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An Exercise in Benford's Law

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Abstract. This article is a generalisation of the results of Thorner etal concerning Benford's law for the Fourier coefficients of a primitive i.e. a normalised newform f of weight k on $\Gamma_0(N)$ without complex multiplication to the Fourier coefficients of $\mathrm{sym}^2 f$, where $\mathrm{sym}^2 f$ is the symmetric square lift of f.

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1. Introduction

In 1881, astronomer Simon Newcomb [New1881] made the observation that pages of logarithmic tables with leading digit 1 were more worn out than the other pages, i.e., these pages were referenced more often, contrary to the naive expectation that all of the logarithms would be referenced uniformly. In 1938, Benford made the same observation for a variety of sequences.

This bias, now known as Benford's law, is given as follows. Let $I \subset \mathbb{N}$ be an infinite subset. For every integer base $b \geq 3$ and an initial string of digits S in base b. For a given arithmetical function $g : \mathbb{N} \to \mathbb{R}$, define

$$A_g(b, S) = \{i \in \mathbb{N} : \text{the first digits of } g(i) \text{ in base } b \text{ are given by } S\}. \tag{1.1}$$

We define the *arithmetic density* of $A_g(b, S)$ within I by

$$\delta_I(A_g(b, S)) = \lim_{x \to \infty} \frac{\#\{i \leq x : i \in I \cap A_g(b, S)\}}{\#\{i \leq x : i \in I\}}. \tag{1.2}$$

We say that the sequence $\{g(i)\}_{i \in I}$ satisfies Benford's law or is Benford if

$$\delta_I(A_g(b, S)) = \log_b(1 + S^{-1}). \tag{1.3}$$

It is easy to show [ ] that $\{g(i)\}_{i \in I}$ is Benford if and only if the set $\{\log_b g(i) : i \in I\}$ is equidistributed modulo 1 for each base b. Here, of course, we set $\log_b g(i) = 0$ if $g(i) \leq 0$. For some general survey's on Benford's law, we recommend the articles [JTY16], [Dia97].

It is well known that the set of positive integers $\mathbb{N}$ is not Benford. By this we mean, using the above notation that $g(n) = n$ and $I = \aleph$ However, if we change our notion of density, then the initial digits still satisfy Benford's law with the new notion of density. We define the *logarithmic density* of $A_g(b, S)$ in I by,

$$\tilde{\delta}_I(A_g(b, S)) = \lim_{x \to \infty} \left(\frac{\displaystyle\sum_{\substack{i \leq x \\ i \in I \cap A_g(b,S)}} i^{-1}}{\displaystyle\sum_{\substack{i \leq x \\ i \in I}} i^{-1}} \right).$$

With this modified notion of density, $\tilde{\delta}_{\mathbb{N}}(A_g(b, S))$ exists and equals $\log_b(1 + S^{-1})$ for any base $b \geq 3$ and any initial string S in base b. In the light of this fact, we say that a sequence $\{g(i)\}_{i \in I}$ is logarithmically Benford if $\tilde{\delta}_I(A_g(b, S)) = \log_b(1 + S^{-1})$ for any base b and initial string S in base b.

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We note that if a set has an arithmetic density, then it also has a logarithmic density, and the two densities are equal. In their article, [JTY16], Thorner et al. investigate the case when $i \in \mathscr{P} =$ the set of rational primes and $g(p) = a_f(p)$ for $f \in S_k^{prim}(\Gamma_0(N))$, i.e., $a_f(p)$ are the Fourier coefficients of $f(z) = \sum\limits_{n=1}^{\infty} a_f(n)e^{2\pi inz}$, $z \in \mathcal{H}$, f being primitive, i.e., normalized newform of weight k on $\Gamma_0(N)$, $k \geq 2$, even. We assume that f is a non CM cusp form. They prove two results, [JTY16, Thm. 1, Thm. 2]. Theorem 1 says that $\delta_{\mathscr{P}}(A_{a_f(p)}(b, S))$ does not exist for any base $b \geq 3$. Theorem 2, says that $\tilde{\delta}_{\mathscr{P}}(A_{a_f(p)}(b, S)) = \log_b(1 + S^{-1})$ for any base $b \geq 3$, i.e., the sequence $\{a_f(p) : p \in \mathcal{P}\}$ is logarithmically Benford.

The aim of this article is to prove the exact analogues of theorems 1 and 2 of Thorner et. al. with $a_f(p)$ replaced by $a_{sym^2 f}(p)$ where $sym^2 f$ is the symmetric square lift of f.

The title of this article has been deliberately chosen. We basically mimic the proof by Thorner et.al. in our content. Thus, we make no claim to originality. We do use the Hecke relations to relate $a_{sym^2 f}(p) = a_f(p^2)$ with $a_f^2(p)$, this in turn reduces the calculation of the Sato-Tate measure for $sym^2 f$, $\mu_{sym^2 f, ST}$ to that of $\mu_{ST,f}$.

2. Main results

Theorem 2.1. *Let* $f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi inz} \in S_k^{prim}(\Gamma_0(N))$ *of even weight* $k \geq 2$, *be a primitive, i.e., normalized newform on* $\Gamma_0(N)$ *without complex multiplication. The arithmetic density* $\delta_{\mathscr{P}}(A_{a_{sym^2 f}}(b, 1))$ *does not exist for any base* $b \geq 3$. *Thus, the sequence* $\{a_{sym^2 f}(p) : p \in \mathscr{P}\}$ *is not Benford.*

Theorem 2.2. *Let* f *be as in Theorem 1. We also retain all the notation of theorem 1. Then we have* $\{a_{sym^2 f}(p)\}_{p \in \mathscr{P}}\}$ *is logarithmically Benford, i.e.,* $\tilde{\delta}_p(A_{a_{sym^2 f}}(S, b)) = \log_b(1 + S^{-1})$.

Corollary 2.3. *Let* f *be as in Theorem 1. Let* $I \subset \left[-\frac{1}{3}, 1\right]$ *be an interval. As* $x \to \infty$ *we have*

$$\sum_{\substack{p \leq x \\ \cos\phi_p \in I}} \mu_{sym^2 f, ST}(I) \sim \sum_{p \leq x} p^{-1}.$$

3. Some preliminary calculations

We write as usual $\lambda_{sym^2 f}(p) = \lambda_f(p^2) = \frac{a_f(p^2)}{p^{k-1}}$. By Deligne, we have $|\lambda_{sym^2 f}(p)| = |\lambda_f(p^2)| \leq d(p^2) = 3$ where $d(\cdot)$ is the divisor function. Hence, we can write $\lambda_{sym^2 f}(p) = 3\cos\phi_p$ for all $p \nmid N$.

Now the Hecke relation is

$$\lambda_f(p^2) = \lambda_{sym^2 f}(p) = \lambda_f^2(p) - 1. \tag{3.4}$$

Writing $\lambda_f(p) = 2\cos\theta_p$ we get,

$$\lambda_{sym^2 f}(p) = \lambda_f(p^2) = 3\cos\phi_p = 4\cos^2\theta_p - 1 \quad \text{or} \quad |\cos\theta_p| = \frac{\sqrt{1 + 3\cos\phi_p}}{2}. \tag{3.5}$$

Note that (3.4) implies $\lambda_{sym^2 f}(p) \in [-1, 3]$, i.e., $\cos\phi_p \in [-\frac{1}{3}, 1]$, i.e., $\theta_p \in [\cos^{-1} 1, \cos^{-\frac{1}{3}}] = [0, (\frac{\pi}{2} + 0.108173\pi)\, rad]$. We want for all $b \geq 3$,

$$\left| \lim_{c \to \infty} \mu_{sym^2 f, ST}\left[\frac{1}{2(1 - b^{-c})}, 1\right] - \mu_{sym^2 f, ST}\left[\frac{b^{-1}}{1 - 2b^{-c}}, \frac{2b^{-1}}{1 - b^{-c}}\right] \right| > \alpha$$

where $\alpha > 0$ will be given explicitly.

$$\mu_{sym^2 f,ST}\left[\frac{1}{2},1\right]-\mu_{sym^2 f,ST}\left[\frac{1}{b},\frac{2}{b}\right]=\alpha. \tag{3.6}$$

For $b=3$, we have $\mu_{sym^2 f,ST}\left[\frac{1}{3},\frac{2}{3}\right]=\frac{2}{\pi}\times 0.1944>0.012=\frac{3}{250}$.

Now for $b\geq 4$, $2b^{-1}\leq\frac{1}{2}$. So, the intersection of these two sets is $\{\frac{1}{2}\}$ or Φ and $\mu_{sym^2 f,ST}(\{1/2\})=0$. Also $|\cos\phi_p|\in\left[-\frac{1}{2},-1\right]$, i.e., $\cos\phi_p\in\left[-1,-\frac{1}{2}\right]$ or $\left[\frac{1}{2},1\right]$ but $\cos\phi_p\in\left[\frac{1}{2},1\right]$ or $\left[-1,-\frac{1}{2}\right]\cap\left[-\frac{1}{3},0\right]=\phi$. Therefore, we only need to consider $\cos\phi_p\in\left[\frac{1}{2},1\right]$ which in turn implies

$$|\cos\theta_p|=\frac{\sqrt{1+3\cos\phi_p}}{2}\in\left[\frac{\sqrt{1+3/2}}{2},\frac{\sqrt{1+3}}{2}\right]=[0.79055,1].$$

Therefore, the Sato-Tate measure, $\mu_{sym^2 f,ST}\left[\frac{1}{2},1\right]$, is

$$\begin{aligned}\frac{4}{\pi}\int_0^{\cos^{-1}0.79055}\sin^2\theta d\theta=&\frac{4}{\pi}\left[\frac{1}{2}\cos^{-1}0.79055-\frac{1}{2}\sin(\cos^{-1}0.79055)\times 0.79055\right]\\=&\frac{2}{\pi}\times 0.174959.\end{aligned}$$

Now $|\lambda_{sym^2 f}(p)|=3|\cos\phi_p|$ or $|\cos\theta_p|=\frac{\sqrt{1+3\cos\phi_p}}{2}$. So if $|\cos\theta_p|\in[\tilde{\alpha},\tilde{\beta}]$ where $-\frac{1}{3}\leq\alpha<\beta\leq 1$, then

$$|\cos\theta_p|^2=\frac{1+3\cos\phi_p}{4}\in\frac{1+3[\tilde{\alpha},\tilde{\beta}]}{4}$$

or

$$|\cos\theta_p|=\frac{1+3\cos\phi_p}{4}\in\frac{\sqrt{1+3[\tilde{\alpha},\tilde{\beta}]}}{2}.$$

Here note that $3\tilde{\alpha}\geq -1$ implies $1+3\tilde{\alpha}\geq 0$ so the quantity under the square root sign is nonnegative.

4. Proof of Theorem 2.1

The proof runs in the same way as in Thorner et.al. namely constructing large intervals on which the proportion of primes p for which $|a_{sym^2 f(p)}|$ has leading digit 1 in a given base $b\geq 3$ differs from the Benford expectation. This will show that $\{a_{sym^2 f(p)}\}_{p\in\mathscr{P}}$ is not Benford in an base. To do this we first state a lemma on $\mu_{sym^2 f,ST}$ of the same set of intervals used by Thorner et.al. in their Lemma 1.

Lemma 4.1. *Fix $b\geq 3$ and let $c\in\mathbb{N}$ be sufficiently large. For $d=1,2$, set*

$$I_{d,1}(c)=\bigcup_{j\in\mathbb{Z}}\left[\frac{b^{-j}}{b^c-2},\frac{2b^{-j}}{db^c-1}\right]\cap[0,1].$$

Then,

$$\left|\mu_{sym^2 f,ST}(I_{2,1}(c)-\mu_{sym^2 f,ST}(I_{1,1}(c)\right|>\alpha,$$

where $\alpha>0$ will be given explicitly.

Proof. We begin by making some remarks about the proof, in particular its departure from the proof of Lemma 1 of [2]. in [2] , For brevity we denote by $\nu,\mu_{sym^2 f,ST}$

For $c\geq 3$, we have

$$\frac{1}{b}<\frac{b^{c-1}}{b^c-2}<\frac{2b^{c-1}}{b^c-1}<1\quad\text{and}\quad\frac{1}{b}<\frac{b^c}{2b^c-2}<1<\frac{2b^c}{2b^c-1}$$

Thus,

$$I_{1,1}(c)=\bigcup_{j\geq-(c-1)}\left[\frac{b^{-j}}{b^c-2},\frac{2b^{-j}}{b^c-1}\right]=\bigcup_{\ell\in\mathbb{N}}\left[\frac{b^{-\ell}}{1-2b^{-c}},\frac{2b^{-\ell}}{1-b^{-c}}\right]$$

and

$$\begin{aligned}I_{2,1}(c)&=\left[\frac{b^c}{2b^c-2},1\right]\bigcup_{j>-c}\left[\frac{b^{-j}}{2b^c-2},\frac{2b^{-j}}{2b^c-1}\right]\\&=\left[\frac{1}{2-2b^{-c}},1\right]\bigcup_{m>0}\left[\left[\frac{b^{-m}}{2-2b^{-c}},\frac{2b^{-m}}{2-b^{-c}}\right]\right].\end{aligned}$$

Now for all $b\geq 3$, we have

$$\begin{aligned}\lim_{c\to\infty}&\left|\left(\nu\left(\left[\frac{1}{2-b^{-c}},1\right]\right)-\nu\left(\left[\frac{b^{-1}}{1-2b^{-c}},\frac{2b^{-1}}{1-b^{-c}}\right]\right)\right)\right|\\&=\left|\nu\left(\left[\frac{1}{2},1\right]\right)-\nu\left(\left[b^{-1},2b^{-1}\right]\right)\right|=\tilde{\alpha}>0.\end{aligned}\tag{4.7}$$

Thus for all c sufficiently large, we have

$$\left|\nu\left(\left[\frac{1}{2-2b^{-c}},1\right]\right)-\nu\left(\left[\frac{b^{-1}}{1-2b^{-c}},\frac{2b^{-1}}{1-b^{-c}}\right]\right)\right|>\alpha$$

where $0<\alpha<\tilde{\alpha}$ is chosen and fixed once for all. Note that we can choose α as close to $\tilde{\alpha}$ as we want. For later use we write $\beta=\left[\frac{1}{\alpha}\right]+1$.

Now for $b=3$, we have $\nu\left(\left[\frac{1}{3},\frac{2}{3}\right]\right)=\frac{2}{\pi}\times 0.2894$. We also have $\nu[\frac{1}{2},1]=(\frac{2}{\pi}\times 0.2786)$. Therefore, the quantity within the absolute value sign in (4.7) $=(\frac{2}{\pi}\times -0.0108=-0.0069$.
Furthermore, the sum of the series over m which is given below for $b=3$ is $\geq\frac{2}{\pi}\times .2780$. Hence the sum of these two quantities is $\geq\frac{2}{\pi}\times .2672=.1701$
Now for $b=4$ we have $\mu_{sym^2f,ST}\left(\left[\frac{1}{4},\frac{1}{2}\right]\right)=\frac{2}{\pi}\times .0220)$. Hence the quantity inside the absolute value sign in

$$(4.7)=(\frac{2}{\pi}\times +.2566)=+.1633$$

For $b=5$, we have $\mu_{sym^2f,ST}\left(\left[\frac{1}{5},\frac{2}{5}\right]\right)=(\frac{2}{\pi}\times .8835)=0.5624$ and hence again the quantity within the absolute value sign in

$$(4.7)=\frac{2}{\pi}\times -0.605=-.3851$$

Now from $b=11$ onwards we have that the quantity inside the absolute value sign is positive. In fact it's value for $b=11$ is $\frac{2}{\pi}\times .2894$ which is

$$>\frac{2}{\pi}\times .2786$$

and this value is .0069. Finally,

$$\sum_{m=1}^{\infty}\mu_{sym^2f,ST}\left(\left[\frac{b^m}{2},b^{-m}\right]\right)-\mu_{sym^2f,ST}\left(\left[\frac{b^{-1-m}}{1},2b^{-1-m}\right]\right)>0,$$

since $b\geq 3$. We therefore obtain (4.7) $\geq .0069$, for all b.

We now tabulate the nth partial sum of the series above up to $n = 10, \quad 4 \leq b \leq 20$.

[4, .1391]
[5, .1115]
[6, .0944]
[7, .0821]
[8, .0727]
[9, .0653]
[10, .0593]
[11, .0543]
[12, ..0501]
[13, .0465]
[14, .0439]
[15, .0407]
[16, .0383]
[17, .0361]
[18, .0342]
[19, .0325]
[20, .0309]

Therefore we can choose $\alpha = .0065$ and hence $\beta = 16$. This finishes the proof of Lemma 1.

Remark 1. In contrast with the statement of Lemma 1 in [2] , it is safer for us to have the absolute value sign for the same quantity as in [2] since $\mu_{sym^2 f,ST}\left(\left[\frac{1}{2},1\right]\right) - \mu_{sym^2 f,ST}\left(\left[b^{-1},2b^{-1}\right]\right)$ changes sign when b varies from 3 to 11. It should also be noted that because of this, in our case we need to add the (positive) contribution of the series occurring earlier. We also need to observe that for fixed b, the nth partial sum of the series is a *decreasing* function of n.

We now resume the proof of Theorem 1.

Consider the intervals $I_\ell = \left[\frac{b^{c-\ell}}{b^c-2}, \frac{2b^{c-\ell}}{2b^c-1}\right]$. The first interval is

$$I_1 = \left[\frac{b^{c-1}}{b^c-2}, \frac{2b^{c-1}}{2b^c-1}\right]$$

and we know that

$$\frac{1}{b} < \frac{b^{c-1}}{b^c-2} < \frac{2b^{c-1}}{b^c-1} \quad \text{and} \quad \frac{1}{b} < \frac{b^c}{2(1-b^{-c})} < 1 < \frac{2b^c}{2b^c-1}.$$

Fix $0 < \epsilon < \frac{\alpha}{2}$, where $\alpha = \mu_{sym^2 f,ST}[\frac{1}{2},1] - \mu_{sym^2 f,ST}[b^{-1},2b^{-1}] > 0$ and let $c \in \mathbb{N}$ be a sufficiently large positive integer so that Lemma 4.1 holds and $\mu_{sym^2 f,ST}\left([0,1] - \bigcup_{S=1}^{b-1} I_{d,s}(c)\right) < \frac{\epsilon}{4}$ for $d = 1, 2$. We consider the primes p

$$(b^c-2)b^n \leq 3p^{k-1} \leq (b^c-1)b^n. \tag{4.8}$$

We note that if p is bounded as in (4.8) and $|\cos\phi_p| \in I_{1,S}$. Let $\beta = b^{\frac{1}{k-1}}$, $\alpha_1 = \left(\frac{b^c-2}{2}\right)^{\frac{1}{k-1}}$ and $r_1 = \left(\frac{b^c-1}{2}\right)^{\frac{1}{k-1}}$. Then

$$|a_{sym^2 f}(p)| \in [Sb^{n-j}, (S+1)b^{n-j}]$$

for some $j \in \mathbb{Z}$, i.e., its first digits are given by S. By letting c be sufficiently large and setting $S = 1$, the Sato-Tate conjecture implies that

$$\left| \lim_{n\to\infty} \frac{\#\left\{\alpha_1\beta^n \le p < r_1\beta^n : p \in A_{a_f}(b,1)\right\}}{\#\left\{\alpha_1\beta^n \le p < r_1\beta^n\right\}} - \mu_{sym^2 f,ST}(I_{1,1}(c)) \right| < \frac{\epsilon}{2}. \tag{4.9}$$

Similarly, by letting $\alpha_2 = \left(\frac{2b^c-2}{2}\right)^{\frac{1}{k-1}}$ and $r_2 = \left(\frac{2b^c-1}{2}\right)^{\frac{1}{k-1}}$,

$$\left| \lim_{n\to\infty} \frac{\#\left\{\alpha_2\beta^n \le p < r_2\beta^n : p \in A_{a_f}(b,1)\right\}}{\#\left\{\alpha_2\beta^n \le p < r_2\beta^n\right\}} - \mu_{sym^2 f,ST}(I_{2,1}(c)) \right| < \frac{\epsilon}{2}. \tag{4.10}$$

Now, suppose on the contrary that $\delta_{\mathscr{P}}(A_{a_{sym^2 f}}(b,1))$ exsits. It follows from (4.9) and (4.10) that,

$$\left|\mu_{sym^2 f,ST}(I_{1,1}(c)) - \mu_{sym^2 f,ST}(I_{2,1}(c))\right| < \epsilon < \frac{\alpha}{2}$$

which contradicts Lemma 4.1. The theorem now follows for bases $b \ge 3$.

5. Proof of Theorem 2.2

Let $b \ge 3$ be a given base and let S be an initial string of digits in base b. By the definition of a logarithmically Benford sequence and the estimate,

$$\sum_{p\le x} p^{-1} \sim \log\log x \quad \text{as} \quad x \to \infty,$$

a proof of Theorem 2.2 will follow from proving that

$$\sum_{\substack{p\le x \\ p\in sym^2 f(b,S)}} p^{-1} \sim \log_b(1+S^{-1}) \log\log x \quad \text{as} \quad x \to \infty.$$

Lemma 5.1. *Let $f \in S_k(\Gamma_0^{prim}(N))$ of weight $k \ge 2$, without CM. Let $b \ge 3$ be a given base. Let S be an initial string of digits in base b and let $\ell > max\{S, \beta\}$ be an integer. As $x \to \infty$ we have*

$$\begin{aligned}(1+o(1))(\log(1+S^{-1})) - \log(1+\ell^{-1}) \log\log x \;&\le \sum_{\substack{p\le x \\ p\in sym^2 f(b,S) \\ |\cos\phi_p| > \ell^{-1}}} p^{-1} \\ &\le (1+o(1))(\log_b(1+S^{-1})) + \log(1+\ell^{-1}) \log\log x + 2\log x.\end{aligned}$$

Proof. We prove the upper bound in the above. The lower bound is proven similarly. Write

$a_{sym^2 f}(p) = 3$. We first observe that

$$\sum_{\substack{p\le x \\ p\in sym^2 f(b,S) \\ |\cos\phi_p|>\ell^{-1}}} p^{-1} = \sum_{t=-\infty}^{\infty}\sum_{i=\ell}^{\ell^2-1} \sum_{\substack{p\le x \\ Sb^t\le |a_{sym^2 f}(p)|<(S+1)b^t \\ \frac{i}{\ell^2}<|\cos\phi_p|\le\frac{i+1}{\ell^2}}} p^{-1}$$

$$\le \sum_{t=-\infty}^{\infty}\sum_{i=\ell}^{\ell^2-1} \sum_{\substack{p\le x \\ \left(\frac{S\ell^2}{3(i+1)}b^t\right)^{\frac{1}{k-1}}\le p\le\left(\frac{(S+1)\ell^2}{3i}b^t\right)^{\frac{1}{k-1}} \\ \frac{i}{\ell^2}<|\cos\phi_p|\le\frac{i+1}{\ell^2}}} p^{-1}.$$

To bound the contribution when $t<0$, all of the primes in the sum are at most $\left(\frac{(S+1)\ell^2}{3\ell}b^{-1}\right)^{\frac{1}{k-1}}$. Therefore

$$\sum_{\substack{p\le x \\ p\in sym^2 f(b,S) \\ |\cos\phi_p|>\ell^{-1}}} p^{-1} \le \sum_{p\le\left(\frac{(S+1)\ell}{3b}\right)^{\frac{1}{k-1}}} p^{-1}.$$

Now $(S+1)\le\ell$. The above sum is $\le \sum_{p\le\ell} p^{-1} \le 2\log\log\ell$.

Using corollary 1, we get that $\sum_{\left(\frac{S\ell^2}{3(i+1)}b^t\right)^{\frac{1}{k-1}}\le p\le\left(\frac{(S+1)\ell^2}{3i}b^t\right)^{\frac{1}{k-1}}} p^{-1}$

$$= (1+o(1))\mu_{sym^2 f,ST}\left(\left[\frac{i}{\ell^2},\frac{i+1}{\ell^2}\right]\right) \sum_{\left(\frac{S\ell^2}{3(i+1)}b^t\right)^{\frac{1}{k-1}}\le p\le\left(\frac{(S+1)\ell^2}{3i}b^t\right)^{\frac{1}{k-1}}} p^{-1}$$

$$= (1+o(1))\mu_{sym^2 f,ST}\left(\left[\frac{i}{\ell^2},\frac{i+1}{\ell^2}\right]\right) \log\left(\frac{\log\left(\frac{(S+1)\ell^2}{3_i}b^t\right)}{\log\left(\frac{S\ell^2}{3(i+1)}b^t\right)}\right).$$

Let us write the last term as $\left(\frac{\log x}{\log y}\right) = \left(\frac{\log_b x}{\log_b y}\right)$. Therefore we have,

$$\log_b\left(\frac{(S+1)\ell^2}{3i}b^t\right) = \log_b\left(\frac{S+1}{3i}\right) + 2\log_b\ell + t = \log_b\left(\frac{\ell^2}{3i}\right) + 2\log_b(S+1) + t$$

and similarly,

$$\log_b\left(\frac{S\ell^2}{3(i+1)}b^t\right) = \log_b\left(\frac{S}{3(i+1)}\right) + 2\log_b\ell + t = \log_b\left(\frac{\ell^2}{3(i+1)}\right) + 2\log_b S + t.$$

Finally we have $\sum_{\left(\frac{S\ell^2}{3(i+1)}b^t\right)^{\frac{1}{k-1}}\le p\le\left(\frac{(S+1)\ell^2}{3i}b^t\right)^{\frac{1}{k-1}}} p^{-1}$

$$= (1+o(1))\mu_{sym^2 f,ST}\left(\left[\frac{i}{\ell^2},\frac{i+1}{\ell^2}\right]\right) \frac{\log_b\left(\frac{\ell^2}{3i}\right)+2\log_b(S+1)+t}{\log_b\left(\frac{\ell^2}{3(i+1)}\right)+2\log_b S+t}.$$

Let $\beta_0 = \log_b\left(\frac{\ell^2}{3(i+1)}\right) + 2\log_b S$ and $\beta_1 = \log_b\left(\frac{\ell^2}{3i}\right) + 2\log_b(S+1)$. Then

$$\beta_1 > \beta_0.$$

We have as $N \to \infty$, using $\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{\prod_{i=0}^{n}(x+i)}$, $x > 0$,

$$\sum_{t=0}^{N} \sum_{\left(\frac{S\ell^2}{3(i+1)}b^t\right)^{\frac{1}{k-1}} \le p \le \left(\frac{(S+1)\ell^2}{3i}b^t\right)^{\frac{1}{k-1}}} p^{-1} \sim \log\left(\prod_{t=0}^{N} \frac{\beta_1 + t}{\beta_0 + t}\right).$$

Switching the order of summation, we get

$$\begin{aligned}
&\sum_{t=0}^{\infty} \sum_{i=\ell}^{\ell^2-1} \sum_{\substack{p\le x \\ Sb^t \le |a_{sym^2 f}(p)| < (S+1)b^t \\ \frac{i}{\ell^2} < |\cos\phi_p| \le \frac{i+1}{\ell^2}}} p^{-1}, \text{ where } \left(\frac{(S+1)\ell^2}{3i}b^t\right)^{\frac{1}{k-1}} \le x \\
&\le \sum_{i=\ell}^{\ell^2-1} \mu_{sym^2 f,ST}\left(\left[\frac{i}{\ell^2}, \frac{i+1}{\ell^2}\right]\right)(1+o(1)) \sum_{t=0}^{\infty} p^{-1} \\
&\le \sum_{i=\ell}^{\ell^2-1} \mu_{sym^2 f,ST}\left(\left[\frac{i}{\ell^2}, \frac{i+1}{\ell^2}\right]\right)(1+o(1)) \log\left(\prod_{0\le t\le \log_b\left(\frac{3ix^{k-1}}{(S+1)\ell^2}\right)} \frac{\beta_1+t}{\beta_0+t}\right) \\
&= \sum_{i=\ell}^{\ell^2-1} \mu_{sym^2 f,ST}\left(\left[\frac{i}{\ell^2}, \frac{i+1}{\ell^2}\right]\right)(1+o(1)) \left(\sum_{t=0}^{\log_b\left(\frac{3ix^{k-1}}{(S+1)\ell^2}\right)} \log\left(\frac{\beta_1+t}{\beta_0+t}\right) + 2\log\log\ell\right).
\end{aligned}$$

Setting $\beta_0 = \log_b\left(\frac{\ell^2}{3(i+1)}\right) + \log_b S$ and $\beta_1 = \log_b\left(\frac{\ell^2}{3i}\right)$.

Now using Euler's formula for the Gamma function, which has been stated earlier, we find that the contribution from $t \le 0$ is at most

$$\sum_{i=\ell}^{\ell^2-1} \mu_{sym^2 f,ST}(1+o(1))\left(\frac{\log\Gamma(\beta_0)}{\log\Gamma(\beta_1)} + (\beta_1-\beta_0)\log\log x\right)$$

which is further

$$\le (1+o(1))\left(\log_b(1+\ell^{-1}) + \log_b(1+S^{-1})\right)\log\log x.$$

This proves the desired upper bound. Using the inequality

$$\sum_{\substack{p\le x \\ Sb^t \le 3p^{k-1}|\cos\phi_p| \le (S+1)b^t \\ \frac{i}{\ell^2} \le |\cos\phi_p| \le \frac{i+1}{\ell^2}}} \ge \sum_{\left(\frac{S\ell}{i}b^t\right)^{\frac{1}{k-1}} \le p \le \left(\frac{(S+1)\ell}{i+1}b^t\right)^{\frac{1}{k-1}}} p^{-1}$$

for $\ell \le i \le \ell^2 - 1$, the lower bound is proven in a similar way. For the final step, let $0 < \epsilon < \log_b 2$ be fixed but arbitrary, let $\ell > max\left\{\frac{1}{b^\epsilon - 1}, \beta\right\}$ be an integer, and let $x > exp((\log\ell)^{\frac{2}{\epsilon}})$. Now we have,

$$\sum_{\substack{p\le x \\ p\in A_{sym^2 f}(b,S)}} p^{-1} = \sum_{\substack{p\le x \\ p\in A_{sym^2 f}(b,S) \\ |\cos\phi_p| \le \ell^{-1}}} p^{-1} + \sum_{\substack{p\le x \\ p\in A_{sym^2 f}(b,S) \\ |\cos\phi_p| \ge \ell^{-1}}} p^{-1}$$

By Corollary 1, the first term is at most

$$\sum_{\substack{p\le x\\ |\cos\phi_p|\le \ell^{-1}}} p^{-1} = (1+o(1))\mu_{sym^2f,ST}([0,\ell^{-1}])\log\log x.$$

Using Lemma 5.1, the right hand side becomes

$$(1+o(1))\left(\log_b(1+S^{-1})-\log_b(1+\ell^{-1})\right)\log\log x \le \sum_{\substack{p\le x\\ p\in A_{sym^2f}(b,S)}} p^{-1}.$$

The right hand side of the above inequality is $\le$

$$(1+o(1))\left(\log_b(1+S^{-1})+\log_b(1+\ell^{-1})+\mu_{sym^2f,ST}([0,\ell^{-1}])\right)\log\log x + 2\log\log\ell.$$

Thus

$$\begin{aligned}(1+o(1))\left(\log_b(1+S^{-1})-\epsilon\right)\log\log x &\le \sum_{\substack{p\le x\\ p\in A_{sym^2f}(b,S)}} p^{-1}\\ &\le (1+o(1))\left(\log_b(1+S^{-1})+9\log(b)t\right)\log\log x.\end{aligned}$$

Letting $\epsilon\to 0$, we obtain the result of Theorem 2.2.
Remark 2. A similar proof should work for the symmetric cube of f, although the calculation of the corresponding Sato-Tate measure for $sym^3 f$ will be considerably more complicated.

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