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ON THE RIESZ MEANS OF $\frac{n}{\phi(n)}$

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In honor of Professor M. Ram Murty on his sixtieth birthday

Abstract. Let $\phi(n)$ denote the Euler-totient function. We study the error term of the general k-th Riesz mean of the arithmetical function $\frac{n}{\phi(n)}$ for any positive integer $k \geq 1$, namely the error term $E_k(x)$ where

$$\frac{1}{k!}\sum_{n\leq x}\frac{n}{\phi(n)}\left(1-\frac{n}{x}\right)^k = M_k(x)+E_k(x).$$

The upper bound for $|E_k(x)|$ established here thus improves the earlier known upper bound when $k=1$.

1. Introduction

Investigating the growth (or decay) of the absolute value of the error term of the summatory function of an arithmetical function is a classical question in number theory. Many results on such interesting questions are available in the literature (for some of them, the readers may refer to chapter 14 of [3]). Let $\phi(n)$ denote the Euler-totient function defined to be the number of positive integers $\leq n$ which are co-prime to n. Let us write

$$\sum_{n\leq x}\frac{1}{\phi(n)} = A\left(\log x + B\right) + E_0^*(x) \tag{1}$$

and

$$\sum_{n\leq x}\frac{n}{\phi(n)} = Ax - \ \log x + E_1^*(x) \tag{2}$$

where

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$$A = \frac{315\zeta(3)}{2\pi^4}, \quad B = \gamma_0 - \sum_p \frac{\log p}{p^2 - p + 1}. \tag{3}$$

Here $\zeta(s)$ and γ_0 denote the Riemann zeta-function and the Euler's constant respectively. The sum defining B extends over all primes p. In [4] (see p.184), E. Landau proved that

$$E_0^*(x) \ll \frac{\log x}{x} \tag{4}$$

as $x \to \infty$. Using a theorem of Walfisz based on Weyl's inequality, in [6], R. Sitaramachandrarao established (by elementary methods) that

$$E_0^*(x) \ll \frac{(\log x)^{\frac{2}{3}}}{x} \tag{5}$$

as $x \to \infty$. In an another paper [7], R. Sitaramachandrarao studied the discrete average and integral average of these error terms $E_j^*(x)$ for $j = 0, 1$. In particular, he proved that

$$\int_1^x E_1^*(t)dt = -\frac{D}{2}x + O(x^{\frac{4}{5}}). \tag{6}$$

by elementary methods where

$$D = \gamma_0 + \log(2\pi) + \sum_p \frac{\log p}{p(p-1)}. \tag{7}$$

As a consequence of (2) and (6) (see Remark 4.1 of [7]), he derived that

$$\begin{aligned} \sum_{n\le x} \frac{n}{\phi(n)}(x-n) &= \int_1^x \left(\sum_{n\le u} \frac{n}{\phi(n)} \right) du \\ &= \frac{A}{2}x^2 - \frac{1}{2}x\log x + \left(\frac{1-D}{2} \right) x + O\left(x^{\frac{4}{5}} \right) \end{aligned} \tag{8}$$

Equivalently, he established that the first Riesz mean satisfies the asymptotic relation

$$\sum_{n\le x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x} \right) = \frac{A}{2}x - \frac{1}{2}\log x + \left(\frac{1-D}{2} \right) + O\left(x^{-\frac{1}{5}} \right). \tag{9}$$

If we denote the error term of the first Riesz mean related to the arithmetic function $\frac{n}{\phi(n)}$ in (9) by $E_1(x)$, then *a conjecture of Sitaramachandrarao* (see Remark 4.1 of [7]) asserts that

$$E_1(x) \ll \frac{1}{x^{\frac{3}{4}-\delta}} \tag{10}$$

for every small fixed positive δ.

The aim of this article is to study the error term of the general k-th Riesz mean related to the arithmetic function $\frac{n}{\phi(n)}$ for any positive integer $k \geq 1$. More precisely, we write

$$\frac{1}{k!}\sum_{n\leq x}\frac{n}{\phi(n)}\left(1-\frac{n}{x}\right)^k = M_k(x) + E_k(x) \tag{11}$$

where $M_k(x)$ is the main term and $E_k(x)$ is the error term of the sum under investigation. It should be mentioned here that in [1], K. Chandrasekharan and R. Narasimhan have developed a general method to study Omega and O-results for the error term of the general k-th Riesz mean whenever the generating function (i.e the Dirichlet series) corresponding to the coefficients satisfies a functional equation (with multiple gamma factors) analogous to the functional equation of the Riemann zeta-type. Though the Lemma 3.1 in the sequel suggests that the generating function in our case do have some nice factors (essentially $\zeta(s)$ and its translate), in totality nothing can be drawn about its functional equation. Therefore, such problems need to be treated in a different way and of course we can make use of the presence of these nice factors in the generating function.
We prove

Main Theorem. *Let $x \geq x_0$ where x_0 is a sufficiently large positive number. For any integer $k \geq 1$, we have*

$$\frac{1}{k!}\sum_{n\leq x}\frac{n}{\phi(n)}\left(1-\frac{n}{x}\right)^k = c_1(k)x + c_2(k)\log x + c_3(k) + E_k(x)$$

where $c_1(k), c_2(k)$ and $c_3(k)$ are certain specific constants (depend only on k) and

$$E_k(x) \ll \frac{1}{x^{\frac{1}{2}-\delta}}$$

for any small fixed positive constant δ satisfying $\delta < \frac{1}{100}$ and the implied constant is independent of k.

Remark: One expects the error term of the sum

$$\sum_{n\leq x} \frac{n}{\phi(n)}(x-n)$$

(whose average is the first Riesz mean of $\frac{n}{\phi(n)}$) to behave like the error term of the sum $\sum_{n\leq x} d(n)$ (where $d(n)$ is the number of positive divisors of n) since the generating function related to $\frac{n}{\phi(n)}$ behaves almost like $\zeta(s)\zeta(s+1)$ (see for example Lemma 3.1 in the sequel). This justifies the conjecture of Sitaramachandrarao. We also observe that for every integer $k \geq 1$, we have

$$\frac{1}{k!}\sum_{n\leq x} \frac{n}{\phi(n)}\left(1-\frac{n}{x}\right)^k = \frac{1}{k!}\sum_{n\leq x} \frac{n}{\phi(n)}\left(1+O\left(2^k\frac{n}{x}\right)\right)$$

Thus, it is reasonable to expect the error $E_1(x)$ to dominate over all other errors $E_k(x)$ in absolute value. In view of the above main theorem, we propose the following :

Conjecture. *For every integer $k \geq 1$,*

$$E_k(x) \ll \frac{1}{x^{\frac{3}{4}-\delta}}$$

for any small fixed positive constant δ and the implied constant is independent of k.

Remark: The constants $c_1(k), c_2(k)$ and $c_3(k)$ are determined explicitly in the last section 5. With $k=1$, we find that

$$c_1(1) = \frac{315\zeta(3)}{2\pi^4}, \quad c_2(1) = -\frac{1}{2}$$

and

$$c_3(1) = \frac{1}{2} - \frac{\gamma_0}{2} - \frac{1}{2}\log(2\pi) - \frac{1}{2}\sum_p \frac{\log p}{p(p-1)} = \frac{1-D}{2}.$$

Thus, the main theorem with $k=1$ improves Sitaramachandrarao's bound on $|E_1(x)|$ in (9) considerably though his conjecture is still far from being resolved.

2. Notations and Preliminaries

Notations: 1. Throughout the paper, $s = \sigma + it$; the parameters T and x are sufficiently large real numbers and k is an integer ≥ 1.
2. δ, ϵ always denote sufficiently small positive constants.
3. As usual $\zeta(s)$ denotes the Riemann zeta-function and γ_0 is Euler's constant.

3. Some Lemmas

Lemma 3.1. *For $\Re s > 1$, we have*

$$F(s) := \sum_{n=1}^{\infty} \frac{n}{\phi(n) n^s} = \zeta(s)\zeta(s+1)g(s)$$

where

$$g(s) := \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1-1/p)}\right)$$

with $g(s)$ is absolutely and uniformly convergent in any compact set in the half-plane $\Re s \geq -\frac{1}{2} + 2\delta$ for any small fixed positive δ satisfying $0 < \delta < \frac{1}{100}$.

Proof. For $\Re s > 1$, we have

$$\begin{aligned}
F(s) &:= \sum_{n=1}^{\infty} \frac{n}{\phi(n) n^s} \\
&= \prod_p \left(1 + \frac{p}{\phi(p)p^s} + \frac{p^2}{\phi(p^2)p^{2s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{p}{(p-1)p^s} \frac{1}{(1-1/p^s)}\right) \\
&= \prod_p \left(\frac{(1-1/p)(1-1/p^s) + 1/p^s}{(1-1/p)(1-1/p^s)}\right) \\
&= \zeta(s) \prod_p \left(1 + \frac{1}{p^{s+1}(1-1/p)}\right) \\
&= \zeta(s)\zeta(s+1) \prod_p \left(1 + \frac{1}{p^{s+1}(1-1/p)}\right)\left(1 - \frac{1}{p^{s+1}}\right) \\
&= \zeta(s)\zeta(s+1) \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1-1/p)}\right) \\
F(s) &= \zeta(s)\zeta(s+1)g(s)
\end{aligned}$$

where

$$g(s) := \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1-1/p)}\right).$$

We observe that $g(s)$ is an infinite product of the form $\prod_p (1 + a_p)$ which is absolutely convergent if and only if the sum $\sum |a_p|$ is convergent. Thus, the sum

$$\sum_p \left| \frac{1/p^{s+2} - 1/p^{2s+2}}{(1-1/p)} \right| \le 2\sum_p \frac{1}{p^{\sigma+2}} + 2\sum_p \frac{1}{p^{2\sigma+2}}$$

is absolutely and uniformly convergent in any compact set in the half-plane $\sigma > -\frac{1}{2} + 2\delta$ for any fixed δ satisfying $0 < \delta < \frac{1}{100}$. □

We prove the following lemma 3.2 adapting the arguments of A.E. Ingham (see p.31 Theorem B of [2]) with the dependence of the implied constants on k explicit.

Lemma 3.2. *Let c and y be any positive real numbers and $T \ge T_0$ where T_0 is sufficiently large. Then we have,*

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s(s+1)...(s+k)} ds = \begin{cases} \frac{1}{k!}\left(1-\frac{1}{y}\right)^k + O\left(\frac{4^k y^c}{T^k}\right) & \text{if } \ y \ge 1, \\ O\left(\frac{1}{T^k}\right) & \text{if } \ 0 < y \le 1. \end{cases}$$

Proof. If $y \ge 1$, we move the line of integration to the far left say to the line $\Re s = -R$ (with $R \ge 10k$, sufficiently large). Then, in the rectangular contour formed by the line segments joining the points $c - iT, c + iT, -R + iT, -R - iT$ and $c - iT$ in this anti-clockwise order, we find that $s = 0, -1, ... - k$ are the simple poles. The residue at $s = -r$ is $\frac{(-1)^r}{r!(k-r)!} y^{-r}$ and hence the sum of the residues namely

$$\sum_{r=0}^{k} \frac{(-1)^r}{r!(k-r)!} y^{-r} = \frac{1}{k!}\left(1-\frac{1}{y}\right)^k. \tag{12}$$

The sum of the horizontal lines contributions in absolute value is

$$\ll \int_{-R}^{c} \frac{y^\sigma}{T^{k+1}}\, d\sigma \ll \frac{(R+c)y^c}{T^{k+1}}. \tag{13}$$

The left vertical line contribution in absolute value is

$$\ll \int_{-T}^{T} \frac{y^{-R}}{|(-R+it)(-R+1+it)\dots(-R+k+it)|}dt$$
$$\ll \int\limits_{|t|\le R} \dots + \int\limits_{T\ge|t|>R} \dots$$
$$\ll \frac{Ry^{-R}}{R(R-1)(R-2)\cdots(R-k)} + \frac{y^{-R}}{kR^k}$$
$$\ll \frac{2^k y^{-R}}{R^k} + \frac{y^{-R}}{kR^k}$$
$$\ll \frac{2^k y^{-R}}{R^k}. \tag{14}$$

With $R = \frac{T}{2}$, we obtain the desired asymptotic when $y \ge 1$.

If $0 < y \le 1$, then we move the line of integration to the far-right namely to the line $\Re s = R_1$ (say with R_1 sufficiently large). Since there are no poles in the rectangular contour formed by the line segments joining the points $c-iT, c+iT, R_1+iT, R_1-iT$ and $c-iT$ in the anti-clockwise order and the integrand is analytic, by Cauchy's theorem for analytic function of a rectangular contour, we obtain the main term to be zero. However, the horizontal lines together contribute an error which is in absolute value

$$\ll \int_c^{R_1} \frac{y^\sigma}{T^{k+1}}\, d\sigma \ll \frac{R_1 y^{R_1}}{T^{k+1}} \ll \frac{R_1}{T^{k+1}}, \tag{15}$$

and the vertical line contributes an error which is in absolute value

$$\ll \int_{-T}^{T} \frac{y^{R_1}}{|(R_1+it)(R_1+1+it)\cdots(R_1+k+it)|}\, dt \ll \frac{Ty^{R_1}}{{R_1}^{k+1}} \ll \frac{T}{{R_1}^{k+1}}. \tag{16}$$

We choose $R_1 = T$. This proves the lemma. □

Lemma 3.3. *The Riemann zeta-function is extended as a meromorphic function in the whole complex plane $\mathbb{C}$ with a simple pole at $s = 1$ and it satisfies a functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ where*

$$\chi(s) = \frac{\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)}{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)}.$$

Also, in any bounded vertical strip, using Stirling's formula, we have

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i\left(t+\frac{\pi}{4}\right)}\left(1+O\left(t^{-1}\right)\right)$$

as $|t| \to \infty$. *Thus, in any bounded vertical strip,*

$$|\chi(s)| \asymp t^{1/2-\sigma}\left(1+O\left(t^{-1}\right)\right)$$

as $|t| \to \infty$.

Proof. See for example p.116 of [8] or p.8-12 of [3]. □

Lemma 3.4. *For any fixed* σ *satisfying* $\frac{1}{2} < \sigma < 1$, *we have*

$$\int_1^T |\zeta(\sigma+it)|^2 dt = \zeta(2\sigma)T + O\left(T^{2-2\sigma}\log T\right).$$

Proof. See for example p.151 of [8]. □

Lemma 3.5. *Let* $U \geq U_0$ *where* U_0 *is sufficiently large. Then, unconditionally there exists a point* $t^* \in [U, U+U^{1/3}]$ *such that the estimate*

$$\max_{1/2\leq\sigma\leq 2} |\zeta(\sigma+it^*)| \ll \exp(c^*(\log\log U)^2)$$

holds where c^* *is an absolute positive constant.*

Proof. This is part of theorem 2 of [5]. See for example Lemma 2 in p.73 of [5]. □

4. Proof of the Main Theorem

We first choose the free large parameter T such that

$$\max_{1/2\leq\sigma\leq 2} |\zeta(\sigma+iT)| \ll \exp\left(c^*(\log\log T)^2\right). \tag{17}$$

The existence of such a T is ensured by Lemma 3.5.

From Lemma 3.2, with $c = 1 + \frac{1}{\log x}$ and writing $F(s) := \zeta(s)\zeta(s+1)g(s)$, we have

$$\begin{aligned} S &:= \frac{1}{k!}\sum_{n\leq x}\frac{n}{\phi(n)}\left(1-\frac{n}{x}\right)^k \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^s}{s(s+1)(s+2)\cdots(s+k)}\,ds \\ &= \int_{c-iT}^{c+iT} F(s)\frac{x^s}{s(s+1)\cdots(s+k)}ds + O\left(\frac{4^k x^c\log x}{T^k}\right) \qquad (18) \end{aligned}$$

Now move the line of integration in the above integral to $\Re s = -1/2 + 2\delta$ (where δ is any fixed positive constant $< \frac{1}{100}$). In the rectangular contour formed by the line segments joining the points $c - iT$, $c + iT$, $-\frac{1}{2} + 2\delta + iT$, $-\frac{1}{2} + 2\delta - iT$ and $c - iT$ in the anticlockwise order, we observe that $s = 1$ is a simple pole and $s = 0$ is a double pole of the integrand, thus we get the main term from the sum of the residues coming from the poles $s = 1$ and $s = 0$, namely $c_1(k)x + c_2(k)\log x + c_3(k)$. We note that

$$\begin{aligned}
&\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s(s+1)....(s+k)}ds \\
&= \int_{-\frac{1}{2}+2\delta+iT}^{c+iT} \cdots + \int_{-\frac{1}{2}+2\delta-iT}^{-\frac{1}{2}+2\delta+iT} \cdots + \int_{c-iT}^{-\frac{1}{2}+2\delta-iT} \cdots + \text{sum of the residues.}
\end{aligned} \tag{19}$$

The left vertical line segment contributes the quantity:

$$\begin{aligned}
Q_1 &= \frac{1}{2\pi}\int_{-T}^{T} F(-\frac{1}{2}+2\delta+it)\frac{x^{-\frac{1}{2}+2\delta+it}}{(-\frac{1}{2}+2\delta+it)(\frac{1}{2}+2\delta+it)\cdots(k-\frac{1}{2}+2\delta+it)}\,dt \\
&= \frac{1}{2\pi}\left(\int_{-1}^{1} + \int\limits_{1<|t|\le T}\right)\frac{x^{-\frac{1}{2}+2\delta+it}\zeta\big(-\frac{1}{2}+2\delta+it\big)\zeta\big(\frac{1}{2}+2\delta+it\big)g\big(-\frac{1}{2}+2\delta+it\big)}{\big(-\frac{1}{2}+2\delta+it\big)\big(\frac{1}{2}+2\delta+it\big)\cdots\big(k-\frac{1}{2}+2\delta+it\big)}\,dt \\
&\ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta}\int\limits_{1<|t|\le T} t^{\frac{1}{2}-(-\frac{1}{2}+2\delta)}|\zeta(3/2+2\delta+it)||\zeta(1/2+2\delta+it)|\frac{dt}{t^k} \\
&\ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta}\int\limits_{1<|t|\le T} \frac{1}{\sqrt{t}}\frac{|\zeta(1/2+2\delta+it)|}{t^{k-\frac{1}{2}}}dt \\
&\ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta}\left(\int_1^T \frac{1}{t}dt\right)^{\frac{1}{2}}\left(\int_1^T \frac{|\zeta(1/2+2\delta+it)|^2}{t^{2k-1}}dt\right)^{\frac{1}{2}} \\
&\ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta}(\log T)
\end{aligned} \tag{20}$$

since, by Lemma 3.4, letting

$$v(T) := \int_1^T |\zeta(1/2+2\delta+it)|^2\ dt\ (\ll_\delta T) \tag{21}$$

we have using integration by parts

$$
\begin{aligned}
\int_1^T \frac{|\zeta(1/2+2\delta+it)|^2}{t^{2k-1}}dt &= \int_1^T \frac{1}{t^{2k-1}}\ dv(T) \\
&= \frac{v(t)}{t^{2k-1}}|_1^T + (2k-1)\int_1^T \frac{v(t)}{t^{2k}}\ dt \\
&\ll_\delta \frac{1}{T^{2k-2}} + 1 + \max(1, \log T) \\
&\ll \log T \qquad (22)
\end{aligned}
$$

(by splitting the cases $k = 1$ and $k \geq 2$ separately) where the implied constant in (22) is independent of k and note that δ is any small fixed positive constant.

Now we will estimate the contributions coming from the upper horizontal line (lower horizontal line is similar). Let

$$
\begin{aligned}
Q_2 &:= \max_{-1/2\leq\sigma\leq1/2} |\zeta(\sigma+iT)| \\
&\ll \max_{-1/2\leq\sigma\leq1/2} |T|^{1/2-\sigma}|\zeta(1-\sigma-iT)| \\
&\ll T \max_{1/2\leq1-\sigma\leq3/2} |\zeta(1-\sigma-iT)| \\
&\ll T\exp(c^*(\log\log T)^2) \qquad (23)
\end{aligned}
$$

Therefore with our choice of T, from (17), we have

$$
\max_{-1/2\leq\sigma\leq2} |\zeta(\sigma+iT)| \ll T\exp(c^*(\log\log T)^2). \qquad (24)
$$

Thus the horizontal lines in total contribute a quantity which is in absolute value

$$
\begin{aligned}
&\ll \int_{-1/2+2\delta}^{c} \left|\zeta(\sigma+iT)\zeta(\sigma+1+iT)\frac{g(\sigma+iT)x^{\sigma+iT}}{(\sigma+iT)(\sigma+1+iT)\cdots(\sigma+k+iT)}\right| d\sigma \\
&\ll_\delta \frac{(x\exp(2c^*(\log\log T)^2)}{T^k} \qquad (25)
\end{aligned}
$$

Collecting all the estimates, we get

$$
E_k(x) \ll \frac{4^k x^c \log x}{T^k} + x^{-1/2+2\delta}(\log T) + \frac{(x\exp(2c^*(\log\log T)^2)}{T^k}. \qquad (26)
$$

Note that

$$
\frac{(x\exp(2c^*(\log\log T)^2)}{T^k} \ll \frac{x}{T^{k-\delta}}
$$

for any fixed small positive constant δ. Now we choose $T := 4x^{\frac{3}{2k}}$ so that from (26), we obtain

$$E_k(x) \ll \frac{1}{x^{\frac{1}{2}-3\delta}} + \frac{1}{4^{k-\delta}} \cdot \frac{x}{x^{\frac{3}{2}-\frac{3\delta}{2k}}}$$
$$\ll \frac{1}{x^{\frac{1}{2}-3\delta}} + \frac{x}{x^{\frac{3}{2}-2\delta}}$$
$$\ll \frac{1}{x^{\frac{1}{2}-3\delta}}. \tag{27}$$

Note that the implied constant in (27) is independent of k. This proves the theorem provided $c_1(k), c_2(k)$ and $c_3(k)$ are precisely determined. This is done in the following section.

5. Evaluation of the constants $c_1(k), c_2(k)$ and $c_3(k)$

We recall that

$$g(s) := \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1-1/p)}\right)$$

is absolutely and uniformly convergent in any compact set contained in the half-plane $\Re s \geq -\frac{1}{2}+2\delta$ for any fixed small positive δ and thus, we observe that, in any compact region in the half plane $\sigma \geq -\frac{1}{2}+2\delta$ (taking the logarithmic derivative of $g(s)$) we have

$$g'(s) = g(s)\left(\sum_p \frac{1}{1+\frac{1/p^{s+2}(1-1/p^s)}{1-1/p}} \frac{1}{1-1/p}\left(-\frac{\log p}{p^{s+2}}(1-1/p^s) + \frac{1}{p^{s+2}}\frac{\log p}{p^s}\right)\right).$$

We note that $g(0) = 1$ and

$$g(1) = \prod_p \left(1 + \frac{1/p^3 - 1/p^4}{1-1/p}\right) = \prod_p \left(1 + \frac{1}{p^3}\right) = \prod_p \left(\frac{(1-\frac{1}{p^6}}{(1-\frac{1}{p^3}}\right) = \frac{\zeta(3)}{\zeta(6)}$$

and $g'(0) = \sum_p \frac{\log p}{p(p-1)}$.

The Bernoulli numbers B_n are defined to be the coefficients of the exponential generating function, precisely by the relation

$$\frac{t}{e^t - 1} := \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

We note that $\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!}$ for any integer $n \geq 1$, $B_2 = \frac{1}{6}$ and $B_6 = \frac{1}{42}$ so that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(6) = \frac{(2\pi)^6}{2\cdot 6!\cdot 42}$. Since $s = 1$ is a simple pole inside the rectangular contour, we obtain

$$\text{Res}_{s=1}\left(F(s)\frac{x^s}{s(s+1)(s+2)\cdots(s+k)}\right)=\frac{\zeta(2)\zeta(3)}{k!\zeta(6)}\ x=\frac{315\zeta(3)}{2\pi^4\ k!}x. \tag{28}$$

Hence,

$$c_1(k)=\frac{315\zeta(3)}{2\pi^4\ k!}. \tag{29}$$

We observe that in the neighbourhood of $s=0$, we have

$$\zeta(s+1)-\frac{1}{s}=\gamma_0+\gamma_1 s+\gamma_2 s^2+\cdots$$

and hence

$$\lim_{s\to 0}\left(\zeta(s+1)-\frac{1}{s}\right)=\gamma_0.$$

If we write

$$h(s):=\prod_{j=1}^{k}\frac{1}{(s+j)},$$

then

$$h'(s)=h(s)\left(-\sum_{j=1}^{k}\frac{1}{(s+j)}\right)$$

and hence

$$h'(0)=h(0)\left(-\sum_{j=1}^{k}\frac{1}{j}\right)=\frac{1}{k!}\left(-\sum_{j=1}^{k}\frac{1}{j}\right).$$

Now $s=0$ is a double pole in the rectangular contour and hence the residue is

$$\begin{aligned}
Q_3 &:= \text{Res}_{s=0}\left[\frac{F(s)x^s}{s(s+1)(s+2)\cdots(s+k)}\right]\\
&=\lim_{s\to 0}\ \frac{d}{ds}\left(s^2\frac{F(s)x^s}{s(s+1)(s+2)\cdots(s+k)}\right)\\
&=\lim_{s\to 0}\ \frac{d}{ds}\left(s\zeta(s)\zeta(s+1)g(s)\frac{x^s}{(s+1)(s+2)\cdots(s+k)}\right)\\
&=\lim_{s\to 0}\ \frac{d}{ds}\left[s\zeta(s)\left(\zeta(s+1)-\frac{1}{s}\right)g(s)\frac{x^s}{(s+1)(s+2)\cdots(s+k)}\right]\\
&+\lim_{s\to 0}\ \frac{d}{ds}\left[\zeta(s)g(s)\frac{x^s}{(s+1)(s+2)\cdots(s+k)}\right]\\
&=\frac{1}{k!}\left\{\zeta(0)\gamma_0 g(0)+\zeta'(0)g(0)+\zeta(0)g'(0)+\zeta(0)g(0)\log x\right\}+\zeta(0)g(0)h'(0)\\
&=\frac{1}{k!}\left\{-\frac{\gamma_0}{2}-\frac{1}{2}\log(2\pi)-\frac{1}{2}\sum_p\frac{\log p}{p(p-1)}-\frac{1}{2}\log x+\frac{1}{2}\left(\sum_{j=1}^{k}\frac{1}{j}\right)\right\}.
\end{aligned}$$

Therefore,

$$c_2(k) = -\frac{1}{2(k!)} \tag{30}$$

and

$$c_3(k) = \frac{1}{k!}\left\{\frac{1}{2}\left(\sum_{j=1}^{k}\frac{1}{j}\right) - \frac{\gamma_0}{2} - \frac{1}{2}\log(2\pi) - \frac{1}{2}\sum_{p}\frac{\log p}{p(p-1)}\right\}. \tag{31}$$

This completes the proof of the main theorem.

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