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On the algebraic relations and dissections for certain continued fractions of order 104

Shruthi C. Bhat and B. R. Srivatsa Kumar

Abstract. In the present article, we obtain thirteen special cases of Ramanujan's continued fraction and establish 2-, 4-dissections of those continued fractions. Also, we derive algebraic relations connecting continued fractions with their respective reciprocals and we validate our results establishing the coloured partition identities.

Keywords. algebraic relations, coloured partitions, dissections, Ramanujan's general continued fraction, theta functions.

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1. Introduction

Throughout the paper, let $q \in \mathbb{C}$, such that $|q| < 1$. Then q-shifted factorial [Ber91] is customarily defined as

$$(\gamma; q)_m := \prod_{k=1}^{m} (1 - \gamma q^{k-1}), \qquad \text{and} \qquad (\gamma; q)_\infty := \prod_{k=0}^{\infty} (1 - \gamma q^k),$$

where γ is a complex number and m is a positive integer. Also, $(\gamma; q)_0 := 1$. We use the following notation for convenience.

$$(\gamma_1, \gamma_2, \ldots, \gamma_m; q)_\infty = (\gamma_1; q)_\infty (\gamma_2; q)_\infty \ldots (\gamma_m; q)_\infty.$$

Following is the renowned Rogers-Ramanujan continued fraction [Ram00]

$$R(q) := \frac{q^{1/5}}{1\;+}\,\frac{q}{1\;+}\,\frac{q^2}{1\;+}\,\frac{q^3}{1\;+\cdots} = q^{1/5}\frac{f(-q, -q^4)}{f(-q^2, -q^3)},$$

where

$$f(\gamma, \delta) = \sum_{n=-\infty}^{\infty} \gamma^{n(n+1)/2} \delta^{n(n-1)/2} = (-\gamma, -\delta, \gamma\delta; \gamma\delta)_\infty, \quad |\gamma\delta| < 1, \tag{1.1}$$

is Ramanujan's general theta function [Ber91, p. 35, Entry 19], [ABBN85]. The following are the theta functions [Ber91, p. 36, Entry 22 (i)-(iii)] arising from $f(\gamma, \delta)$ which are useful in the latter sections.

$$f(q, q) = \varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_\infty}{(q; -q)_\infty}, \tag{1.2}$$

$$f(q, q^3) = \psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}. \tag{1.3}$$

The Entry 12 [Ber91, p.24], in his second notebook gives the following general form of continued fraction. Let $k, l, q \in \mathbb{C}$ with $|kl| < 1$ and $|q| < 1$ or that $k = l^{2m+1}$ for some $m \in \mathbb{Z}$. Then

$$\frac{(k^2q^3; q^4)_\infty (l^2q^3; q^4)_\infty}{(k^2q; q^4)_\infty (l^2q; q^4)_\infty} = \frac{1}{(1-kl)\;+}\,\frac{(k-lq)(l-kq)}{(1-kl)(q^2+1)\;+}\,\frac{(k-lq^3)(l-kq^3)}{(1-kl)(q^4+1)\;+\cdots}. \tag{1.4}$$

The above continued fraction can also be found in [ABBN85]. Ramanujan [Ram57, Ram88] dealt with various special cases of the above expression and recorded algebraic relations on the same. Rajkhowa and Saikia [RaSa23b] studied certain continued fractions of order ten and established algebraic relations on the same. For recent work on algebraic relations connecting the special cases of (1.4) with their respective reciprocals, [RaSa23a, RaSa24a, RaSa24b, RSK23] can be referred.

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The m-dissection [LeLi00] of the power series $P = \sum_{n=0}^{\infty} a(n)q^n$ is the representation of P as $P = P_0 + P_1 + \ldots + P_{m-1}$, where $P_k = \sum_{n=0}^{\infty} a(mn+k)q^{mn+k}$, $0 \leq k \leq m-1$. The very first person to offer dissections of q-series identities is Ramanujan. The following are the 2-dissections of the continued fraction $R^*(q) := q^{-1/5}R(q)$ and its reciprocal given by Ramanujan [ABBN85, p. 50].

$$R^*(q) = \frac{(q^4, q^4, q^{16}, q^{16}; q^{20})_\infty}{(q^2, q^{10}, q^{10}, q^{18}; q^{20})_\infty} - q\frac{(q^4, q^6, q^{14}, q^{16}; q^{20})_\infty}{(q^8, q^{10}, q^{10}, q^{12}; q^{20})_\infty}$$

and

$$\frac{1}{R^*(q)} = \frac{(q^8, q^8, q^{12}, q^{12}; q^{20})_\infty}{(q^6, q^{10}, q^{10}, q^{14}; q^{20})_\infty} + q\frac{(q^2, q^8, q^{12}, q^{18}; q^{20})_\infty}{(q^4, q^{10}, q^{10}, q^{16}; q^{20})_\infty}.$$

Hirschhorn [Hir98] demonstrated the 5-dissections of $R^*(q)$ and its reciprocal, as reported by Ramanujan [Ram88]. Lewis and Liu [LeLi00] resolved a conjecture on 4-dissections of $R^*(q)$ and its reciprocal that was also provided by Hirschhorn [Hir98]. The 2-dissections of the reciprocal of the Rogers-Ramanujan continued fraction were examined by Andrews [And81]. The 2-, 3-, 4-, 6-, and 12-dissections of a continued fraction of order twelve were covered by Lin [Lin13]. Adiga et al. [ASV15] examined the 2- and 4-dissections of an order six continued fraction. Surekha [Sur17] derived algebraic relations and obtained 2-, 4-, 8-, 16-dissections of a general continued fraction of order sixteen which is a special case of (1.4). Recently, Chetry and Saikia [ChSa23] established modular relations satisfied by a continued fraction of order thirty two obtained from (1.4) and proved 2-, 4-, 8- and 16-dissections of the reciprocal of the continued fraction.

Motivated by the wonderful work in the literature, we deduce thirteen continued fractions from (1.4), each of order 104. We establish the algebraic relations satisfied by the continued fractions with their respective reciprocals in Section 2. We obtain 2-, 4-dissections of the continued fractions in Section 3. As the algebraic relations bear application in the theory of coloured partition, in Section 4 we give coloured partition identity for one of the algebraic relations established in Section 2. We illustrate it with an example as well. We give vanishing coefficients in the expansion of power series of the continued fractions in Section 5.

We conclude this section deducing thirteen continued fractions. Replacing q by q^{26} in (1.4), we obtain

$$\frac{(k^2q^{78}; q^{104})_\infty (l^2q^{78}; q^{104})_\infty}{(k^2q^{26}; q^{104})_\infty (l^2q^{26}; q^{104})_\infty} = \frac{1}{(1-kl)\,+}\,\frac{(k-lq^{26})(l-kq^{26})}{(1-kl)(q^{52}+1)\,+}\,\frac{(k-lq^{78})(l-kq^{78})}{(1-kl)(q^{104}+1)\,+\ldots}. \tag{1.5}$$

Setting $k = q^{77/2}$ and $l = q^{-25/2}$ in (1.5) and using the definition of $f(\gamma, \delta)$, we arrive at the following continued fraction of order 104 on simplification.

$$\begin{aligned}\frac{1}{\Gamma_1(q)} &:= \frac{q^{-25/2}(1-q^{51})}{(1-q^{26})\,+}\,\frac{q^{26}(1-q^{77})(1-q^{25})}{(1-q^{26})(q^{52}+1)\,+}\,\frac{q^{26}(1-q^{129})(1-q^{27})}{(1-q^{26})(q^{104}+1)\,+\ldots} \\ &= q^{-25/2}\frac{f(-q^{51}, -q^{53})}{f(-q, -q^{103})}.\end{aligned} \tag{1.6}$$

Similarly, for $(k, l) = (q^{(78-i)/2}, q^{-(26-i)/2})$, for an odd number i, $3 \leq i \leq 25$, (1.5) yields twelve more continued fractions $\Gamma_i(q)$ that are given by

$$\begin{aligned}\frac{1}{\Gamma_i(q)} &:= \frac{q^{-(26-i)/2}(1-q^{52-i})}{(1-q^{26})\,+}\,\frac{q^{26}(1-q^{78-i})(1-q^{26-i})}{(1-q^{26})(q^{52}+1)\,+}\,\frac{q^{26}(1-q^{130-i})(1-q^{26+i})}{(1-q^{26})(q^{104}+1)\,+\ldots} \\ &= q^{-(26-i)/2}\frac{f(-q^{i}, -q^{104-i})}{f(-q^{52-i}, -q^{52+i})}.\end{aligned} \tag{1.7}$$

2. Algebraic relations on $\Gamma_i(q)$

The algebraic relations satisfied by $\Gamma_i(q)$ and $1/\Gamma_i(q)$, for $i = 1, 3, \ldots, 25$, are given in this section.

Theorem 2.1. *We have*

$$\frac{1}{\Gamma_i(q)} \mp \Gamma_i(q) = \frac{f(\mp q^{26-i}, \mp q^{26+i})\varphi(\pm q^{26})}{q^{(26-i)/2}f(-q^{i}, -q^{52-i})\psi(q^{52})}, \tag{2.8}$$

for $i = 1, 3, \ldots, 25$.

Proof. Use (1.6) to get

$$\frac{1}{\sqrt{\Gamma_1(q)}} - \sqrt{\Gamma_1(q)} = \frac{f(-q^{51}, -q^{53}) - q^{25/2}f(-q, -q^{103})}{\sqrt{q^{25/2}f(-q, -q^{103})f(-q^{51}, -q^{53})}}. \tag{2.9}$$

The following identity can be found in Entry 30 (ii) & (iii) of [Ber91, p. 46]:

$$f(x,y)=f(x^3y,xy^3)+xf(y/x,x^5y^3). \tag{2.10}$$

Set $x=-q^{25/2}$ and $y=q^{27/2}$ in (2.10) to obtain

$$f(-q^{25/2},q^{27/2})=f(-q^{51},-q^{53})-q^{25/2}f(-q,-q^{103}). \tag{2.11}$$

Using the above, (2.9) becomes

$$\frac{1}{\sqrt{\Gamma_1(q)}}-\sqrt{\Gamma_1(q)}=\frac{f(-q^{25/2},q^{27/2})}{\sqrt{q^{25/2}f(-q,-q^{103})f(-q^{51},-q^{53})}}. \tag{2.12}$$

Similarly, setting $x=q^{25/2}$ and $y=-q^{27/2}$ in (2.10), gives

$$\frac{1}{\sqrt{\Gamma_1(q)}}+\sqrt{\Gamma_1(q)}=\frac{f(q^{25/2},-q^{27/2})}{\sqrt{q^{25/2}f(-q,-q^{103})f(-q^{51},-q^{53})}}. \tag{2.13}$$

Multiply the above two equations to get

$$\frac{1}{\Gamma_1(q)}-\Gamma_1(q)=\frac{f(-q^{25/2},q^{27/2})f(q^{25/2},-q^{27/2})}{q^{25/2}f(-q,-q^{103})f(-q^{51},-q^{53})}. \tag{2.14}$$

The following identity can be found in Entry 78 (iv) of [Ber91, p. 46]:

$$f(x,y)f(-x,-y)=f(-x^2,-y^2)\varphi(-xy). \tag{2.15}$$

Set $x=-q^{25/2}$ and $y=q^{27/2}$ in (2.15) to get

$$f(-q^{25/2},q^{27/2})f(q^{25/2},-q^{27/2})=f(-q^{25},-q^{27})\varphi(q^{26}). \tag{2.16}$$

Employ the above expression in (2.14) to obtain

$$\frac{1}{\Gamma_1(q)}-\Gamma_1(q)=\frac{f(-q^{25},-q^{27})\varphi(q^{26})}{q^{25/2}f(-q,-q^{103})f(-q^{51},-q^{53})}. \tag{2.17}$$

The following identity can be found in Entry 30 (i) of [Ber91, p. 46]:

$$f(x,xy^2)f(y,x^2y)=f(x,y)\psi(xy). \tag{2.18}$$

Set $x=-q$ and $y=-q^{51}$ in (2.18) to get

$$f(-q,-q^{103})f(-q^{51},-q^{53})=f(-q,-q^{51})\psi(q^{52}).$$

Employ the above equation in (2.17) to obtain

$$\frac{1}{\Gamma_1(q)}-\Gamma_1(q)=\frac{f(-q^{25},-q^{27})\varphi(q^{26})}{q^{25/2}f(-q,-q^{51})\psi(q^{52})}. \tag{2.19}$$

Squaring on both the sides of (2.13) gives

$$\frac{1}{\Gamma_1(q)}+\Gamma_1(q)=\frac{f^2(q^{25/2},-q^{27/2})}{q^{25/2}f(-q,-q^{103})f(-q^{51},-q^{53})}-2. \tag{2.20}$$

Also, the following identity can be found in Entry 30 (v) & (vi) of [Ber91, p. 46]:

$$f^2(x,y)=f(x^2,y^2)\varphi(xy)+2xf(y/x,x^3y)\psi(x^2y^2). \tag{2.21}$$

Set $x=q^{25/2}$ and $y=-q^{27/2}$ in (2.21) to get

$$f^2(q^{25/2},-q^{27/2})=f(q^{25},q^{27})\varphi(-q^{26})+2q^{25/2}f(-q,-q^{51})\psi(q^{52}).$$

Employ the above equation in (2.20) to obtain

$$\frac{1}{\Gamma_1(q)}+\Gamma_1(q)=\frac{f(q^{25},q^{27})\varphi(-q^{26})}{q^{25/2}f(-q,-q^{51})\psi(q^{52})}.$$

This proves the algebraic relations on $\Gamma_1(q)$ and the proofs of remaining results are left to the interest of the reader.

The previous theorem directly leads to the following result.

Corollary 2.2. *We have*

$$\left(\frac{1}{\Gamma_i(q)}-\Gamma_i(q)\right)\left(\frac{1}{\Gamma_{26-i}(q)}-\Gamma_{26-i}(q)\right)=\left(\frac{1}{\Gamma_{13}(q)}-\Gamma_{13}(q)\right)^2=\frac{\varphi^2(q^{26})}{q^{13}\psi^2(q^{52})},$$

for $i=1,3,\ldots,11$.

3. 2-, 4-dissections of $\Gamma_i^*(q)$

In the current section, we prove 2-, 4-dissections of $\Gamma_i^*(q)$, given by

$$\Gamma_i^*(q) = \frac{f(-q^i, -q^{104-i})}{f(-q^{52-i}, -q^{52+i})} = \sum_{n=0}^{\infty} a_{(i,n)} q^n,$$

for $i = 1, 3, \ldots, 25$.

Theorem 3.1. *The 2-dissections of* $\sum_{n=0}^{\infty} a_{(i,n)} q^n$ *are given by*

$$\sum_{n=0}^{\infty} a_{(i,2n)} q^n = \frac{f(-q^{26+i}, -q^{78-i})\ \psi(-q^{26})}{f(-q^{52-i}, -q^{52+i})\ \varphi(-q^{52})} \tag{3.22}$$

and

$$\sum_{n=0}^{\infty} a_{(i,2n+1)} q^n = -q^{(i-1)/2} \frac{f(-q^{26-i}, -q^{78+i})\ \psi(-q^{26})}{f(-q^{52-i}, -q^{52+i})\ \varphi(-q^{52})}, \tag{3.23}$$

for $i = 1, 3, \ldots, 25$.

Proof. Consider

$$\Gamma_1^*(q) = \sum_{n=0}^{\infty} a_{(1,n)} q^n = \frac{f(-q, -q^{103}) f(q^{51}, q^{53})}{f(-q^{51}, -q^{53}) f(q^{51}, q^{53})}. \tag{3.24}$$

From [Ber91, Entry 29, p. 45], for $xy = zw$,

$$f(x, y)f(z, w) = f(xz, yw)f(xw, yz) + x f(y/z, xz^2w) f(y/w, xzw^2). \tag{3.25}$$

Assign $x = -q$, $y = -q^{103}$, $z = q^{51}$ and $w = q^{53}$ in (3.25) to obtain

$$\begin{aligned} f(-q, -q^{103}) f(q^{51}, q^{53}) =& f(-q^{52}, -q^{156}) f(-q^{54}, -q^{154}) \\ & - q f(-q^{52}, -q^{156}) f(-q^{50}, -q^{158}). \end{aligned} \tag{3.26}$$

Assign $x = -q^{51}$, $y = -q^{53}$ (2.15) to obtain

$$f(-q^{51}, -q^{53}) f(q^{51}, q^{53}) = f(-q^{102}, -q^{106}) \varphi(-q^{104}). \tag{3.27}$$

Now, using (3.26), (3.27) and (1.3), (3.24) becomes

$$\sum_{n=0}^{\infty} a_{(1,n)} q^n = \frac{f(-q^{54}, -q^{154}) \psi(-q^{52})}{f(-q^{102}, -q^{106}) \varphi(-q^{104})} - q \frac{f(-q^{50}, -q^{158}) \psi(-q^{52})}{f(-q^{102}, -q^{106}) \varphi(-q^{104})}. \tag{3.28}$$

From (3.28), one can deduce

$$\sum_{n=0}^{\infty} a_{(1,2n)} q^{2n} = \frac{f(-q^{54}, -q^{154}) \psi(-q^{52})}{f(-q^{102}, -q^{106}) \varphi(-q^{104})}$$

and

$$\sum_{n=0}^{\infty} a_{(1,2n+1)} q^{2n} = -\frac{f(-q^{50}, -q^{158}) \psi(-q^{52})}{f(-q^{102}, -q^{106}) \varphi(-q^{104})}.$$

Replacing q^2 by q in the above two equations, the following 2-dissections of $\Gamma_1^*(q)$ can be obtained.

$$\sum_{n=0}^{\infty} a_{(1,2n)} q^n = \frac{f(-q^{27}, -q^{77}) \psi(-q^{26})}{f(-q^{51}, -q^{53}) \varphi(-q^{52})} \tag{3.29}$$

and

$$\sum_{n=0}^{\infty} a_{(1,2n+1)} q^n = -\frac{f(-q^{25}, -q^{79}) \psi(-q^{26})}{f(-q^{51}, -q^{53}) \varphi(-q^{52})}. \tag{3.30}$$

This completes the proof for the case $i = 1$ and the proofs for the remaining results follow the same pattern, which are left to the interest of the reader.

Theorem 3.2. *The 4-dissections of* $\sum_{n=0}^{\infty} a_{(i,n)}q^n$ *are given by*

$$\sum_{n=0}^{\infty} a_{(i,4n)}q^n = \frac{f(-q^{39},-q^{65})\ f(-q^{39+i},-q^{65-i})\ \psi(-q^{13})}{f(-q^{52-i},-q^{52+i})\ \varphi(-q^{26})\ \varphi(-q^{52})}, \tag{3.31}$$

$$\sum_{n=0}^{\infty} a_{(i,4n+1)}q^n = -q^{(i-1)/4}\frac{f(-q^{39},-q^{65})\ f(-q^{39-i},-q^{65+i})\ \psi(-q^{13})}{f(-q^{52-i},-q^{52+i})\ \varphi(-q^{26})\ \varphi(-q^{52})}, \tag{3.32}$$

$$\sum_{n=0}^{\infty} a_{(i,4n+2)}q^n = -q^{(25+i)/2}\frac{f(-q^{13},-q^{91})\ f(-q^{13-i},-q^{91+i})\ \psi(-q^{13})}{f(-q^{52-i},-q^{52+i})\ \varphi(-q^{26})\ \varphi(-q^{52})}, \tag{3.33}$$

and

$$\sum_{n=0}^{\infty} a_{(i,4n+3)}q^n = q^{(49-i)/4}\frac{f(-q^{13},-q^{91})\ f(-q^{13+i},-q^{91-i})\ \psi(-q^{13})}{f(-q^{52-i},-q^{52+i})\ \varphi(-q^{26})\ \varphi(-q^{52})}. \tag{3.34}$$

for $i = 1, 3, \ldots, 25$.

Proof. Consider

$$\sum_{n=0}^{\infty} a_{(1,2n)}q^{2n} = \frac{\psi(-q^{26})\ f(-q^{27},-q^{77})\ f(q^{51},q^{53})}{\varphi(-q^{52})\ f(-q^{51},-q^{53})\ f(q^{51},q^{53})}. \tag{3.35}$$

Assign $x = -q^{27}$, $y = -q^{77}$, $z = q^{51}$ and $w = q^{53}$ in (3.25) to obtain

$$\begin{aligned} f(-q^{27},-q^{77})f(q^{51},q^{53}) =& f(-q^{78},-q^{130})f(-q^{80},-q^{128}) \\ &- q^{27}f(-q^{26},-q^{182})f(-q^{24},-q^{184}). \end{aligned} \tag{3.36}$$

Now, using (3.36) and (3.27), (3.35) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} a_{(1,2n)}q^n =& \frac{f(-q^{78},-q^{130})\ f(-q^{80},-q^{128})\ \psi(-q^{26})}{f(-q^{102},-q^{106})\ \varphi(-q^{52})\ \varphi(-q^{104})} \\ &- q^{27}\ \frac{f(-q^{26},-q^{182})\ f(-q^{24},-q^{184})\ \psi(-q^{26})}{f(-q^{102},-q^{106})\ \varphi(-q^{52})\ \varphi(-q^{104})}. \end{aligned} \tag{3.37}$$

Similarly, by (3.37) and using (3.30), 4-dissections of Γ_1^* can be obtained:

$$\sum_{n=0}^{\infty} a_{(1,4n)}q^n = \frac{f(-q^{39},-q^{65})\ f(-q^{40},-q^{64})\ \psi(-q^{13})}{f(-q^{51},-q^{53})\ \varphi(-q^{26})\ \varphi(-q^{52})},$$

$$\sum_{n=0}^{\infty} a_{(1,4n+1)}q^n = -\frac{f(-q^{39},-q^{65})\ f(-q^{38},-q^{66})\ \psi(-q^{13})}{f(-q^{51},-q^{53})\ \varphi(-q^{26})\ \varphi(-q^{52})},$$

$$\sum_{n=0}^{\infty} a_{(1,4n+2)}q^n = -q^{13}\frac{f(-q^{13},-q^{91})\ f(-q^{12},-q^{92})\ \psi(-q^{13})}{f(-q^{51},-q^{53})\ \varphi(-q^{26})\ \varphi(-q^{52})},$$

and

$$\sum_{n=0}^{\infty} a_{(1,4n+3)}q^n = q^{12}\frac{f(-q^{13},-q^{91})\ f(-q^{14},-q^{90})\ \psi(-q^{13})}{f(-q^{51},-q^{53})\ \varphi(-q^{26})\ \varphi(-q^{52})}.$$

This completes the proof for the case $i = 1$ and the proofs for the remaining results follow the same pattern, which are left to the interest of the reader.

4. Coloured partition identities

The algebraic identities established in previous section bear applications in coloured partitions, which are illustrated in this section. We establish coloured partition identity for case $i = 1$ in Theorem 2.1 here. Agarwal and Andrews [AgAn87] proposed theory of coloured partitions. "A non-increasing sequence of positive integers less than n, whose sum is n is known as a partition of n. A positive integer n is said to have l colours, if there are l copies of n available colours, all of which are considered distinct. Partitions of a positive integer n into parts with colours are called coloured partitions of n." This can be illustrated as follows:

If 1 is assigned with 2 colours, 2 with 2 colours and 3 with 3 colours, then the coloured partitions of 3 can be listed as $1_v + 1_v + 1_v$, $1_b + 1_b + 1_b$, $1_v + 1_b + 1_b$, $1_b + 1_v + 1_v$, $2_v + 1_v$, $2_v + 1_b$, $2_b + 1_v$, $2_b + 1_b$, 3_v, 3_b and 3_r, where v, b and r denote violet, blue and red colours respectively to distinguish.

The generating function for number of partitions of k into parts congruent to r modulo s with l colours is given by

$$\sum_{k=0}^{\infty} A(k) q^k = (q^r; q^s)_{\infty}^{-l}.$$

By convention, we set $A(0) = 1$. Also, for convenience we define

$$(q_k^{\pm x}; q^y)_{\infty} := (q^x, q^{y-x}; q^y)_{\infty}^k, \qquad x < y, \quad x, y \in \mathbb{N}. \tag{4.38}$$

Theorem 4.1. *Let $\kappa_1(k)$ denote the total partitions of k being partitioned into the units congruent to ± 25 and ± 27 modulo 104 with 1 colour each and $\pm 1$and ± 52 modulo 104 with 2 colours each. Let $\kappa_2(k)$ represent the total partitions of k into the fragments congruent to ± 25 and ± 27 modulo 104 with 1 colour each and ± 51 and ± 52 modulo 104 with 2 colours each. Let $\kappa_3(k)$ serve as the total partitions of k into the fragments congruent to ± 1, ± 26 and ± 51 modulo 104 with 2 colours each. Then $\kappa_1(k), \kappa_2(k)$ and $\kappa_3(k)$ satisfy the following expression:*

$$\kappa_1(k) - \kappa_2(k-25) - \kappa_3(k) = 0, \qquad k \geq 25.$$

Proof. For $i = 1$, (2.1) reduces to (2.19). Upon using (1.1), (1.2), (1.3) and (1.6) in (2.19) and making use of (4.38), multiplying by $q^{25/2}$, the following expression is obtained.

$$\frac{(q^{\pm 51}; q^{104})_{\infty}}{(q^{\pm 1}; q^{104})_{\infty}} - q^{25} \frac{(q^{\pm 1}; q^{104})_{\infty}}{(q^{\pm 51}; q^{104})_{\infty}} - \frac{(q^{\pm 25}, q^{\pm 27}, q_2^{\pm 52}; q^{104})_{\infty}}{(q^{\pm 1}, q^{\pm 51}, q_2^{\pm 26}; q^{104})_{\infty}} = 0. \tag{4.39}$$

Divide (4.39) by $(q^{\pm 1}, q^{\pm 25}, q^{\pm 27}, q^{\pm 51}, q_2^{\pm 52}; q^{104})_{\infty}$ to get

$$\frac{1}{(q_2^{\pm 1}, q^{\pm 25}, q^{\pm 27}, q_2^{\pm 52}; q^{104})_{\infty}} - \frac{q^{25}}{(q^{\pm 25}, q^{\pm 27}, q_2^{\pm 51}, q_2^{\pm 52}; q^{104})_{\infty}} \\ - \frac{1}{(q_2^{\pm 1}, q_2^{\pm 26}, q_2^{\pm 51}; q^{104})_{\infty}} = 0.$$

It is observed that, the expression atop gives rise to three generating functions $\kappa_1(k), \kappa_2(k)$ and $\kappa_3(k)$ to get

$$\sum_{k=0}^{\infty} \kappa_1(k) q^k - q^{25} \sum_{k=0}^{\infty} \kappa_2(k) q^k - \sum_{k=0}^{\infty} \kappa_3(k) q^k = 0.$$

Further, on extracting the coefficients of q^k in the expression atop, the result is obtained.

Verification of Theorem 4.1 for $k = 25$ is as follows:

$\kappa_1(25) = 27 : 25_v, 1_v + 1_v + \ldots + 1_v$ (25 terms), $1_v + 1_v + \ldots + 1_v + 1_i$ (25 terms), $1_v + 1_v + \ldots + 1_v + 1_i + 1_i$ (25 terms), $\ldots$, $1_i + 1_i + \ldots + 1_i$ (25 terms)
$\kappa_2(0) = 1.$
$\kappa_3(25) = 26 : 1_v + 1_v + \ldots + 1_v$ (25 terms), $1_v + 1_v + \ldots + 1_v + 1_i$ (25 terms), $1_v + 1_v + \ldots + 1_v + 1_i + 1_i$ (25 terms), $\ldots$, $1_i + 1_i + \ldots + 1_i$ (25 terms)

From the above table, it can be seen that $\Gamma_1(25) + \Gamma_2(0) - \Gamma_3(25) = 0$.

5. Vanishing coefficients

One can see that certain coefficients in the power series expansion of certain infinite product vanish. In [And79], Andrews and Bressoud gave general result for the vanishing coefficients of certain family of power series:

Theorem 5.1. ([And79]) *If $1 \leq r \leq k$ are relatively prime integers of opposite parity and*

$$\frac{(q^r, q^{2k-r}; q^{2k})_\infty}{(q^{k-r}, q^{k+r}; q^{2k})_\infty} = \sum_{n=0}^{\infty} a_n q^n,$$

then $a_{kn+r(k-r+1)/2}$ is always zero.

From this result, one can observe that the vanishing coefficients in the expansion of $\Gamma_i(q) = \sum_{n=0}^{\infty} a_{(i,n)} q^n$ are given by

$$a_{(i,52n+(i+1)(105-i)/8)} = 0,$$

where $i = 2r - 1$, for $1 \leq r \leq 13$, such that $\gcd(r, 52) = 1$.

6. Conclusion

In this article, we have deduced special cases $1/\Gamma_i(q)$, of Ramanujan's continued fraction, each of order 104, for $i = 1, 3, \ldots, 25$. We have documented the algebraic relations connecting $\Gamma_i(q)$ and $1/\Gamma_i(q)$ for $i = 1, 3, \ldots, 25$. Further, we have established 2-dissections of $\Gamma_i(q)$, for $i = 1, 3, \ldots, 25$ and 4-dissections of $\Gamma_1(q)$. The algebraic relation on $\Gamma_1(q)$ is validated by the coloured partition identity obtained in the latter section. We also have given the vanishing coefficients for certain continued fractions. We believe that 4-dissections can be established for many more continued fractions $\Gamma_i(q)$, for different values of i, following the proof given for 4-dissections of $\Gamma_1(q)$ in the article. Moreover, many more coloured partition identities can be obtained using the algebraic relations on $\Gamma_i(q)$, for different i.

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Shruthi C. Bhat
Department of Mathematics,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal-576104, Karnataka, India.

e-mail: shruthi.research1@gmail.com, shruthi.u@learner.manipal.edu

B. R. Srivatsa Kumar
Department of Mathematics,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal-576104, Karnataka, India.

e-mail: srivatsa.kumar@manipal.edu