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Moments of norm-counting functions over integer representations as a sum of eight squares

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Abstract. Let $\ell, k \geq 2$ be integers. In this article, we investigate the ℓth power moments of the arithmetic function $\tau_{k,\mathbb{K}}(n)$, which counts the number of ways n can be expressed as a product of norms of k non-zero ideals in the ring of integers of a non-normal cubic field $\mathbb{K}$, evaluated over integers represented as sums of eight squares.

Keywords. Automorphic L-functions; Dedekind-zeta function; generalized divisor problem; non-normal cubic field

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1. Introduction

The analysis of divisor-type functions lies at the heart of analytic number theory and has long served as a gateway to understanding the distribution of arithmetic functions. One of the most basic yet profound examples is the *divisor function* $\tau(n)$, which enumerates the number of positive integers dividing n, i.e.,

$$\tau(n) = \sum_{d|n} 1.$$

Although $\tau(n)$ behaves quite erratically from point to point, its cumulative sum up to a large bound, $\sum_{n\leq X} \tau(n)$, reveals a remarkably regular growth pattern. This summatory function admits the asymptotic expansion:

$$\sum_{n\leq X} \tau(n) = X \log X + (2\gamma - 1)X + \mathcal{O}(X^{\theta}),$$

where γ denotes Euler's constant and θ is a positive real number strictly less than 1. Refining the exponent θ in the error term constitutes the celebrated *Dirichlet divisor problem*, a central question that continues to attract deep investigation.

Analogous questions arise in algebraic number fields. For a number field $\mathbb{K}$, the function $\tau_{k,\mathbb{K}}(n)$ counts the number of ways to write n as a product of norms of k-ideals in the ring of integers $\mathcal{O}_{\mathbb{K}}$:

$$\tau_{k,\mathbb{K}}(n) = \sum_{\substack{0\neq \mathfrak{a}_1,\ldots,\mathfrak{a}_k \subseteq \mathcal{O}_{\mathbb{K}} \\ N(\mathfrak{a}_1\ldots\mathfrak{a}_k)=n}} 1 = \sum_{n=n_1 n_2\ldots n_k} a_{\mathbb{K}}(n_1)\ldots a_{\mathbb{K}}(n_k), \tag{1.1}$$

where $a_{\mathbb{K}}(n)$ denotes the number of integral ideals in $\mathbb{K}$ with norm n. Here, the absolute norm of an ideal $\mathfrak{u}$ in $\mathcal{O}_{\mathbb{K}}$ is given by

$$N(\mathfrak{u}) := [\mathcal{O}_{\mathbb{K}} : \mathfrak{u}].$$

Note that $\tau_{1,\mathbb{K}}(n) = a_{\mathbb{K}}(n)$. The generating Dirichlet series of $a_{\mathbb{K}}(n)$ is the well-known Dedekind zeta function $\zeta_{\mathbb{K}}(s)$.

Understanding the asymptotic behaviour of the summatory function of $\tau_{k,\mathbb{K}}(n)$ is a natural and significant problem in number theory. One can easily check that $\tau_{k,\mathbb{K}}(n)$ is a multiplicative function of n. The problem of finding an asymptotic formula for

$$\sum_{n\leq X} \tau_{k,\mathbb{K}}(n)$$

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is known as the *k-dimensional divisor problem* in the number field $\mathbb{K}$.

The study of moments of arithmetic functions is a classical problem in mathematics. In 1949, Landau [Lan49] investigated the first moment of $\tau_{1,\mathbb{K}}(n)$ for a number field $\mathbb{K}$ of degree $d \geq 2$ and proved that

$$\sum_{n \leq X} \tau_{1,\mathbb{K}}(n) = c_{\mathbb{K}} X + \mathcal{O}(X^{1-\frac{2}{d+1}+\epsilon})$$

for any $\epsilon > 0$, where

$$c_{\mathbb{K}} = \lim_{s \to 1^+} (s-1)\zeta_{\mathbb{K}}(s) = \frac{2^{r_1}(2\pi)^{r_2} hR}{\omega\sqrt{|D_{\mathbb{K}}|}}. \tag{1.2}$$

Here, r_1 is the number of real embeddings of $\mathbb{K}$, and $2r_2$ is the number of complex embeddings; h denotes the class number; R is the regulator; ω is the number of roots of unity in $\mathbb{K}$; and $D_{\mathbb{K}}$ is the discriminant of $\mathbb{K}$.

Later, Chandrasekharan and Narasimhan [ChNa63] studied the second moment of $\tau_{1,\mathbb{K}}(n)$ and proved that

$$\sum_{n \leq X} \tau_{1,\mathbb{K}}^2(n) \ll X(\log X)^{d-1}.$$

For a Galois extension $\mathbb{K}/\mathbb{Q}$ of degree $d > 1$, Chandrasekharan and Good [ChGo83] established the following estimate. For any $\ell \geq 2$ and $\epsilon > 0$,

$$\sum_{n \leq X} \tau_{1,\mathbb{K}}^{\ell}(n) = X P_{\ell}(\log X) + \mathcal{O}(X^{1-2d^{-\ell}+\epsilon}),$$

where $P_\ell(X)$ is a polynomial in X of degree $d^{\ell-1} - 1$.

In the case of $\mathbb{K} = \mathbb{Q}$, several authors have studied this problem, including Voronoi, Landau, Hardy and Littlewood, Ivić, Heath-Brown, Rieger, and others. In [Pan88, Pan94], Panteleeva considered this problem for quadratic fields $\mathbb{K} = \mathbb{Q}(\sqrt{D_{\mathbb{K}}})$, where $D_{\mathbb{K}}$ is a square-free integer, as well as for cyclotomic fields $\mathbb{K} = \mathbb{Q}(\xi)$, where ξ is an ℓth root of unity.

For a non-normal cubic extension $\mathbb{K}$ over $\mathbb{Q}$, Fomenko [Fom07] studied the second and third moments of $\tau_{1,\mathbb{K}}(n)$, providing asymptotic formulae for these cases. Later, Lü [Lü13] refined error terms, and Liu [Liu21] further improved Fomenko's results. More recently, Chakraborty and Krishnamoorthy [ChKr22] considered a more general setting. In particular, for the Galois closure $\mathbb{K}'$ of a non-normal extension $\mathbb{K}/\mathbb{Q}$, they derived an asymptotic formula for

$$\sum_{n \leq X} \tau_{1,\mathbb{K}}^{\ell}(n)$$

under certain conditions on the Galois group $\mathrm{Gal}(\mathbb{K}'/\mathbb{Q})$ for $\ell \geq 3$. The k-dimensional divisor problem associated with non-normal cubic fields has also been investigated over sequences of positive integers corresponding to reduced forms of negative discriminant D.

Let $\mathbb{K}_3$ be a non-normal cubic field over $\mathbb{Q}$, defined by an irreducible polynomial

$$h(x) = x^3 + Ax^2 + Bx + C$$

with discriminant $D_h < 0$. A key motivation for studying the distribution of $\tau_{1,\mathbb{K}_3}(n)$ is its connection to the Dedekind zeta function $\zeta_{\mathbb{K}_3}(s)$ and the L-function $L(f,s)$ associated with a cusp form f of weight 1. In particular, Fomenko [Fom07] established the following identity:

$$\zeta_{\mathbb{K}_3}(s) = \zeta(s)L(f,s), \tag{1.3}$$

where f is a holomorphic cusp form of weight 1 for the congruence subgroup $\Gamma_0(|D_h|)$. We refer to Section 2. for relevant definitions and results concerning classical modular forms.

In recent years, the arithmetic function $a_{\mathbb{K}_3}(n)$ has attracted considerable attention; see [ShSa23, Hua24, TaWa24, GoTi25a, GoTi25b] for relevant developments. Building upon this line of inquiry, the present work investigates sums associated with $a_{\mathbb{K}_3}(n)$.

Main results. For a given positive integer m, let $R_m(n)$ denote the set of ordered m-tuples of integers $(x_1, \ldots, x_m)$ such that $x_1^2 + \cdots + x_m^2 = n$, i.e.,

$$R_m(n) := \left\{(x_1, \ldots, x_m) \in \mathbb{Z}^m : x_1^2 + \cdots + x_m^2 = n\right\}$$

denote the set of representations of a positive integer n as a sum of m squares. The cardinality of the set $R_m(n)$ is denoted by $r_m(n)$.

For given integers $k, \ell \geq 2$, consider the ℓth power moment of the k-dimensional divisor function associated with the coefficients of the Dedekind zeta function in $\mathbb{K}_3$ over the sequence $\left\{n \in \mathbb{N} : n = x_1^2 + x_2^2 + \cdots + x_8^2\right\}$ given by

$$S_{k,\ell}(X) := \sum_{n \leq X} \tau_{k,\mathbb{K}_3}^{\ell}(n) \cdot r_8(n), \tag{1.4}$$

where $\tau_{k,\mathbb{K}_3}(n)$ denotes the k-dimensional divisor function (as defined in (1.1)) associated with the non-normal cubic field $\mathbb{K}_3$ and X is a sufficiently large real number. In particular, we establish the following asymptotic formulae for $S_{k,\ell}(X)$.

Theorem 1.1. *Let $\mathbb{K}_3$ be a non-normal cubic field over $\mathbb{Q}$, and let $k \geq 2$ be a fixed integer. Then, for any $\epsilon > 0$ and sufficiently large X, we have*

$$S_{k,\ell}(X) = \begin{cases} X^4 P_{k,2}(\log X) + \mathcal{O}\big(X^{4 - \frac{105}{331k^2+15} + \epsilon}\big) & \text{for } \ell = 2, \\ \\ X^4 P_{k,3}(\log X) + \mathcal{O}\big(X^{4 - \frac{35}{256k^3+5} + \epsilon}\big) & \text{for } \ell = 3, \end{cases}$$

where $P_{k,2}(t)$ and $P_{k,3}(t)$ are polynomials of degree $2k^2 - 1$ and $5k^3 - 1$, respectively.

Theorem 1.2. *Let $\mathbb{K}_3$ be a non-normal cubic field over $\mathbb{Q}$, and let $k \geq 2$ be an integer. Then, for an integer $\ell \geq 4$, any $\epsilon > 0$, and sufficiently large X, we have*

$$S_{k,\ell}(X) = X^4 P_{k,\ell}(\log X) + \mathcal{O}\left(X^{4 - \frac{42}{k^{\ell}(7 \times 3^{\ell+1} - 8a_{0,\ell}) + 6} + \epsilon}\right),$$

where $P_{k,\ell}(t)$ is a polynomial of degree $k^{\ell}(a_{0,\ell} + a_{3,\ell}) - 1$, and the coefficients $a_{0,\ell}$ and $a_{3,\ell}$ are given by

$$a_{0,\ell} = 1 + \sum_{i=1}^{\lfloor \ell/2 \rfloor} \binom{\ell}{2i} \frac{(2i)!}{i!(i+1)!}, \quad \text{and} \quad a_{3,\ell} = \sum_{i=1}^{\lceil (\ell-2)/2 \rceil} \binom{\ell}{2i+1} \frac{4(2i+1)!}{(i-1)!(i+3)!}.$$

This article is structured as follows. Section 2. presents foundational material on the Dedekind zeta function and automorphic L-functions associated with Hecke cusp forms, along with key results on subconvexity bounds and integral moment estimates. In Section 3., we give a decomposition of the Dirichlet series corresponding to $\tau_{k,\mathbb{K}}(n) \cdot r_8(n)$ in terms of lower-degree general L-functions. Section 4. is devoted to the proofs of the main theorems.

Throughout this paper, ϵ denotes an arbitrarily small positive constant, though not necessarily the same in different instances. Any implied constant may depend on ϵ.

2. Preliminaries

In this section, we provide a brief review of the Dedekind zeta function, cusp forms, and associated L-functions. We then define symmetric power L-functions corresponding to Hecke eigenforms and discuss their fundamental properties. We conclude this section by listing several lemmas essential for proving the main theorems.

Let $\mathbb{K}$ be a number field of degree $[\mathbb{K}:\mathbb{Q}] = d$ and let $\mathcal{O}_\mathbb{K}$ denote its ring of integers. The Dedekind zeta function associated with $\mathbb{K}$ is defined as

$$\zeta_\mathbb{K}(s) := \prod_{\substack{\mathfrak{p}\subseteq\mathcal{O}_\mathbb{K}\\ \mathfrak{p}\neq 0}} \left(1-\frac{1}{(N\mathfrak{p})^s}\right)^{-1} = \sum_{\substack{\mathfrak{u}\subseteq\mathcal{O}_\mathbb{K}\\ \mathfrak{u}\neq 0}} \frac{1}{(N\mathfrak{u})^s}, \quad \text{for } \operatorname{Re}(s)>1.$$

This can also be expressed in terms of the Dirichlet series

$$\zeta_\mathbb{K}(s) = \sum_{n=1}^{\infty} \frac{a_\mathbb{K}(n)}{n^s}, \tag{2.5}$$

where $a_\mathbb{K}(n)$ denotes the number of integral ideals in $\mathbb{K}$ with norm n. In the special case $\mathbb{K}=\mathbb{Q}$, we have $a_\mathbb{K}(n) = 1$ for all positive integers n, reducing (2.5) to the Riemann zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The function $\zeta_\mathbb{K}(s)$ admits an analytic continuation to the entire complex plane, except for a simple pole at $s = 1$, whose residue is given by the analytic class number formula (1.2). Moreover, $\zeta_\mathbb{K}(s)$ satisfies a functional equation analogous to that of the Riemann zeta function:

$$\xi_\mathbb{K}(s) := \left(\frac{\sqrt{|D_\mathbb{K}|}}{2^{r_2}\pi^{d/2}}\right)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2}\zeta_\mathbb{K}(s) = \xi_\mathbb{K}(1-s).$$

It is well known that the arithmetic function $a_\mathbb{K}(n)$, associated with the Dedekind zeta function in (2.5), is multiplicative and satisfies the bound

$$a_\mathbb{K}(n) \le \tau(n)^{[\mathbb{K}:\mathbb{Q}]} \ll n^\epsilon$$

for any $\epsilon > 0$, where $\tau(n)$ denotes the divisor function.

Let $M_k(\Gamma_0(N), \chi)$ and $C_k(\Gamma_0(N), \chi)$ denote the spaces of modular forms and cusp forms, respectively, of integral weight k for the congruence subgroup $\Gamma_0(N)$ with nebentypus χ. When χ is trivial, these spaces are denoted by $M_k(\Gamma_0(N))$ and $C_k(\Gamma_0(N))$, respectively. A cusp form $f \in C_k(\Gamma_0(N))$ is called a Hecke eigenform (or newform) if it is a simultaneous eigenfunction of all Hecke operators and the Atkin-Lehner W-operators.

Let $k \ge 1$ be an integer, and let $f \in C_k(\Gamma_0(N))$ be a Hecke eigenform with the Fourier expansion at the cusp ∞ given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} q^n,$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$, the Poincaré upper half-plane. The function f is said to be normalized if $\lambda_f(1) = 1$. The normalized Fourier coefficient $\lambda_f(n)$ is a multiplicative function and satisfies the Hecke relation [Iwa04, Eq. (6.83)]:

$$\lambda_f(m)\lambda_f(n) = \sum_{d|\gcd(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

for all positive integers m and n coprime to N. By Deligne's bound, we have

$$|\lambda_f(n)| \leq \tau(n) \ll_\epsilon n^\epsilon,$$

for any arbitrarily small $\epsilon > 0$.
For $\mathrm{Re}(s) > 1$, the Hecke L-function associated with a normalized Hecke eigenform f is defined as

$$\begin{aligned} L(f,s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p\mid N} \left(1 - \frac{\lambda_f(p)}{p^s}\right)^{-1} \prod_{p\nmid N} \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \\ &= \prod_{p} \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1}, \end{aligned}$$

where $\alpha_p + \beta_p = \lambda_f(p)$, $\alpha_p\beta_p = 1$ for all primes $p \nmid N$, and $\alpha_p = \lambda_f(p), \beta_p = 0$ for $p \mid N$. The function $L(f,s)$ extends analytically to the entire complex plane and satisfies a nice functional equation.

For a given Dirichlet character χ of modulus m, the twisted form of f is defined as

$$f \otimes \chi(z) := \sum_{n=1}^{\infty} \lambda_f(n)\chi(n)n^{\frac{k-1}{2}} q^n \in C_k(\Gamma_0(M), \chi^2),$$

where $M \mid Nm^2$. The corresponding twisted Hecke L-function is given by

$$L(f\otimes\chi, s) = \sum_{n\geq 1} \frac{\lambda_f(n)\chi(n)}{n^s} = \prod_{p} \left(1 - \frac{\alpha_p\chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_p\chi(p)}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. This function converges absolutely and uniformly and remains non-vanishing for $\mathrm{Re}(s) > 1$. It extends analytically to the entire complex plane and satisfies a functional equation. For further details, see [Iwa04, Section 7.2].

For $j \geq 2$, the jth symmetric power L-function of degree $(j+1)$ is defined as

$$L(\mathrm{sym}^j f, s) := \prod_{p} \prod_{i=0}^{j} \left(1 - \alpha_p^{j-i}\beta_p^i p^{-s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \quad \text{for } \mathrm{Re}(s) > 1. \tag{2.6}$$

Since $\lambda_{\mathrm{sym}^j f}(n)$ is a real-valued multiplicative function, the jth symmetric power L-function has the Euler product expansion

$$L(\mathrm{sym}^j f, s) = \prod_{p} \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \frac{\lambda_{\mathrm{sym}^j f}(p^2)}{p^{2s}} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^m)}{p^{ms}} + \cdots\right)$$

for $\mathrm{Re}(s) > 1$.
The coefficients $\lambda_{\mathrm{sym}^j f}(n)$ of the Dirichlet series in (2.6) and the Fourier coefficients $\lambda_f(n)$ satisfy the relation

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \frac{\alpha_p^{j+1} - \beta_p^{j+1}}{\alpha_p - \beta_p}.$$

From Deligne's bound [Del74], we obtain

$$\left|\lambda_{\mathrm{sym}^j f}(n)\right| \leq d_{j+1}(n) \ll_\epsilon n^\epsilon,$$

for any $\epsilon > 0$, where $d_{j+1}(n)$ denotes the number of ways n can be expressed as a product of $j+1$ factors.

Remark 2.1. *From Fomenko's paper* [Fom07]*, we know that* $L(\mathrm{sym}^3 f, s)$ *has an analytic continuation to the half-plane* $Re(s) > \frac{1}{2}$*, except for a simple pole at* $s = 1$*.*

For $j \geq 2$, we define the twisted jth symmetric power L-function as

$$L(\mathrm{sym}^j f \otimes \chi, s) := \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)\chi(n)}{n^s} \quad \text{for } \mathrm{Re}(s) > 1.$$

Both the L-functions $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^j f \otimes \chi, s)$ admit analytic continuations to the entire complex plane and satisfy nice functional equations (for more details, see [NeTh21a, NeTh21b]).

Let $\zeta(s)$ and $L(s, \chi)$ (for a Dirichlet character χ of modulus N) denote the Riemann zeta function and Dirichlet L-function, respectively, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

for $\mathrm{Re}(s) > 1$. We adopt the following conventions:

$$L(\mathrm{sym}^0 f, s) = \zeta(s), \qquad L(\mathrm{sym}^1 f, s) = L(f, s).$$

In recent times, the asymptotic behaviour of certain arithmetic functions related to the Fourier coefficients of Hecke eigenforms has received much attention; see [GJS25, TiGo25, CGV25].

We now state the following two key lemmas, which are crucial for handling the error terms in our main theorems. For Lemma 2.2, we refer to [Hea78], [Bou17, Theorem 5], and [Jut87], while Lemma 2.3 follows from [Goo82] and [LNQ23]. Their proofs are omitted.

Lemma 2.2. *For any* $\epsilon > 0$*, we have*

$$\zeta(\gamma + \imath t) \ll_\epsilon (1 + |t|)^{\max\left\{\frac{13}{42}(1-\gamma), 0\right\} + \epsilon}$$

uniformly for $\frac{1}{2} \leq \gamma \leq 1$ *and* $|t| \geq 1$*. Additionally, we have*

$$\int_1^T \left|\zeta\left(\frac{1}{2} + \imath t\right)\right|^{12} dt \ll T^{2+\epsilon} \quad \textit{and} \quad \int_1^T \left|L\left(f, \frac{1}{2} + \imath t\right)\right|^{6} dt \ll T^{2+\epsilon}$$

uniformly for $T \geq 1$*.*

Lemma 2.3. *For any* $\epsilon > 0$*, we have*

$$L(f, \gamma + \imath t) \ll (1 + |t|)^{\max\left\{\frac{2}{3}(1-\gamma), 0\right\} + \epsilon}$$

and

$$L(\mathrm{sym}^2 f, \gamma + \imath t) \ll (1 + |t|)^{\max\left\{\frac{6}{5}(1-\gamma), 0\right\} + \epsilon}$$

uniformly for $\frac{1}{2} \leq \gamma \leq 1$ *and* $|t| \geq 1$*.*

For an L-function (in the sense of Perelli [Per82]), we now state the following result, which provides a convexity bound and an integral moment in a more general context (cf. [Iwa04, JiLü14]).

Lemma 2.4. *Let* $\mathfrak{L}(G, s)$ *be a general* L*-function of degree* $2A$ *given by*

$$\mathfrak{L}(G, s) = \sum_{n=1}^{\infty} \frac{\lambda_G(n)}{n^s} \quad \textit{for } Re(s) > 1.$$

For any $\epsilon > 0$*, we have*

$$\mathfrak{L}(G, \gamma + \imath t) \ll (1 + |t|)^{A(1-\gamma)+\epsilon}$$

uniformly for $\frac{1}{2} \leq \gamma \leq 1 + \epsilon$ *and* $|t| \geq 1$*. If* $\sum_{n \leq X} |\lambda_G(n)|^2 \ll X^{1+\epsilon}$*, we have*

$$\int_T^{2T} |\mathfrak{L}(G, \gamma + \imath t)|^2 \, dt \ll T^{\max\{2A(1-\gamma), 1\}+\epsilon}$$

uniformly for $\frac{1}{2} \leq \gamma \leq 1 + \epsilon$ *and* $T \geq 1$*.*

3. Intermediate results

This section forms the core of the article, where we establish key results concerning the Dirichlet series associated with the function $\tau^{\ell}_{k,\mathbb{K}_3}(n) \cdot r_8(n)$. We begin with the following result, which is a refined version of a lemma from [ChDa19], and serves as a pivotal step in deriving the L-function decomposition of the generating Dirichlet series for $\tau^{\ell}_{k,\mathbb{K}_3}(n) \cdot r_8(n)$, expressed in terms of known automorphic L-functions.

Lemma 3.1. *Let $a \geq 0$ be a real number, and let*

$$R(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad L(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

be two Dirichlet series, both absolutely convergent for Re$(s) > a+1$, satisfying the following conditions:
(i) *both a_n and b_n are multiplicative functions of n,*
(ii) *for any $\epsilon > 0$, we have $a_n \ll_\epsilon n^{a+\epsilon}$ and $b_n \ll_\epsilon n^{a+\epsilon}$,*
(iii) *$a_p = b_p$ for all but finitely many primes p.*
Then, we have

$$R(s) = L(s)H(s),$$

where $H(s)$ is a Dirichlet series that converges absolutely and uniformly in Re$(s) \geq \frac{1}{2} + a + \epsilon$ for any $\epsilon > 0$.

Proof. Since a_n and b_n are multiplicative functions, the Dirichlet series $R(s)$ and $L(s)$ have Euler product given by

$$R(s) = \prod_p \left(1 + \frac{a_p}{p^s} + \sum_{m=2}^{\infty} \frac{a_{p^m}}{p^{ms}}\right),$$

$$L(s) = \prod_p \left(1 + \frac{b_p}{p^s} + \sum_{m=2}^{\infty} \frac{b_{p^m}}{p^{ms}}\right)$$

for Re$(s) > a+1$. For $\gamma > a+1$, define $H(s)$ via local Euler factors by

$$H(s) := \prod_p H_p(s),$$

where

$$H_p(s) = \left(1 + \frac{a_p}{p^s} + \sum_{m=2}^{\infty} \frac{a_{p^m}}{p^{ms}}\right)\left(1 + \frac{b_p}{p^s} + \sum_{m=2}^{\infty} \frac{b_{p^m}}{p^{ms}}\right)^{-1}.$$

Observe that

$$\left|\sum_{m=2}^{\infty} \frac{a_{p^m}}{p^{ms}}\right| \ll \sum_{m=2}^{\infty} \frac{p^{(a+\epsilon)m}}{p^{m\gamma}} \leq \sum_{m=2}^{\infty} \frac{p^{(a+\epsilon)m}}{p^{(a+1+2\epsilon)m}} \leq \sum_{m=2}^{\infty} \frac{1}{p^{(1+\epsilon)m}} < 1.$$

Similarly, we obtain

$$\left|\sum_{m=2}^{\infty} \frac{b_{p^m}}{p^{ms}}\right| \ll \sum_{m=2}^{\infty} \frac{p^{(a+\epsilon)m}}{p^{m\gamma}} < 1.$$

Using assumption (iii), we conclude that for all but finitely many primes p,

$$\left(1 + \frac{a_p}{p^s} + \sum_{m=2}^{\infty} \frac{a_{p^m}}{p^{ms}}\right)\left(1 + \frac{b_p}{p^s} + \sum_{m=2}^{\infty} \frac{b_{p^m}}{p^{ms}}\right)^{-1} = 1 + \frac{(a_{p^2} - b_{p^2})}{p^{2s}} + \cdots + \frac{c_{p^m}}{p^{ms}},$$

where $c_n \ll n^{a+\epsilon}$. Hence, for $\gamma > \frac{1}{2} + a$, we have

$$\prod_p \left(1 + \sum_{m=2}^{\infty} \frac{a_{p^m}}{p^{ms}}\right) \left(1 + \sum_{m=2}^{\infty} \frac{b_{p^m}}{p^{ms}}\right)^{-1} \ll 1,$$

which implies

$$H(s) \ll 1.$$

Consequently, $H(s)$ is absolutely and uniformly convergent in the right half-plane $\mathrm{Re}(s) \geq a + \frac{1}{2} + \epsilon$.

Next, we compute the exponents of L-functions appearing in the decomposition of the relevant Dirichlet series. Note that the relation (1.3) implies

$$a_{\mathbb{K}_3}(n) = \sum_{m|n} \lambda_f(m) \tag{3.7}$$

and from (1.1), we see that $\tau_{k,\mathbb{K}_3}(p) = k a_{\mathbb{K}_3}(p)$ for all primes p. Additionally, from (3.7), we have $a_{\mathbb{K}_3}(p) = 1 + \lambda_f(p)$ for all primes p. Thus, we obtain

$$\tau^{\ell}_{k,\mathbb{K}_3}(p) = (k a_{\mathbb{K}_3}(p))^{\ell} = k^{\ell}(1 + \lambda_f(p))^{\ell} = k^{\ell} \sum_{i=0}^{\ell} \binom{\ell}{i} \lambda_f^i(p).$$

Moreover, from [PV24], we have the identity

$$\lambda_f^r(p) = \sum_{n=0}^{\lfloor \frac{r}{2} \rfloor} \left(\binom{r}{n} - \binom{r}{n-1} \right) \lambda_{\mathrm{sym}^{r-2n} f}(p).$$

This allows us to express $\tau^{\ell}_{k,\mathbb{K}_3}(p)$ as

$$\tau^{\ell}_{k,\mathbb{K}_3}(p) = k^{\ell} \left(a_{0,\ell} + a_{1,\ell} \lambda_f(p) + \sum_{i=2}^{\ell} a_{i,\ell} \lambda_{\mathrm{sym}^i f}(p) \right), \tag{3.8}$$

where the coefficients $a_{0,\ell}$ and $a_{3,\ell}$ are given by

$$a_{0,\ell} = \begin{cases} 1 + \sum_{i=1}^{\frac{\ell}{2}} \binom{\ell}{2i} \frac{(2i)!}{i!(i+1)!} & \ell \text{ even}, \\ 1 + \sum_{i=1}^{\frac{\ell-1}{2}} \binom{\ell}{2i} \frac{(2i)!}{i!(i+1)!} & \ell \text{ odd}, \end{cases} \quad \text{and} \quad a_{3,\ell} = \begin{cases} \sum_{i=1}^{\frac{\ell}{2}-1} \binom{\ell}{2i+1} \frac{4(2i+1)!}{(i-1)!(i+3)!} & \ell \text{ even}, \\ \sum_{i=1}^{\frac{\ell-1}{2}} \binom{\ell}{2i+1} \frac{4(2i+1)!}{(i-1)!(i+3)!} & \ell \text{ odd}. \end{cases} \tag{3.9}$$

The remaining coefficients $a_{i,\ell}$ can be computed easily; however, for our purposes, we only require $a_{0,\ell}$ and $a_{3,\ell}$.

A closed formula for $r_8(n)$. Representing an integer n as a sum of m squares is an additive number theory problem, yet for an even integer m, the count often involves divisor sums–central objects in multiplicative number theory. As m increases, there exists a natural embedding of solution sets:

$$R_m(n) \hookrightarrow R_{m'}(n) \quad \text{for even } m < m',$$

via

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0),$$

padding with zeros. This preserves the sum-of-squares identity, so every representation in $R_m(n) \subset \mathbb{Z}^m$ canonically extends to higher dimensions. Consequently, the number of representations grows:

$$0 \leq r_2(n) < r_4(n) < r_6(n) < \cdots,$$

reflecting the increased degrees of freedom. Notably, Jacobi's four-square theorem guarantees $r_4(n) > 0$ for all $n \geq 1$.

From [Nat00, Page 445], we have

$$16n^3 < r_8(n) < \frac{128\,\zeta(3)}{7}n^3.$$

This implies $r_8(n) \ll n^{3+\epsilon}$. In fact, for all $n \geq 1$, we have an exact expression for $r_8(n)$ given in [Nat00, Theorem 14.7] as

$$r_8(n) = 16\sum_{d|n}(-1)^{n+d}d^3.$$

This shows that the function

$$r(n) := \frac{r_8(n)}{16} \tag{3.10}$$

is multiplicative and satisfies

$$r(p) = \begin{cases} -1+p^3, & \text{if } p = 2, \\ 1+p^3, & \text{otherwise.} \end{cases} \tag{3.11}$$

Now, from (1.4) and (3.10), we write

$$S_{k,\ell}(X) = \sum_{n\leq X} \tau^{\ell}_{k,\mathbb{K}_3}(n) \cdot r_8(n) = 16 \sum_{n\leq X} \tau^{\ell}_{k,\mathbb{K}_3}(n) \cdot r(n).$$

We proceed to express the Dirichlet series corresponding to $\tau^{\ell}_{k,\mathbb{K}_3}(n) \cdot r(n)$ as a product of general L-functions of lower degrees. This decomposition will play a crucial role in the subsequent analysis. Notice that $\tau^{\ell}_{k,\mathbb{K}_3}(n) \cdot r(n) \ll n^{3+\epsilon}$. Let us consider the Dirichlet series

$$R_{k,\ell}(s) = \sum_{n=1}^{\infty} \frac{\tau^{\ell}_{k,\mathbb{K}_3}(n) \cdot r(n)}{n^s}$$

for $\mathrm{Re}(s) > 4$.

Lemma 3.2. *For the Dirichlet series $R_{k,\ell}(s)$, we have*

$$R_{k,\ell}(s) = \begin{cases} L_{k,2}(s)L_{k,2}(s-3)H_{k,2}(s) & \textit{for } \ell = 2, \\ L_{k,3}(s)L_{k,3}(s-3)H_{k,3}(s) & \textit{for } \ell = 3, \\ L_{k,\ell}(s)L_{k,\ell}(s-3)H_{k,\ell}(s) & \textit{for } \ell \geq 4, \end{cases}$$

where

$$L_{k,2}(s) = \zeta(s)^{2k^2} L(f,s)^{2k^2} L(\mathrm{sym}^2 f, s)^{k^2}, \tag{3.12}$$

$$L_{k,3}(s) = \zeta(s)^{4k^3} L(\mathrm{sym}^3 f, s)^{k^3} L(f,s)^{5k^3} L(\mathrm{sym}^2 f, s)^{3k^3}, \tag{3.13}$$

$$L_{k,\ell}(s) = \zeta(s)^{k^{\ell} a_{0,\ell}} L(\mathrm{sym}^3 f, s)^{k^{\ell} a_{3,\ell}} \prod_{\substack{1\leq t_1\leq \ell \\ t_1\neq 3}} L(\mathrm{sym}^{t_1} f, s)^{a_{k,t_1,\ell}}, \tag{3.14}$$

for a suitable constant $a_{k,t_1,\ell}$, with $a_{0,\ell}$ and $a_{3,\ell}$ given in (3.9). Moreover, $H_{k,\ell}(s)$ converges absolutely and uniformly in the right half-plane $Re(s) > \frac{7}{2}$, and remains nonzero when $Re(s) = 4$.

Proof. Since $\tau^{\ell}_{k,\mathbb{K}_3}(n) \cdot r(n)$ is a multiplicative function, $R_{k,\ell}(s)$ has the Euler product

$$R_{k,\ell}(s) = \prod_p \left(1 + \frac{\tau^{\ell}_{k,\mathbb{K}_3}(p)r(p)}{p^s} + \sum_{j=2}^{\infty} \frac{\tau^{\ell}_{k,\mathbb{K}_3}(p^j)r(p^j)}{p^{js}}\right)$$

for any $\ell \geq 2$. Let $\ell = 3$. From (3.8) and (3.11), we have following for any prime $p \neq 2$:

$$\begin{aligned}
\tau^{3}_{k,\mathbb{K}_3}(p) \cdot r(p) &= k^3 a^3_{\mathbb{K}_3}(p) \cdot r(p) \\
&= k^3(4 + 5\lambda_f(p) + 3\lambda_{\mathrm{sym}^2 f}(p) + \lambda_{\mathrm{sym}^3 f}(p))(1 + p^3) \\
&= 4k^3 + 5k^3\lambda_f(p) + 3k^3\lambda_{\mathrm{sym}^2 f}(p) + k^3\lambda_{\mathrm{sym}^3 f}(p) + 4k^3p^3 + 5k^3\lambda_f(p)p^3 + 3k^3\lambda_{\mathrm{sym}^2 f}(p)p^3 \\
&\quad + k^3\lambda_{\mathrm{sym}^3 f}(p)p^3 \\
&=: c(p).
\end{aligned}$$

In the Euler product of $R_{k,\ell}(s)$, we can separate the product over $p = 2$ and $p \neq 2$.

For $\mathrm{Re}(s) > 4$ the product

$$L_{k,3}(s)L_{k,3}(s-3), \qquad L_{k,3}(s) := \zeta(s)^{4k^3} L(\mathrm{sym}^3 f, s)^{k^3} L(f, s)^{5k^3} L(\mathrm{sym}^2 f, s)^{3k^3},$$

has an Euler product and hence a local expansion at each prime. Writing the local factor as a power series in p^{-s}, we obtain for every prime p the expansion

$$(\text{local factor at } p) \;=\; 1 + \frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \cdots,$$

where the coefficients $c(p^{\alpha})$ are the coefficients of $p^{-\alpha s}$ in that local expansion. Thus, the Euler product may be written as

$$L_{k,3}(s)L_{k,3}(s-3) = \prod_p \left(1 + \frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \cdots\right),$$

and the Dirichlet series $\sum_{n\geq 1} c(n)n^{-s}$ with multiplicatively defined coefficients $c(n) = \prod_{p^{\alpha}\| n} c(p^{\alpha})$ equals this product in its region of absolute convergence. Comparing the linear coefficients $c(p)$ with the coefficients appearing in $R_{k,3}(s)$ (computed above for all but finitely many primes) and applying Lemma 3.1 yields

$$R_{k,3}(s) = L_{k,3}(s)L_{k,3}(s-3)\, H_{k,3}(s),$$

with $H_{k,3}(s)$ absolutely and uniformly convergent in $\mathrm{Re}(s) > 7/2$ and nonzero at $\mathrm{Re}(s) = 4$. The same argument applies to the case $\ell = 2$.

Now let $\ell \geq 4$ be an integer. We can write

$$\tau^{\ell}_{k,\mathbb{K}_3}(p)\, r(p) = a_{k,0,\ell} + a_{k,1,\ell}\lambda_f(p) + \sum_{i=2}^{\ell} a_{k,i,\ell}\lambda_{\mathrm{sym}^i f}(p) + a_{k,0,\ell}\, p^3 + a_{k,1,\ell}\lambda_f(p)p^3 + \sum_{i=2}^{\ell} a_{k,i,\ell}\lambda_{\mathrm{sym}^i f}(p)p^3,$$

where $a_{k,0,\ell} = k^{\ell} a_{0,\ell}$ and $a_{k,3,\ell} = k^{\ell} a_{3,\ell}$. Analogously to the case $\ell = 3$, we define

$$\widetilde{c}(p) := a_{k,0,\ell} + a_{k,1,\ell}\lambda_f(p) + \sum_{i=2}^{\ell} a_{k,i,\ell}\lambda_{\mathrm{sym}^i f}(p) + a_{k,0,\ell}\, p^3 + a_{k,1,\ell}\lambda_f(p)p^3 + \sum_{i=2}^{\ell} a_{k,i,\ell}\lambda_{\mathrm{sym}^i f}(p)p^3.$$

Proceeding in the same manner as in the case $\ell = 3$, we obtain the desired decomposition formula for $R_{k,\ell}(s)$.

4. Proofs of Main Results

Before proceeding with the proofs, we first outline the general framework necessary for deriving asymptotic formulas for $S_{k,\ell}(X)$ with $\ell \geq 2$.

Preparation for the Proofs of Theorems 1.1 and 1.2 Note that $\tau_{k,\mathbb{K}_3}^{\ell}(n) \cdot r(n) \ll n^{3+\epsilon}$. Using the truncated Perron's formula given in [Mur08, Ex. 4.4.16, Pg 67], we relate the partial sum of $\tau_{k,\mathbb{K}_3}^{\ell}(n) \cdot r(n)$ to its associated Dirichlet series, yielding

$$\sum_{n \leq X} \tau_{k,\mathbb{K}_3}^{\ell}(n) \cdot r(n) = \frac{1}{2\pi\iota} \int_{4+\epsilon-\iota T}^{4+\epsilon+\iota T} R_{k,\ell}(s) \frac{X^s}{s}\, ds + \mathcal{O}\left(\frac{X^{4+\epsilon}}{T}\right),$$

where $1 \leq T \leq X$, and T will be chosen later. Let $7/2 < \gamma_0 = \mathrm{Re}(s) < 4$ be a real number. Shifting the line of integration from $\mathrm{Re}(s) = 4 + \epsilon$ to $\mathrm{Re}(s) = \gamma_0 = \frac{7}{2} + \epsilon$ and applying Cauchy's residue theorem, we obtain

$$\begin{aligned}
\sum_{n \leq X} \tau_{k,\mathbb{K}_3}^{\ell}(n) \cdot r(n) &= \mathrm{Res}_{s=4}\left\{ R_{k,\ell}(s) \frac{X^s}{s} \right\} \\
&\quad + \frac{1}{2\pi\iota} \int_{\gamma_0-\iota T}^{\gamma_0+\iota T} R_{k,\ell}(s) \frac{X^s}{s}\, ds \\
&\quad + \frac{1}{2\pi\iota} \left\{ \int_{4+\epsilon-\iota T}^{\gamma_0-\iota T} + \int_{\gamma_0+\iota T}^{4+\epsilon+\iota T} \right\} R_{k,\ell}(s) \frac{X^s}{s}\, ds + \mathcal{O}\left(\frac{X^{4+\epsilon}}{T}\right) \\
&=: \mathrm{Res}_{s=4}\left\{ R_{k,\ell}(s) \frac{X^s}{s} \right\} + \mathcal{V}_{k,\ell} + \mathcal{H}_{k,\ell} + \mathcal{O}\left(\frac{X^{4+\epsilon}}{T}\right).
\end{aligned} \tag{4.15}$$

Since the order of the pole at $s = 4$ of $R_{k,\ell}(s)$ is $k^{\ell}(a_{0,\ell} + a_{3,\ell})$ (see Remark 2.1), we deduce

$$\mathrm{Res}_{s=4} R_{k,\ell}(s) \frac{X^s}{s} = X^4 P_{k,\ell}(\log X), \tag{4.16}$$

where $P_{k,\ell}(X)$ is a polynomial in X of degree $k^{\ell}(a_{0,\ell} + a_{3,\ell}) - 1$. Now, substituting the decomposition $R_{k,\ell}(s) = L_{k,\ell}(s) L_{k,\ell}(s-3) H_{k,\ell}(s)$ and using the absolute convergence of $H_{k,\ell}(s)$ in $\mathrm{Re}(s) \geq 7/2 + \epsilon$, the contributions of the horizontal and vertical integrals are

$$\begin{aligned}
|\mathcal{H}_{k,\ell}| &\ll \max_{1/2+\epsilon \leq \gamma \leq 1+\epsilon} \left(\frac{X^{\gamma+3}}{T} |L_{k,\ell}(\gamma + \iota T)| \right), \\
|\mathcal{V}_{k,\ell}| &\ll X^{(7/2)+\epsilon} + X^{(7/2)+\epsilon} \max_{1 \leq T_1 \leq \frac{T}{2}} \left(\frac{1}{T_1} \int_{T_1}^{2T_1} |L_{k,\ell}(\frac{1}{2} + \epsilon + \iota t)| dt \right).
\end{aligned} \tag{4.17}$$

Thus, (4.15) and (4.16) imply

$$\sum_{n \leq X} \tau_{k,\mathbb{K}_3}^{\ell}(n) \cdot r(n) = X^4 P_{k,\ell}(\log X) + \mathcal{O}(\mathcal{H}_{k,\ell}) + \mathcal{O}(\mathcal{V}_{k,\ell}) + \mathcal{O}\left(\frac{X^{4+\epsilon}}{T}\right). \tag{4.18}$$

Therefore, it remains to estimate $|\mathcal{H}_{k,\ell}|$ and $|\mathcal{V}_{k,\ell}|$ and choose T appropriately to derive the required result.

4.A. Proof of Theorem 1.1

Let $\ell = 2$. From (3.12) in Lemma 3.2, we have

$$L_{k,2}(s-3) = \zeta(s-3)^{2k^2} L(f, s-3)^{2k^2} L(\mathrm{sym}^2 f, s-3)^{k^2}.$$

Using Lemmas 2.2-2.4, Hölder's inequality, and following (4.17), we obtain

$$|\mathcal{H}_{k,2}| \ll \frac{X^{4+\epsilon}}{T} + X^{\frac{7}{2}+\epsilon} T^{\frac{331}{210}k^2-1+\epsilon} \tag{4.19}$$

and

$$\begin{aligned}
|\mathcal{V}_{k,2}| &\ll X^{\frac{7}{2}+\epsilon} + X^{\frac{7}{2}+\epsilon} \max_{1\leq T_1 \leq \frac{T}{2}} \Bigg\{ T_1^{-1} \max_{T_1 \leq t \leq 2T_1} \left|\zeta\left(\frac{1}{2}+\epsilon+\iota t\right)\right|^{2k^2-6} \left|L\left(f, \frac{1}{2}+\epsilon+\iota t\right)\right|^{2k^2-3} \\
&\quad \times \left|L\left(\mathrm{sym}^2 f, \frac{1}{2}+\epsilon+\iota t\right)\right|^{k^2} \\
&\quad \times \left(\int_{T_1}^{2T_1} \left|\zeta\left(\frac{1}{2}+\epsilon+\iota t\right)\right|^{12} dt\right)^{\frac{1}{2}} \left(\int_{T_1}^{2T_1} \left|L\left(f, \frac{1}{2}+\epsilon+\iota t\right)\right|^{6} dt\right)^{\frac{1}{2}} \Bigg\} \\
&\ll X^{\frac{7}{2}+\epsilon} + X^{\frac{7}{2}+\epsilon} T^{\frac{331}{210}k^2+\frac{1}{14}-1+\epsilon} \\
&\ll X^{\frac{7}{2}+\epsilon} T^{\frac{331}{210}k^2+\frac{1}{14}-1+\epsilon}.
\end{aligned} \tag{4.20}$$

Plugging (4.19) and (4.20) in (4.18) for $\ell = 2$, we obtain

$$\sum_{n\leq X} \tau_{k,\mathbb{K}_3}^2(n) \cdot r(n) = X^4 P_{k,2}(\log X) + \mathcal{O}\left(\frac{X^{4+\epsilon}}{T}\right) + \mathcal{O}\left(X^{\frac{7}{2}+\epsilon} T^{\frac{331}{210}k^2+\frac{1}{14}-1+\epsilon}\right).$$

We choose

$$1 \leq T = X^{\frac{105}{331k^2+15}} < X,$$

so that

$$S_{k,2}(X) = X^4 P_{k,2}(\log X) + \mathcal{O}\big(X^{4-\frac{105}{331k^2+15}+\epsilon}\big),$$

where $P_{k,2}(t)$ is a polynomial in t of degree $2k^2-1$.

Let $\ell = 3$. From (3.13) in Lemma 3.2, we have

$$L_{k,3}(s-3) = \zeta(s-3)^{4k^3} L(f, s-3)^{5k^3} L(\mathrm{sym}^2 f, s-3)^{3k^3} L(\mathrm{sym}^3 f, s-3)^{k^3}.$$

Using Lemmas 2.2-2.4 and Hölder's inequality, the contribution of $|\mathcal{H}_{k,3}|$, following (4.17), is

$$\begin{aligned}
|\mathcal{H}_{k,3}| &\ll \int_{\frac{1}{2}+\epsilon}^{1+\epsilon} \left\{ \frac{|\zeta(\gamma+\iota T)^{4k^3} L(f,\gamma+\iota T)^{5k^3} L(\mathrm{sym}^2 f, \gamma+\iota T)^{3k^3} L(\mathrm{sym}^3 f, \gamma+\iota T)^{k^3}|}{T} \right\} X^{\gamma+3} d\gamma \\
&\ll \frac{X^{4+\epsilon}}{T} + X^{\frac{7}{2}+\epsilon} T^{\frac{178}{35}k^3-1+\epsilon}
\end{aligned}$$

and the contribution of $|\mathcal{V}_{k,3}|$ is

$$\begin{aligned}
&\ll X^{\frac{7}{2}+\epsilon} + X^{\frac{7}{2}+\epsilon} \max_{1\leq T_1 \leq \frac{T}{2}} \Bigg\{ T_1^{-1} \max_{T_1 \leq t \leq 2T_1} \left|\zeta\left(\frac{1}{2}+\epsilon+\iota t\right)\right|^{4k^3-6} \left|L\left(f, \frac{1}{2}+\epsilon+\iota t\right)\right|^{5k^3-3} \\
&\quad \times \left|L\left(\mathrm{sym}^2 f, \frac{1}{2}+\epsilon+\iota t\right)\right|^{3k^3} \left|L\left(\mathrm{sym}^3 f, \frac{1}{2}+\epsilon+\iota t\right)\right|^{k^3} \\
&\quad \times \left(\int_{T_1}^{2T_1} \left|\zeta\left(\frac{1}{2}+\epsilon+\iota t\right)\right|^{12} dt\right)^{\frac{1}{2}} \left(\int_{T_1}^{2T_1} \left|L\left(f, \frac{1}{2}+\epsilon+\iota t\right)\right|^{6} dt\right)^{\frac{1}{2}} \Bigg\} \\
&\ll X^{\frac{7}{2}+\epsilon} + X^{\frac{7}{2}+\epsilon} T^{\frac{178}{35}k^3+\frac{31}{140}-1+\epsilon} \\
&\ll X^{\frac{7}{2}+\epsilon} T^{\frac{178}{35}k^3+\frac{1}{14}-1+\epsilon}.
\end{aligned}$$

Therefore,

$$\sum_{n\le X} \tau^3_{k,\mathbb{K}_3}(n)\cdot r(n) = X^4 P_{k,3}(\log X) + \mathcal{O}\left(\frac{X^{4+\epsilon}}{T}\right) + \mathcal{O}\left(X^{\frac{7}{2}+\epsilon} T^{\frac{178}{35}k^3+\frac{1}{14}-1+\epsilon}\right),$$

where $P_{k,3}(t)$ is a polynomial of degree $5k^3-1$.
Choosing

$$T = X^{\frac{35}{256k^3+5}},$$

we get

$$S_{k,3}(X) = X^4 P_{k,3}(\log X) + \mathcal{O}\big(X^{4-\frac{35}{256k^3+5}+\epsilon}\big).$$

This completes the proof of Theorem 1.1. ☐

4.B. Proof of Theorem 1.2

From Lemma 3.2 and (3.14), we have

$$L_{k,\ell}(s-3) = \zeta(s-3)^{k^\ell a_{0,\ell}} \prod_{1\le t_1\le \ell} L(\mathrm{sym}^{t_1} f, s-3)^{a_{k,t_1,\ell}}.$$

Define

$$L^*_{k,\ell}(s-3) := \prod_{1\le t_1\le \ell} L(\mathrm{sym}^{t_1} f, s-3)^{a_{k,t_1,\ell}},$$

which has degree $k^\ell(3^\ell - a_{0,\ell})$. Using Lemmas 2.2-2.4 and Hölder's inequality, the contribution of $|\mathcal{H}_{k,\ell}|$ (following (4.17)) is

$$\begin{aligned} |\mathcal{H}_{k,\ell}| &\ll \int_{\frac{1}{2}+\epsilon}^{1+\epsilon} \frac{\left|\zeta(\gamma+\iota T)^{k^\ell a_{0,\ell}} L^*_{k,\ell}(\gamma+\iota T)\right| X^{\gamma+3} d\gamma}{T} \\ &\ll \frac{X^{4+\epsilon}}{T} + X^{\frac{7}{2}+\epsilon} T^{\frac{(3k)^\ell}{4} - \frac{2k^\ell a_{0,\ell}}{21} - 1 + \epsilon} \end{aligned}$$

and

$$\begin{aligned} |\mathcal{V}_{k,\ell}| \ll X^{\frac{7}{2}+\epsilon} + X^{\frac{7}{2}+\epsilon} \max_{1\le T_1\le \frac{T}{2}} &\left\{ T_1^{-1} \max_{T_1\le t\le 2T_1} \left|\zeta\left(\frac{1}{2}+\epsilon+\iota t\right)\right|^{k^\ell a_{0,\ell}-6} \right. \\ &\left. \times \left(\int_{T_1}^{2T_1} \left|\zeta\left(\frac{1}{2}+\epsilon+\iota t\right)\right|^{12} dt\right)^{\frac{1}{2}} \left(\int_{T_1}^{2T_1} \left|L^*_{k,\ell}(\frac{1}{2}+\epsilon+\iota t)\right|^2 dt\right)^{\frac{1}{2}} \right\} \\ \ll X^{\frac{7}{2}+\epsilon} T^{\frac{(3k)^\ell}{4} - \frac{2k^\ell a_{0,\ell}}{21} + \frac{1}{14} - 1 + \epsilon}. \end{aligned}$$

Therefore, we conclude

$$\sum_{n\le X} \tau^\ell_{k,\mathbb{K}_3}(n)\cdot r(n) = X^4 P_{k,\ell}(\log X) + \mathcal{O}\left(\frac{X^{4+\epsilon}}{T}\right) + \mathcal{O}\left(X^{\frac{7}{2}+\epsilon} T^{\frac{(3k)^\ell}{4} - \frac{2k^\ell a_{0,\ell}}{21} + \frac{1}{14} - 1 + \epsilon}\right).$$

We choose

$$T = X^{\frac{42}{k^\ell(7\times 3^{\ell+1} - 8a_{0,\ell})+6}},$$

which leads to

$$S_{k,\ell}(X) = X^4 P_{k,\ell}(\log X) + \mathcal{O}\left(X^{4-\frac{42}{k^\ell(7\times 3^{\ell+1}-8a_{0,\ell})+6}+\epsilon}\right),$$

where $P_{k,\ell}(t)$ is a polynomial of degree $k^\ell(a_{0,\ell}+a_{3,\ell})-1$. This completes the proof. ☐

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