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Omega Estimate for the Lattice Point Discrepancy of a Body of Revolution Using The Resonance Method

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Abstract. Using a recent method developed by Mahatab, we obtain an improved Ω-bound for the error term arising in lattice counting problem of bodies of revolution in $\mathbb{R}^3$ around a coordinate axis and having smooth boundary with bounded nonzero curvature. This strengthens an earlier result by Kühleitner and Nowak.

Keywords. Omega Bound, Lattice Point

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1. Introduction

The classical Gauss circle problem is to find the number of points with integer coordinates inside a circle. This is equivalent to the number of representations of an integer as the sum of two squares. This problem extends to any closed domain in Euclidean spaces and has connections with number theory. An extensive study can be found in [Krä88]. As a special case, we may consider large homothetic smooth convex bodies. Throughout this article we suppose $\mathfrak{B} \subset \mathbb{R}^3$ is a compact convex body with the origin $(0,0,0) \in \operatorname{int}(\mathfrak{B})$ and the boundary $\partial\mathfrak{B}$ is in class C^∞ with positive Gaussian curvature. We consider the number of lattice points in the "blown up" body $\sqrt{t}\mathfrak{B}$, for a large real t. Therefore, we define

$$\mathcal{N}_{\mathfrak{B}} := \#\{\mathbf{m} \in \mathbb{Z}^3 : \mathbf{m} \in \sqrt{t}\mathfrak{B}\}.$$

Further, we define its *lattice point discrepancy* as

$$P_{\mathfrak{B}}(t) := \mathcal{N}_{\mathfrak{B}} - \operatorname{vol}(\mathfrak{B})t^{3/2}. \tag{1.1}$$

In this direction, Hlawka [Hla50] was the first to study the asymptotic behavior of $\mathcal{N}_{\mathfrak{B}}$ for a general convex body $\mathfrak{B}$. Assuming the body $\mathfrak{B}$ to be a body of revolution with respect to any one of the coordinate axes, Chamizo [Cha98] showed that

$$P_{\mathfrak{B}}(t) \ll t^{11/16}.$$

Extensions of the above asymptotic result to higher dimensions can be found in [Guo15, Müh98]. In the same paper, Hlawka [Hla50] also studied Ω-bound for $P_{\mathfrak{B}}$. Through a series of papers, Kühleitner and Nowak [Küh00, KüNo04, Now85][1] improved the Ω-bound. The final improvement in [KüNo04] is obtained using Soundararajan's [Sou03, Lemma 2.1] method, which proves

$$P_{\mathfrak{B}}(t) = \Omega_-\Big(t^{1/2}(\log t)^{1/3}(\log_2 t)^{\frac{2}{3}(\sqrt{2}-1)}(\log_3 t)^{-2/3}\Big). \tag{1.2}$$

Recently, Mahatab [Mah25] has obtained a sharper Ω-bound for the circle problem and the divisor problems using the resonance method, inspired from a resonator used by Aistleitner, Mahatab and Munsch [AMM19]. In this paper, we use Mahatab's resonator [Mah25, Theorem 6] to obtain an improved lower bound for (1.2). In fact, we improve on the power of $\log_3 t$ from $-2/3$ to $-1/3$.

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[1] Throughout the article, we use $\log_r x = \log\log\ldots\log x$ (r times).

Theorem 1.1. *Let $\mathfrak{B}$ be a compact, convex body in $\mathbb{R}^3$ whose interior contains the origin and is invariant under rotations around one of the coordinate axes. Assume that its boundary $\partial\mathfrak{B}$ is in C^∞ with positive bounded curvature. Then for all sufficiently large t, we have*

$$P_{\mathfrak{B}}(t) = \Omega_-\left(t^{1/2}(\log t)^{1/3}(\log_2 t)^{\frac{2}{3}(\sqrt{2}-1)}(\log_3 t)^{-1/3}\right).$$

In the next section, we collect all required tools to prove Theorem 1.1.

2. Preliminaries

First, we mention a theorem, due to Mahatab [Mah25], as the most important tool for our result. Let $(\lambda_n)_{n=1}^{\infty}$ be a non-decreasing sequence of positive real numbers and α be a positive real parameter. We consider a finite linearly independent set $\mathcal{M}$ over $\mathbb{Q}$ such that $\mathcal{M} \subseteq \{\lambda_n : C_0\alpha \leq \lambda_n \leq 2\alpha\}$, with $0 < C_0 < 2$. We denote the cardinality of the set $\mathcal{M}$ by M. Also, given A_1, we consider real numbers A_2, A_3, and A_4 satisfying $0 < A_4 < A_3 < A_2 < A_1$.

Theorem 2.1. (K. Mahatab) *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers. Then for large T, we have*

$$\max_{T^{A_3/2} \leq t \leq 2A_2^2 T^{A_2}\log^2 T}\left(\sum_{n\leq T^{A_1}} a_n\cos(t\lambda_n)\right) \geq \frac{\pi}{4e}\sum_{n\in\mathcal{M}} a_n + O\left(T^{A_3-A_2}e^{2M/C_0}\left(\sum_{\lambda_n\leq 4\alpha} a_n\right)\right) + O\left(\frac{T^{-A_4}}{\alpha}\sum_{n\leq T^{A_1}} a_n\right).$$

Now we define the *tac function* H for a convex body $\mathfrak{B}$ satisfying the conditions defined before.

Definition 1. *For $\mathbf{u} \in \mathbb{R}^3$, the tac function H of a convex body $\mathfrak{B}$ is defined by*

$$H(\mathbf{v}) = \max_{\mathbf{u}\in\mathfrak{B}}\langle\mathbf{u},\mathbf{v}\rangle,$$

where $\langle,\rangle$ denotes the usual Euclidean inner product. The following properties are immediate.

1. *H is a real positive homogeneous function of degree 1.*
2. *For any $\mathbf{u} \in \mathbb{R}^3$, there exist positive constants a and b such that*
$$a\,\|\mathbf{u}\| \leq H(\mathbf{u}) \leq b\,\|\mathbf{u}\|, \tag{2.3}$$
where $\|\cdot\|$ denotes the Euclidean norm.
3. *For $(u_1, u_2, u_3) \in \mathbb{R}^3$,*
$$H(u_1, u_2, u_3) = H\left(\sqrt{u_1^2+u_2^2}, 0, u_3\right). \tag{2.4}$$

The proof of Theorem 1.1 also uses the next lemma, which can be obtained by taking $s = 3$ in [Now86, Eq. (13)]. For more information, the reader is suggested to see [KüNo04, Section 2].

Lemma 2.2. *For a body $\mathfrak{B}$ satisfying the conditions stated before and a large real parameter t, the Borel mean value of the lattice point discrepancy $P_{\mathfrak{B}}$ is defined as*

$$B(t) := \frac{1}{\Gamma(k+1)} \int_0^\infty e^{-u} u^k P_{\mathfrak{B}}(Xu)\,\mathrm{d}u, \tag{2.5}$$

where $X = X(t) = 1/(\log t)$ and $k = k(t) = t^2 \log t$. Then $B(t)$ has the following asymptotic formula:

$$B(t) = -\frac{1}{2\pi} tS(t) + O(t^{3/4+\epsilon}), \tag{2.6}$$

with

$$S(t) := \sum_{0<\|\mathbf{m}\|\le t^\epsilon \sqrt{\log t}} \frac{\theta(\mathbf{m})}{\|\mathbf{m}\|^2} \exp\Big(-\frac{1}{2}\pi^2 X H(\mathbf{m})^2\Big) \cos\Big(2\pi H(\mathbf{m})t\Big), \tag{2.7}$$

where the coefficients $\theta(\mathbf{m})$ are positive and uniformly bounded from above and $\mathbf{m} \in \mathbb{Z}^3$.

3. Proof of Theorem 1.1

For $\mathbf{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$, we set $l = m_1^2 + m_2^2$ and construct a bijection $\psi : \mathbb{N} \to \mathbb{N} \times \mathbb{Z}$ as $n \mapsto \psi(n) = (l, m_3)$, such that the sequence $(\lambda_n)_{n=1}^\infty$ defined by

$$\lambda_n := H(\mathbf{m})\Big|_{(l,m_3)=\psi(n)} = H(\sqrt{l}, 0, m_3)\Big|_{(l,m_3)=\psi(n)}$$

is non-decreasing. Implementing (2.4) in (2.7), one has

$$S(t) := \sum_{0<l+m_3^2\le t^{2\epsilon}\log t} \frac{f(l,m_3)}{l+m_3^2} \exp\Big(-\frac{1}{2}\pi^2 X H(\sqrt{l},0,m_3)^2\Big) \cos\Big(2\pi H(\sqrt{l},0,m_3)t\Big),$$

where

$$f(l, m_3) := \sum_{m_1^2+m_2^2=l} \theta(m_1, m_2, m_3) \asymp r(l), \tag{3.8}$$

and $r(l)$ denotes the number of ways of writing l as sum of two squares. Now we define a sequence $(a_n)_{n=1}^\infty$ as

$$a_n := \begin{cases} \dfrac{f(l,m_3)}{l+m_3^2} \exp\left(-\dfrac{1}{2}\pi^2 X H(\sqrt{l},0,m_3)^2\right)\Big|_{(l,m_3)=\psi(n)}, & \text{if } l+m_3^2 \le t^{2\epsilon}\log t, \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

We assume a set $\mathcal{M}$ such that $\{\lambda_n : n \in \mathcal{M}\} \subset [C_0\alpha, 2\alpha]$, where $0 < C_0 < 2$. Also, we choose $A_1 = 2$, $A_2 = 3/2, A_3 = 1$ and $A_4 = 7/8$. By Theorem 2.1, there exists t satisfying $\sqrt{T} \le t \le 5T^{3/2}\log^2 T$ such that

$$S(t) \ge \frac{\pi}{4e} \sum_{n\in\mathcal{M}} a_n + O\left(T^{-1/2} e^{2M/C_0} \left(\sum_{\lambda_n \le 4\alpha} a_n\right)\right) + O\left(\frac{T^{-7/8}}{\alpha} \sum_{n\le T^2} a_n\right), \tag{3.10}$$

where α is a real parameter to be determined later.

Since the *tac function* H is homogeneous, there exist constants $0 < c_1 < c_2$ and $0 < c_3 < c_4$ depending on $\mathfrak{B}$ such that the interval $[c_1, c_2] \times [c_3, c_4]$ in (u_1, u_3)-plane lies between the curves

$H(u_1, 0, u_3) = C_0$ and $H(u_1, 0, u_3) = 2$. Therefore, the condition $(\sqrt{l}, m_3) \in [c_1\alpha, c_2\alpha] \times [c_3\alpha, c_4\alpha]$ implies $H(\sqrt{l}, 0, m_3) \in [C_0\alpha, 2\alpha]$ for integers $l > 0$ and m_3.

Now we construct the following resonating set $\widehat{\mathcal{M}}$ such that $\mathcal{M}$ is the preimage of $\widehat{\mathcal{M}}$ under the map ψ:

$$\widehat{\mathcal{M}} = \left\{(l, m_3) \in \mathbb{N}^2 : \ c_1^2\alpha^2 \le l \le c_2^2\alpha^2, \ c_3\alpha \le m_3 \le c_4\alpha, \ \omega(l) = [\lambda \log_2 \alpha] \ \text{ and } l \in \mathcal{A}\right\},$$

where $\mathcal{A} := \{q \in \mathbb{N} : p \equiv 1(\bmod 4), \text{if } \ p|q \ \text{ and } p \text{ is prime; and } q \text{ is square free}\}$, $\omega(l)$ denotes the number of distinct prime factors of l and λ is a positive constant.

Now using (3.8) and the assumption $XH(\sqrt{l}, 0, m_3)^2 \ll 1$ in (3.9), we have

$$\begin{aligned}\sum_{n\in\mathcal{M}} a_n &\gg \frac{1}{\alpha^2} \sum_{c_3\alpha \le m_3 \le c_4\alpha} \ \sum_{\substack{c_1^2\alpha^2 \le l \le c_2^2\alpha^2 \\ \omega(l) = [\lambda \log_2 \alpha], \ l \in \mathcal{A}}} r(l) \\ &\gg \frac{1}{\alpha} \sum_{\substack{c_1^2\alpha^2 \le l \le c_2^2\alpha^2 \\ \omega(l) = [\lambda \log_2 \alpha], \ l \in \mathcal{A}}} r(l). \end{aligned} \tag{3.11}$$

Let S be the cardinality of the set

$$S_{\alpha,\Lambda} = \{l \in \mathbb{N} : a_1^2\alpha \le l \le a_2^2\alpha^2, l \in \mathcal{A} \text{ and } \omega(l) = \Lambda\}.$$

Using Sathe's [Sat54] (See also [Ten15, Section II.6]) and Stirling's formula, one can obtain

$$S \asymp \frac{\alpha^2}{\log \alpha} \frac{(\frac{1}{2}\log_2 \alpha)^{\Lambda - 1}}{(\Lambda - 1)!} \asymp \frac{\alpha^2}{\sqrt{\log_2 \alpha}} (\log \alpha)^{\lambda - 1 - \lambda \log \lambda - \lambda \log 2},$$

where we have chosen $\Lambda = [\lambda \log_2 \alpha]$ and therefore

$$M = |\mathcal{M}| = |\widehat{\mathcal{M}}| \asymp \frac{\alpha^3}{\sqrt{\log_2 \alpha}} (\log \alpha)^{\lambda - 1 - \lambda \log \lambda - \lambda \log 2}.$$

Note that, for $l \in \mathcal{A}$, $r(l) \ge 2^{w(l)} = 2^{[\lambda \log_2 \alpha]} \gg (\log \alpha)^{\lambda \log 2}$. Using this lower bound of $r(l)$ in (3.11), we have

$$\begin{aligned}\sum_{n\in\mathcal{M}} a_n &\gg \frac{(\log \alpha)^{\lambda \log 2}}{\alpha} \frac{\alpha^2}{\sqrt{\log_2 \alpha}} (\log \alpha)^{\lambda - 1 - \lambda \log \lambda - \lambda \log 2} \\ &= \frac{\alpha}{\sqrt{\log_2 \alpha}} (\log \alpha)^{\lambda - 1 - \lambda \log \lambda}. \end{aligned} \tag{3.12}$$

Now we choose α such that M is of order $\log T$. Indeed,

$$\alpha = \frac{1}{C} (\log T)^{1/3} (\log_2 T)^{\frac{1}{3}(1 - \lambda + \lambda \log \lambda + \lambda \log 2)} (\log_3 T)^{1/6},$$

where C is a large positive constant. Note that $\log \alpha \asymp \log_2 T$ and $\log_2 \alpha \asymp \log_3 T$. Since $X \ll (\log T)^{-1}$ and $H(\sqrt{l}, 0, m_3) \ll \alpha$, the assumption $XH(\sqrt{l}, 0, m_3)^2 \ll 1$ is now verified. Substituting the above choice of α in (3.12),

$$\sum_{n\in\mathcal{M}} a_n \gg (\log T)^{1/3} (\log_2 T)^{\frac{2}{3}(\lambda - 1 - \lambda \log \lambda) + \frac{1}{3}\lambda \log 2} (\log_3 T)^{-1/3}.$$

To optimize the exponent of $\log_2 T$, we choose $\lambda = \sqrt{2}$ and we obtain

$$\sum_{n\in\mathcal{M}} a_n \gg (\log T)^{1/3}(\log_2 T)^{\frac{2}{3}(\sqrt{2}-1)}(\log_3 T)^{-1/3}. \tag{3.13}$$

To estimate the error, first we note that

$$M \asymp \frac{\log T}{C^3}.$$

We choose C to be large enough satisfying

$$e^{2M/C_0} \ll T^{1/4-\epsilon}.$$

Now, using (2.3) and (2.4), we calculate[2]

$$\begin{aligned}\sum_{\lambda_n\le 4\alpha} a_n &\ll \sum_{0<H(\sqrt{l},0,m_3)\le 4\alpha} \frac{r(l)}{l+m_3^2}\\ &\le \sum_{0<a\|\mathbf{m}\|\le 4\alpha} \frac{1}{\|\mathbf{m}\|^2}\\ &= \sum_{n\le(16\alpha^2)/a^2} \frac{r_3(n)}{n}\\ &= \int_1^{(16\alpha^2)/a^2} \frac{1}{x}\,\mathrm{d}\left(\sum_{n\le x} r_3(n)\right)\\ &\ll \int_1^{(16\alpha^2)/a^2} \frac{1}{x^{1/2}}\mathrm{d}x \ll \alpha \ll T^{\epsilon},\end{aligned}$$

where we use the bound $\sum_{n\le x} r_3(n) \ll x^{3/2}$.

Similarly,

$$\begin{aligned}\sum_{n\le T^2} a_n &\ll \sum_{0<\|\mathbf{m}\|\le t^{\epsilon}\sqrt{\log t}} \frac{1}{\|\mathbf{m}\|^2}\\ &= \int_1^{t^{2\epsilon_0}\log t} \frac{1}{x}\,\mathrm{d}\left(\sum_{n\le x} r_3(n)\right)\\ &\ll t^{\epsilon}\sqrt{\log t} \ll T^{3\epsilon}.\end{aligned}$$

The above two estimates show that the two error terms in the right hand side of (3.10) are smaller than the main term (3.13). In fact,

$$\begin{aligned}\text{Error} &\ll T^{-1/2}e^{2M/C_0}\left(\sum_{\lambda_n\le 4\alpha} a_n\right) + \frac{T^{-7/8}}{\alpha}\left(\sum_{n\le T^2} a_n\right)\\ &\ll T^{-1/4+\epsilon} + T^{-7/8+\epsilon} \ll T^{-1/4+\epsilon}.\end{aligned}$$

Using (3.13) in (3.10), it follows that for large T, there exists t satisfying $\sqrt{T} \le t \le 5T^{3/2}\log^2 T$ such that

$$S(t) \gg (\log t)^{1/3}(\log_2 t)^{\frac{2}{3}(\sqrt{2}-1)}(\log_3 t)^{-1/3}.$$

[2] The quantity $r_3(n)$ denotes the number of ways of writing n as sum of three squares.

Then from (2.6), we have

$$-B(t) \gg t(\log t)^{1/3}(\log_2 t)^{\frac{2}{3}(\sqrt{2}-1)}(\log_3 t)^{-1/3}. \tag{3.14}$$

Let

$$\mathcal{F}(w) := (\log w)^{1/3}(\log_2 w)^{\frac{2}{3}(\sqrt{2}-1)}(\log_3 w)^{-1/3}.$$

We will prove our Ω-result by the method of contradiction. Suppose for any $\delta > 0$, there exists a constant C_1 such that

$$-P_{\mathfrak{B}}(w) \le C_1 + \delta w^{1/2}\mathcal{F}(w),$$

for all $w > 0$. Now applying (2.5) to get

$$\begin{aligned} -B(t) &\le \frac{1}{\Gamma(k+1)}\int_0^\infty e^{-w}w^k\left(C_1 + \delta w^{1/2}\mathcal{F}(w)\right)\mathrm{d}w \\ &= C_1 + \frac{\delta}{\Gamma(k+1)}\int_0^\infty e^{-w}w^k(Xw)^{1/2}\mathcal{F}(w)\mathrm{d}w, \end{aligned}$$

for all $t > 0$. Estimating the above infinite integral by Hafner's result [Haf82, Lemma 2.3.6], we obtain that

$$-B(t) \le C_1 + C_2\delta(kX)^{1/2}\mathcal{F}(kX) \le C_1 + C_2\,\delta\,t\,\mathcal{F}(t^2),$$

for some constant $C_2 > 0$ and for any $\delta > 0$. However, this contradicts (3.14), which proves our required Ω-result.

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References

[AMM19] C. Aistleitner, K. Mahatab and M. Munsch, *Extreme values of the Riemann zeta function on the 1-line*, Int. Math. Res. Not. **2019**, no. 22, 6924–6932.

[Cha98] F. Chamizo, *Lattice points in bodies of revolution*, Acta Arith. **85** (1998), no. 3, 265–277.

[Guo15] J. Guo, *Lattice points in rotated convex domains*, Rev. Mat. Iberoam. **31** (2015), no. 2, 411–438.

[Haf82] J. L. Hafner, *On the average order of a class of arithmetical functions*, J. Number Theory **15** (1982), 36–76.

[Hla50] E. Hlawka, *Über Integrale auf konvexen Körpern.* I, Monatsh. Math. **54** (1950), 1–36.

[Krä88] E. Krätzel, *Lattice points*, Mathematics and its Applications (East European Series), 33, Kluwer Acad. Publ., Dordrecht, 1988.

[Küh00] M. Kühleitner, *Lattice points in bodies of revolution in* $\mathbf{R}^3$*: an* Ω_-*-estimate for the error term*, Arch. Math. (Basel) **74** (2000), no. 3, 234–240.

[KüNo04] M. Kühleitner and W. G. Nowak, *The lattice point discrepancy of a body of revolution: improving the lower bound by Soundararajan's method,* Arch. Math. (Basel) **83** (2004), 208–216.

[Mah25] K. Mahatab, *Omega Results for The Divisor and Circle Problems Using The Resonance Method*, arXiv, 2025, arXiv: 2504.17032.

[Müh98] W. Müller, *Lattice points in large convex bodies*, Monatsh. Math. **128** (1999), no. 4, 315–330.

[Now85] W. G. Nowak, *On the lattice rest of a convex body in* $\mathbb{R}^s$, Arch. Math. (Basel) **45** (1985), no. 3, 284–288.

[Now86] W. G. Nowak, *On the lattice rest of a convex body in* $\mathbb{R}^s$*,* II. Arch. Math. **47** (1986), 232–237.

[Sat54] L. G. Sathe, *On a problem of Hardy on the distribution of integers having a given number of prime factors. III*, J. Indian Math. Soc. (N.S.) **18** (1954), 27–42.

[Sou03] K. Soundararajan, *Omega results for the divisor and circle problems*, Int. Math. Res. Not. **2003**, no. 36, 1987–1998.

[Ten15] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Graduate Studies in Mathematics 163, Amer. Math. Soc. 2015

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