)}{e(a+c-e)}{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right). \end{aligned} \tag{4.33}$$

In (2.10), we exchange a and b to deduce that

$${}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right) = \frac{d-a}{d}{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) + \frac{a(e-c)}{de}{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right). \tag{4.34}$$

Moreover, in (2.10), we exchange b and c and also d and e, and then replace b by $b+1$ and d by $d+1$. After some rearrangement, we see that

$$\begin{aligned} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right) &= \frac{e}{e-c}{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) \\ &\quad - \frac{c(d-b)}{(d+1)(e-c)}{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ d+2,\, e+1 \end{matrix}; 1\right). \end{aligned} \tag{4.35}$$

Substitute (4.35) into (4.34) to obtain

$$\begin{aligned} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right) &= \frac{d-a}{d}{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) \\ &\quad + \frac{a(e-c)}{de}\left\{\frac{e}{e-c}{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right)\right. \\ &\quad \left. - \frac{c(d-b)}{(d+1)(e-c)}{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ d+2,\, e+1 \end{matrix}; 1\right)\right\} \\ &= {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) - \frac{ac(d-b)}{de(d+1)}{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ d+2,\, e+1 \end{matrix}; 1\right). \end{aligned} \tag{4.36}$$

Next, substitute (4.33) into (4.36) to eliminate ${}_3F_2\left(\begin{smallmatrix}a,b+1,c\\d+1,e\end{smallmatrix};1\right)$. Hence, after considerable simplification,

$$\begin{aligned}
{}_3F_2\left(\begin{matrix}a,\,b,\,c\\d,\,e\end{matrix};1\right) &= \frac{ac(d-b)}{(d+1)e(a+c-e)}\,{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\d+2,\,e+1\end{matrix};1\right)\\
&\quad-\frac{(e-a)(e-c)}{e(a+c-e)}\,{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right)\\
&\quad-\frac{ac(d-b)}{de(d+1)}\,{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\d+2,\,e+1\end{matrix};1\right)\\
&=\frac{ac(d-b)(d+e-a-c)}{de(d+1)(a+c-e)}\,{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\d+2,\,e+1\end{matrix};1\right)\\
&\quad-\frac{(e-a)(e-c)}{e(a+c-e)}\,{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right).
\end{aligned}\tag{4.37}$$

Let $d=a-b-1$ and $e=a-c$ in (2.16) to deduce that

$$(a-b-1)\frac{{}_3F_2\left(\begin{smallmatrix}a,b,c\\a-b-1,a-c\end{smallmatrix};1\right)}{{}_3F_2\left(\begin{smallmatrix}a,b+1,c\\a-b,a-c\end{smallmatrix};1\right)}=(a-b-c-1)-\frac{c^2}{a-b-c-1+\mathop{\mathbf{K}}\limits_{k=1}^{\infty}\frac{r_k}{s_k}},\tag{4.38}$$

where

$$\begin{aligned}
r_{2k-1}&=(k+b)(k-1-b),\\
s_{2k-1}&=a-b-c-1,\\
r_{2k}&=(k+c)(k-c),\\
s_{2k}&=a-b-c-1.
\end{aligned}$$

We now turn to evaluating the two hypergeometric functions on the left-hand side of (4.38). With the aforementioned substitutions, (4.37) takes the shape

$$\begin{aligned}
&{}_3F_2\left(\begin{matrix}a,\,b,\,c\\a-b-1,\,a-c\end{matrix};1\right)\\
&\quad=\frac{a(a-2b-1)(a-b-2c-1)}{2(a-b-1)(a-c)(a-b)}\,{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\a-b+1,\,a-c+1\end{matrix};1\right)\\
&\qquad+\frac{(a-2c)}{2(a-c)}\,{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\a-b,\,a-c+1\end{matrix};1\right),
\end{aligned}\tag{4.39}$$

and (4.33) takes the form

$$\begin{aligned}
{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\a-b,\,a-c\end{matrix};1\right)&=\frac{a(a-2b-1)}{2(a-b)(a-c)}\,{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\a-b+1,\,a-c+1\end{matrix};1\right)\\
&\quad+\frac{(a-2c)}{2(a-c)}\,{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\a-b,\,a-c+1\end{matrix};1\right).
\end{aligned}\tag{4.40}$$

Note that the hypergeometric functions on the right-hand sides of (4.39) and (4.40) are identical. Also, note that both ${}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\a-b+1,\,a-c+1\end{matrix};1\right)$ and ${}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\a-b,\,a-c+1\end{matrix};1\right)$ can be evaluated using Dixon's Identity (2.1). Thus,

$$\begin{aligned}
&\frac{a(a-2b-1)}{2(a-b)(a-c)}\,{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\a-b+1,\,a-c+1\end{matrix};1\right)\\
&=\frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac12(a+1))\Gamma(\frac12(a+1)-b-c-1)}{2\Gamma(\frac12(a+1)-b-1)\Gamma(\frac12(a+1)-c)\Gamma(a)\Gamma(a-b-c)}
\end{aligned}\tag{4.41}$$

and

$$\frac{(a-2c)}{2(a-c)}{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\ a-b,\,a-c+1\end{matrix};1\right)=\frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(a)\Gamma(a-b-c)}. \tag{4.42}$$

Hence, putting (4.39) and (4.40) into (4.38), and then utilizing (4.41) and(4.42), we deduce that

$$\begin{aligned}
&(a-b-1)\frac{{}_3F_2\left(\begin{smallmatrix}a,b,c\\ a-b-1,a-c\end{smallmatrix};1\right)}{{}_3F_2\left(\begin{smallmatrix}a,b+1,c\\ a-b,a-c\end{smallmatrix};1\right)}\\
&=(a-b-1)\frac{\frac{a(a-2b-1)(a-b-2c-1)}{2(a-b-1)(a-c)(a-b)}{}_3F_2\left(\begin{smallmatrix}a+1,b+1,c+1\\ a-b+1,a-c+1\end{smallmatrix};1\right)+\frac{(a-2c)}{2(a-c)}{}_3F_2\left(\begin{smallmatrix}a,b+1,c\\ a-b,a-c+1\end{smallmatrix};1\right)}{\frac{a(a-2b-1)}{2(a-b)(a-c)}{}_3F_2\left(\begin{smallmatrix}a+1,b+1,c+1\\ a-b+1,a-c+1\end{smallmatrix};1\right)+\frac{(a-2c)}{2(a-c)}{}_3F_2\left(\begin{smallmatrix}a,b+1,c\\ a-b,a-c+1\end{smallmatrix};1\right)}\\
&=(a-b-1)\frac{\frac{(a-b-2c-1)\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}{2(a-b-1)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c)}+\frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(a)\Gamma(a-b-c)}}{\frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}{2\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c)}+\frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(a)\Gamma(a-b-c)}}\\
&=(a-b-1)\frac{\frac{(a-b-2c-1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}{(a-b-1)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)}+\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)}}{\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}{\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)}+\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)}}\\
&=(a-b-1)\frac{\frac{a-b-2c-1}{a-b-1}+\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}}{1+\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}}\\
&=(a-b-1)\frac{\frac{a-b-2c-1}{a-b-1}+R}{1+R}\\
&=(a-b-1)\left\{-\frac{\frac{2c}{a-b-1}}{1+R}+1\right\}\\
&=-\frac{2c}{1+R}+(a-b-1)\\
&=(a-b-c-1)-c\frac{1-R}{1+R},
\end{aligned}$$

where

$$R=\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}. \tag{4.43}$$

Thus, (4.38) becomes

$$(a-b-c-1)-c\frac{1-R}{1+R}=(a-b-c-1)-\frac{c^2}{a-b-c-1+}\mathop{\mathbf{K}}_{k=1}^{\infty}\frac{r_k}{s_k},$$

or

$$\frac{1-R}{1+R}=\frac{c}{a-b-c-1+}\mathop{\mathbf{K}}_{k=1}^{\infty}\frac{r_k}{s_k}, \tag{4.44}$$

where

$$\begin{cases} r_{2k-1} &= (k+b)(k-1-b),\\ s_{2k-1} &= a-b-c-1,\\ r_{2k} &= (k+c)(k-c),\\ s_{2k} &= a-b-c-1. \end{cases} \tag{4.45}$$

Now let

$$a = \tfrac{1}{4}x + b + c + 1, \quad b = \tfrac{1}{4}(n-2), \quad \text{and} \quad c = \tfrac{1}{4}m$$

in (4.43). Then R has the form

$$\begin{aligned} R &= \frac{\Gamma(\frac{1}{8}(x+m+n+2))\Gamma(\frac{1}{8}(x-m-n+6))\Gamma(\frac{1}{8}(x+m-n+2))\Gamma(\frac{1}{8}(x-m+n+6))}{\Gamma(\frac{1}{8}(x+m-n+6))\Gamma(\frac{1}{8}(x-m+n+2))\Gamma(\frac{1}{8}(x+m+n+6))\Gamma(\frac{1}{8}(x-m-n+2))} \\ &= P, \end{aligned} \tag{4.46}$$

by (4.32). Also, from (4.45),

$$\begin{aligned} s_{2k-1} = s_{2k} &= \left(\frac{1}{4}x + b + c + 1\right) - b - c - 1 = \frac{1}{4}x, \\ r_{2k-1} &= \left(k + \frac{1}{4}(n-2)\right)\left(k - 1 - \frac{1}{4}(n-2)\right) = \frac{1}{16}\left((4k-2)^2 - n^2\right), \\ r_{2k} &= \left(k + \frac{1}{4}m\right)\left(k - \frac{1}{4}m\right) = \frac{1}{16}\left((4k)^2 - m^2\right). \end{aligned}$$

Thus, by (4.44) and (4.46),

$$\begin{aligned} \frac{1-P}{1+P} &= \frac{\frac{1}{4}m}{x/4+}\,\frac{\frac{1}{16}(2^2-n^2)}{x/4}\,+\,\frac{\frac{1}{16}(4^2-m^2)}{x/4}\,+\,\frac{\frac{1}{16}(6^2-n^2)}{x/4}\,+\,\frac{\frac{1}{16}(8^2-m^2)}{x/4}\,+\cdots \\ &= \frac{m}{x+}\,\frac{2^2-n^2}{x}\,+\,\frac{4^2-m^2}{x}\,+\,\frac{6^2-n^2}{x}\,+\,\frac{8^2-m^2}{x}\,+\cdots, \end{aligned}$$

as desired.

Lastly, we examine the convergence claimed for Theorem 4.1. If either $m \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$, the continued fraction terminates, so it converges. For the more difficult region, we apply a theorem of Jacobsen [Jac89, Theorem 2.3]. Since the application is direct, we forego the details.

Theorem 4.1 can actually be shown to be equivalent to Entry 34 by replacing n by $2n$, m by $2m$, and x by $2x$ in Theorem 4.1 and then simplifying. However, the format given in Theorem 4.1 makes the corollary, Theorem 5.1, more obvious. The substitutions $d = a - b + 1$, $e = a - c + 2$ and $d = a - b - 2$, $e = a - c - 1$ in the main continued fraction (2.16), followed by an appropriate second substitution, also give identities that are equivalent to Entry 34.

5. A Corollary of Theorem 4.1

Theorem 5.1. *Suppose that N is an even integer, or that $\Re(x) > 0$, where N is an arbitrary complex number. Define*

$$R = \frac{\Gamma(\frac{1}{4}(x+N+1))\Gamma(\frac{1}{4}(x-N+3))}{\Gamma(\frac{1}{4}(x+N+3))\Gamma(\frac{1}{4}(x-N+1))}. \tag{5.47}$$

Then

$$\frac{1-R}{1+R} = \frac{N}{2x+}\,\frac{2^2-N^2}{2x}\,+\,\frac{4^2-N^2}{2x}\,+\,\frac{6^2-N^2}{2x}\,+\,\frac{8^2-N^2}{2x}\,+\cdots. \tag{5.48}$$

This theorem is a nice companion to Ramanujan's Entry 25 in Chapter 12 of Ramanujan's second notebook [Ramanujan], [Ber89, p. 140]. We emphasize that Entry 25 is originally due to Euler; a discussion of other proofs of Entry 25 with references can be found in the first author's book [Ber89, p. 141].

Theorem 5.2. (Entry 25) *Suppose that either n is an odd integer or x is an arbitrary complex number. Or, suppose that n is an arbitrary complex number and $\Re(x) > 0$. Then,*

$$\frac{\Gamma(\frac{1}{4}(x+n+1))\Gamma(\frac{1}{4}(x-n+1))}{\Gamma(\frac{1}{4}(x+n+3))\Gamma(\frac{1}{4}(x-n+3))} = \frac{4}{x-}\,\frac{n^2-1^2}{2x}\,-\,\frac{n^2-3^2}{2x}\,-\,\frac{n^2-5^2}{2x}\,-\cdots.$$

Note that Entry 25 contains the same Gamma functions as in Theorem 5.1, but in a different configuration. Two proofs are provided; one shows this theorem is an easy corollary of Theorem 4.1, while the other proves it directly using the method that Ramanathan employed in [Ram87] to prove Entry 25. The region of converge for Entry 5.2 is established in Jacobsen's paper [Jac89, p. 430].

Proof 1 of Theorem 5.1. In Theorem 4.1, replace x by $2x$ and let $m = n = N$. Theorem 5.1 follows immediately as a corollary.

Proof 2 of Theorem 5.1. The continued fraction used in this proof is Entry 22 from Chapter 12 in Ramanujan's second Notebook [Ramanujan], [Ber89, p. 136], which is given by

$$\begin{aligned}&\frac{\beta x}{\gamma}\frac{{}_2F_1\left({-\alpha,\beta+1 \atop \gamma+1};-x\right)}{{}_2F_1\left({-\alpha,\beta \atop \gamma};-x\right)}\\ &\quad = \frac{\beta x}{\gamma-(\alpha+\beta+1)x+}\,\frac{(\beta+1)(\alpha+\gamma+1)x}{\gamma+1-(\alpha+\beta+2)x+}\,\frac{(\beta+2)(\alpha+\gamma+2)x}{\gamma+2-(\alpha+\beta+3)x+\cdots}.\end{aligned} \tag{5.49}$$

(At the end of our proof, we discuss convergence.) Let $\alpha = -a$, $\beta = b$, $\gamma = a - b + 2$, and $x = 1$ in (5.49) to deduce that

$$\begin{aligned}&\frac{{}_2F_1\left({a,b+1 \atop a-b+3};-1\right)}{{}_2F_1\left({a,b \atop a-b+2};-1\right)}\\ &\quad = \frac{a-b+2}{2a-2b+1+}\,\frac{(b+1)(3-b)}{2a-2b+1}\,+\,\frac{(b+2)(4-b)}{2a-2b+1}\,+\,\frac{(b+3)(5-b)}{2a-2b+1}\,+\cdots.\end{aligned} \tag{5.50}$$

We will evaluate ${}_2F_1\left({a,\,b+1 \atop a-b+3};-1\right)$ and ${}_2F_1\left({a,\,b \atop a-b+2};-1\right)$ by rewriting them in a form that allows the use of Kummer's Identity (2.2). To do this, we use the relations

$${}_2F_1\left({\alpha+1,\beta+1 \atop \gamma+1};x\right) = \frac{\gamma}{\beta(1-x)}{}_2F_1\left({\alpha,\beta \atop \gamma};x\right) - \frac{\gamma-\beta}{\beta(1-x)}{}_2F_1\left({\alpha+1,\beta \atop \gamma+1};x\right), \tag{5.51}$$

$${}_2F_1\left({\alpha,\beta \atop \gamma};x\right) = {}_2F_1\left({\alpha,\beta+1 \atop \gamma};x\right) - \frac{\alpha x}{\gamma}{}_2F_1\left({\alpha+1,\beta+1 \atop \gamma+1};x\right), \tag{5.52}$$

and

$$\begin{aligned}{}_2F_1\left({\alpha+1,\beta+1 \atop \gamma+2};x\right) &= \frac{\gamma(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)x}{}_2F_1\left({\alpha,\beta \atop \gamma};x\right)\\ &\quad - \frac{\gamma(\gamma+1)(1-x)}{(\gamma-\alpha)(\gamma-\beta)x}{}_2F_1\left({\alpha+1,\beta+1 \atop \gamma+1};x\right).\end{aligned} \tag{5.53}$$

The identity (5.52) is due to C. F. Gauss [Gau1876, Equation (17), p. 133]. The identities (5.51) and (5.53) can be found in Ramanathan's paper [Ram87, Equations (9), (11), p. 279], and were obtained from other results of Gauss in [Gau1876].

In (5.51), let $\alpha = a-1$, $\beta = b-1$, $\gamma = a-b+1$, and $x=-1$ to obtain

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a,\, b \\ a-b+2 \end{matrix}; -1\right) &= \frac{a-b+1}{2(b-1)}\, {}_2F_1\left(\begin{matrix} a-1,\, b-1 \\ a-b+1 \end{matrix}; -1\right) \\ &\quad - \frac{a-2b+2}{2(b-1)}\, {}_2F_1\left(\begin{matrix} a,\, b-1 \\ a-b+2 \end{matrix}; -1\right). \end{aligned} \tag{5.54}$$

Both of the hypergeometric series on the right side of (5.54) can be evaluated by Kummer's Theorem (2.2). After simplification, we find that

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} a,\, b \\ a-b+2 \end{matrix}; -1\right) \\ &= \frac{\Gamma(a-b+2)\Gamma(\frac{1}{2}(a+1))}{2(b-1)\Gamma(\frac{1}{2}(a+1)-b+1)\Gamma(a)} - \frac{(a-2b+2)\Gamma(a-b+2)\Gamma(\frac{1}{2}a)}{2(b-1)(a-2b+2)\Gamma(\frac{1}{2}a-b+1)\Gamma(a)} \\ &= \frac{\Gamma(a-b+2)}{2(b-1)\Gamma(a)} \left\{ \frac{\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b+1)} - \frac{\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)} \right\}. \end{aligned} \tag{5.55}$$

To evaluate ${}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+3 \end{matrix}; -1\right)$, we begin with (5.53). Let $\alpha = a-1$, $\beta = b$, $\gamma = a-b+1$, and $x=-1$ to obtain

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+3 \end{matrix}; -1\right) &= \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\, {}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b+1 \end{matrix}; -1\right) \\ &\quad - \frac{2(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\, {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+2 \end{matrix}; -1\right). \end{aligned} \tag{5.56}$$

We now need to evaluate ${}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b+1 \end{matrix}; -1\right)$ and ${}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+2 \end{matrix}; -1\right)$. For the first of these, we let $\alpha = b$, $\beta = a-1$, $\gamma = a-b+1$, and $x=-1$ in (5.52). Therefore,

$${}_2F_1\left(\begin{matrix} b,\, a-1 \\ a-b+1 \end{matrix}; -1\right) = {}_2F_1\left(\begin{matrix} b,\, a \\ a-b+1 \end{matrix}; -1\right) + \frac{b}{a-b+1}\, {}_2F_1\left(\begin{matrix} b+1,\, a \\ a-b+2 \end{matrix}; -1\right). \tag{5.57}$$

To evaluate ${}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+2 \end{matrix}; -1\right)$, we apply (5.53) with $\alpha = a-1$, $\beta = b$, $\gamma = a-b$, and $x=-1$ to deduce that

$${}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+2 \end{matrix}; -1\right) = \frac{(a-b)(a-b+1)}{(b-1)(a-2b)} \left\{ {}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b \end{matrix}; -1\right) - 2\, {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+1 \end{matrix}; -1\right) \right\}. \tag{5.58}$$

Next, (5.52) with $\alpha = a$, $\beta = b$, $\gamma = a-b$, and $x=-1$ yields

$${}_2F_1\left(\begin{matrix} a,\, b \\ a-b \end{matrix}; -1\right) = {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right) + \frac{a}{a-b}\, {}_2F_1\left(\begin{matrix} a+1,\, b+1 \\ a-b+1 \end{matrix}; -1\right) \tag{5.59}$$

and (5.51) with $\alpha = b$, $\beta = a$, $\gamma = a-b$, and $x=-1$ gives

$${}_2F_1\left(\begin{matrix} b+1,\, a+1 \\ a-b+1 \end{matrix}; -1\right) = \frac{a-b}{2a}\, {}_2F_1\left(\begin{matrix} b,\, a \\ a-b \end{matrix}; -1\right) + \frac{b}{2a}\, {}_2F_1\left(\begin{matrix} b+1,\, a \\ a-b+1 \end{matrix}; -1\right). \tag{5.60}$$

Isolating ${}_2F_1\left(\begin{smallmatrix} a, b+1 \\ a-b+1 \end{smallmatrix}; -1\right)$ in (5.60), we deduce that

$$\begin{aligned}
{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b+1 \end{matrix}; -1\right) &= \frac{2a}{b}{}_2F_1\left(\begin{matrix} a+1, b+1 \\ a-b+1 \end{matrix}; -1\right) - \frac{a-b}{b}{}_2F_1\left(\begin{matrix} a, b \\ a-b \end{matrix}; -1\right) \\
&= \frac{2a}{b}{}_2F_1\left(\begin{matrix} a+1, b+1 \\ a-b+1 \end{matrix}; -1\right) - \frac{a-b}{b}\Bigg\{{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b \end{matrix}; -1\right) \\
&\quad + \frac{a}{a-b}{}_2F_1\left(\begin{matrix} a+1, b+1 \\ a-b+1 \end{matrix}; -1\right)\Bigg\} \\
&= \frac{a}{b}{}_2F_1\left(\begin{matrix} a+1, b+1 \\ a-b+1 \end{matrix}; -1\right) - \frac{a-b}{b}{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b \end{matrix}; -1\right),
\end{aligned} \tag{5.61}$$

where in the second equality we used (5.59). Substituting (5.61) into (5.58), we find that

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b+2 \end{matrix}; -1\right) \\
&= \frac{(a-b)(a-b+1)}{(b-1)(a-2b)}\Bigg\{{}_2F_1\left(\begin{matrix} a-1, b \\ a-b \end{matrix}; -1\right) - 2\Bigg[\frac{a}{b}{}_2F_1\left(\begin{matrix} a+1, b+1 \\ a-b+1 \end{matrix}; -1\right) \\
&\quad - \frac{a-b}{b}{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b \end{matrix}; -1\right)\Bigg]\Bigg\} \\
&= \frac{(a-b)(a-b+1)}{(b-1)(a-2b)}\Bigg\{{}_2F_1\left(\begin{matrix} a-1, b \\ a-b \end{matrix}; -1\right) - \frac{2a}{b}\frac{(a-b)}{2a}{}_2F_1\left(\begin{matrix} a-1, b \\ a-b \end{matrix}; -1\right) \\
&\quad + \frac{2(a-b)}{b}{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b \end{matrix}; -1\right)\Bigg\} \\
&= \frac{(a-b)(a-b+1)}{(b-1)(a-2b)}\Bigg\{\frac{2b-a}{b}{}_2F_1\left(\begin{matrix} a-1, b \\ a-b \end{matrix}; -1\right) + \frac{2(a-b)}{b}{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b \end{matrix}; -1\right)\Bigg\},
\end{aligned} \tag{5.62}$$

where in the second equality we used the representation

$$\begin{aligned}
{}_2F_1\left(\begin{matrix} a+1, b+1 \\ a-b+1 \end{matrix}; -1\right) &= \frac{\Gamma(a-b+1)\Gamma(\frac{1}{2}(a+1)+1)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(a+2)} \\
&= \frac{(a-b)\Gamma(a-b)\frac{1}{2}(a+1)\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b)(a+1)a\Gamma(a)} \\
&= \frac{a-b}{2a}{}_2F_1\left(\begin{matrix} a-1, b \\ a-b \end{matrix}; -1\right),
\end{aligned}$$

which is obtained by applying Kummer's Identity (2.2) twice. Substituting (5.57) into (5.56), we find that

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b+3 \end{matrix}; -1\right) \\
&= \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\Bigg\{{}_2F_1\left(\begin{matrix} a, b \\ a-b+1 \end{matrix}; -1\right) + \frac{b}{a-b+1}{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b+2 \end{matrix}; -1\right)\Bigg\} \\
&\quad - \frac{2(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b+2 \end{matrix}; -1\right) \\
&= \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}{}_2F_1\left(\begin{matrix} a, b \\ a-b+1 \end{matrix}; -1\right) \\
&\quad + \frac{(a-b+2)(3b-2a-2)}{(b-2)(a-2b+1)}{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b+2 \end{matrix}; -1\right).
\end{aligned}$$

Next, substituting (5.62) into the last result, we find that

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b+3\end{matrix};-1\right)\\
&\quad=\frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\,{}_2F_1\left(\begin{matrix}a,\, b\\ a-b+1\end{matrix};-1\right)\\
&\qquad+\frac{(a-b+2)(3b-2a-2)}{(b-2)(a-2b+1)}\left\{\frac{(a-b)(a-b+1)}{(b-1)(a-2b)}\left[\frac{2b-a}{b}\,{}_2F_1\left(\begin{matrix}a-1,\, b\\ a-b\end{matrix};-1\right)\right.\right.\\
&\qquad\left.\left.+\frac{2(a-b)}{b}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b\end{matrix};-1\right)\right]\right\}\\
&\quad=\frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\left\{{}_2F_1\left(\begin{matrix}a,\, b\\ a-b+1\end{matrix};-1\right)\right.\\
&\qquad-\frac{(a-b)(3b-2a-2)}{b(b-1)}\,{}_2F_1\left(\begin{matrix}a-1,\, b\\ a-b\end{matrix};-1\right)\\
&\qquad\left.+\frac{2(a-b)^2(3b-2a-2)}{b(b-1)(a-2b)}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b\end{matrix};-1\right)\right\}.
\end{aligned}
\tag{5.63}
$$

By Kummer's Identity (2.2),

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}a,\, b\\ a-b+1\end{matrix};-1\right)&=\frac{\Gamma(a-b+1)\Gamma(\frac{1}{2}a+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(a+1)}\\
&=\frac{(a-b)\Gamma(a-b)\Gamma(\frac{1}{2}a+1)}{(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-b)\Gamma(a+1)}\\
&=\frac{2(a-b)}{a-2b}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b\end{matrix};-1\right).
\end{aligned}
$$

Hence, (5.63) takes the form

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b+3\end{matrix};-1\right)\\
&\quad=\frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\left\{\frac{2(a-b)}{a-2b}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b\end{matrix};-1\right)\right.\\
&\qquad-\frac{(a-b)(3b-2a-2)}{b(b-1)}\,{}_2F_1\left(\begin{matrix}a-1,\, b\\ a-b\end{matrix};-1\right)\\
&\qquad\left.+\frac{2(a-b)^2(3b-2a-2)}{b(b-1)(a-2b)}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b\end{matrix};-1\right)\right\}\\
&\quad=\frac{2(a-b)(a-b+1)(a-b+2)(b-2a)}{b(b-1)(b-2)(a-2b)}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b\end{matrix};-1\right)\\
&\qquad-\frac{(a-b)(a-b+1)(a-b+2)(3b-2a-2)}{b(b-1)(b-2)(a-2b+1)}\,{}_2F_1\left(\begin{matrix}a-1,\, b\\ a-b\end{matrix};-1\right).
\end{aligned}
\tag{5.64}
$$

Applying Kummer's Identity (2.2) in (5.64), we deduce that

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b+3\end{matrix};-1\right)\\
&\quad=\frac{2(a-b)(a-b+1)(a-b+2)(b-2a)}{b(b-1)(b-2)(a-2b)}\,\frac{\Gamma(a-b)\Gamma(\frac{1}{2}a+1)}{\Gamma(\frac{1}{2}a-b)\Gamma(a+1)}\\
&\qquad-\frac{(a-b)(a-b+1)(a-b+2)(3b-2a-2)}{b(b-1)(b-2)(a-2b+1)}\,\frac{\Gamma(a-b)\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(a)}
\end{aligned}
$$

$$= \frac{\Gamma(a-b+3)}{2b(b-1)(b-2)\Gamma(a)} \left\{ \frac{(b-2a)\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)} + \frac{(2a-3b+2)\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b+1)} \right\}. \tag{5.65}$$

Then, if we set

$$\begin{cases} P &= \dfrac{\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b+1)}, \\ Q &= \dfrac{\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)}, \end{cases} \tag{5.66}$$

we finally find from (5.55) and (5.65) that

$$\begin{aligned} \frac{{}_2F_1\left(\begin{matrix} a, b+1 \\ a-b+3 \end{matrix}; -1\right)}{{}_2F_1\left(\begin{matrix} a, b \\ a-b+2 \end{matrix}; -1\right)} &= \frac{\dfrac{\Gamma(a-b+3)}{2b(b-1)(b-2)\Gamma(a)} \left\{ \dfrac{(b-2a)\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)} + \dfrac{(2a-3b+2)\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b+1)} \right\}}{\dfrac{\Gamma(a-b+2)}{2(b-1)\Gamma(a)} \left\{ \dfrac{\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b+1)} - \dfrac{\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)} \right\}} \\ &= \frac{\dfrac{(a-b+2)\Gamma(a-b+2)}{b(b-2)} \{(b-2a)Q + (2a-3b+2)P\}}{\Gamma(a-b+2)\,(P-Q)} \\ &= \frac{(a-b+2)}{b(b-2)} \frac{\{(2a-b)\,(P-Q) + (2-2b)P\}}{P-Q} \\ &= \frac{(a-b+2)}{b(b-2)} \left\{ 2a-b + \frac{2(1-b)P}{P-Q} \right\}. \end{aligned}$$

Thus, by (5.50),

$$\begin{aligned} &\frac{(a-b+2)}{b(b-2)} \left\{ 2a-b + \frac{2(1-b)P}{P-Q} \right\} \\ &= \frac{a-b+2}{2a-2b+1+} \, \frac{(b+1)(3-b)}{2a-2b+1} \, {}_{+} \, \frac{(b+2)(4-b)}{2a-2b+1} \, {}_{+} \, \frac{(b+3)(5-b)}{2a-2b+1} \, {}_{+ \cdots}. \end{aligned} \tag{5.67}$$

Multiplying both sides of (5.67) by $-b(b-2)/(a-b+2)$ and adding $2a-b$ to both sides of the result yields

$$\begin{aligned} &\frac{2(b-1)P}{P-Q} \\ &= 2a-b + \frac{b(2-b)}{2a-2b+1+} \, \frac{(b+1)(3-b)}{2a-2b+1} \, {}_{+} \, \frac{(b+2)(4-b)}{2a-2b+1} \, {}_{+} \, \frac{(b+3)(5-b)}{2a-2b+1} \, {}_{+ \cdots}. \end{aligned}$$

Now let $a = \frac{1}{2}(x+N+1)$ and $b = \frac{1}{2}N+1$, and employ (5.66), to obtain

$$\begin{aligned} &\frac{N\Gamma(\frac{1}{4}(x+N+3))\Gamma(\frac{1}{4}(x-N+1))}{\Gamma(\frac{1}{4}(x+N+3))\Gamma(\frac{1}{4}(x-N+1)) - \Gamma(\frac{1}{4}(x+N+1))\Gamma(\frac{1}{4}(x-N+3))} \\ &= x + \frac{N}{2} + \frac{\frac{1}{4}(2^2-N^2)}{x} \, {}_{+} \, \frac{\frac{1}{4}(4^2-N^2)}{x} \, {}_{+} \, \frac{\frac{1}{4}(6^2-N^2)}{x} \, {}_{+ \cdots}. \end{aligned} \tag{5.68}$$

If we define

$$\begin{cases} S &= \Gamma\left(\frac{1}{4}(x+N+3)\right)\Gamma\left(\frac{1}{4}(x-N+1)\right), \\ T &= \Gamma\left(\frac{1}{4}(x+N+1)\right)\Gamma\left(\frac{1}{4}(x-N+3)\right), \end{cases}$$

we see that (5.68) can be written in the form

$$x + \frac{\frac{1}{4}(2^2-N^2)}{x} \, {}_{+} \, \frac{\frac{1}{4}(4^2-N^2)}{x} \, {}_{+} \, \frac{\frac{1}{4}(6^2-N^2)}{x} \, {}_{+ \cdots} = \frac{NS}{S-T} - \frac{N}{2}$$

$$= \frac{N(1+\frac{T}{S})}{2(1-\frac{T}{S})}.$$

Taking reciprocals of both sides, we arrive at

$$\begin{aligned}\frac{2(1-\frac{T}{S})}{N(1+\frac{T}{S})} &= \frac{1}{x+}\,\frac{\frac{1}{4}(2^2-N^2)}{x\quad+}\,\frac{\frac{1}{4}(4^2-N^2)}{x\quad+}\,\frac{\frac{1}{4}(6^2-N^2)}{x\quad+\cdots}\\ &= \frac{2}{2x+}\,\frac{2^2-N^2}{2x\quad+}\,\frac{4^2-N^2}{2x\quad+}\,\frac{6^2-N^2}{2x\quad+\cdots}.\end{aligned} \tag{5.69}$$

Multiplying both sides of (5.69) by $\frac{1}{2}N$ and noting from (5.47) that $R = T/S$, we complete the proof of (5.48).

The domain of validity for Theorem 5.1 follows directly from that of Theorem 4.1. We let $m = n = N$ and replace x by $2x$ in Theorem 4.1. If $N = m \equiv 0 (\mathrm{mod}\ 4)$, N is even, and if $N = n \equiv 2 (\mathrm{mod}\ 4)$, N is even. In both cases the identity holds. If N is an arbitrary complex number, we must have $\Re(2x) > 0$, so $\Re(x) > 0$ is the necessary condition.

The original results and proofs in this paper first appeared in the doctoral thesis of the second author in 2014 [Reu14].

We are grateful to the referee for suggestions regarding Lemma 2.4.

References

[AAR99] G.E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, 1999.

[Ber89] B.C. Berndt, *Ramanujan's notebooks, Part II*, Springer-Verlag, New York, 1989.

[Gau1876] C.F. Gauss, *Disquisitiones generales circa seriem infinitam* $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots$, Pars prior, Comm. Soc. Regiae Sci. Gottingensis rec. 2:1–46, 1812; in C. F. Gauss, Werke, Vol 3, Königlichen Gesellschaft der Wissenschaften, Göttingen, 1876, pp. 123–162.

[Jac89] L. Jacobsen, Domains of validity for some of Ramanujan's continued fraction formulas, *J. Math. Anal. Appl.* **143**, 1989, 412–437.

[Ram87] K.G. Ramanathan, Hypergeometric series and continued fractions, *Proc. Indian Acad. Sci. Math. Sci.* **97** (1987), 277–296.

[Ramanujan] S. Ramanujan, *Notebooks of Srinivasa Ramanujan* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957; second ed., 2012.

[Reu14] V.J. Reuter, *Hypergeometric functions, continued fractions for products of gamma functions, and q-analogues*, Ph. D. Thesis, University of Illinois at Urbana-Champaign, 2014.

Bruce C. Berndt
Department of Mathematics, University of Illinois
1409 West Green Street
Urbana, IL 61801, USA
e-mail: berndt@illinois.edu

and

Victoria J. Reuter
Department of Mathematics, 1 University Plaza
University of Wisconsin–Platteville
Platteville, WI 53818, USA
e-mail: reutervi@uwplatt.edu