Hardy-Ramanujan Journal 48 (2025) Part A, 61-76 submitted 02/03/2025, accepted 07/12/2025, revised 07/12/2025

Some determinants in the semi-stable Langlands program

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Abstract. We evaluate some determinants involving harmonic numbers that are needed in order to provide solutions to certain matrix equations occurring in our earlier paper [ChGh24]. That paper determined the mod p reductions of all two-dimensional semi-stable representations $V_{k,\mathcal{L}}$ of the Galois group of $\mathbb{Q}_p$ of weights $3 \leq k \leq p+1$ and $\mathcal{L}$-invariants $\mathcal{L}$ for primes $p \geq 5$. The present paper computes these determinants with the aid of two computer packages.

Keywords. Local Langlands Correspondence, Partial Harmonic sums, Sigma, and fastZeil

2010 Mathematics Subject Classification. 05E10 (Combinatorial aspects of representation theory) and 11F80 (Galois representations)

Introduction

Let $p \geq 5$ be a prime and E be a finite extension of $\mathbb{Q}_p$ containing $\sqrt{p}$. In our previous paper [ChGh24], we computed the semi-simplification of the reduction mod p of the irreducible two-dimensional semi-stable representation $V_{k,\mathcal{L}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over E with Hodge-Tate weights $(0, k-1)$ for $k \in [3, p+1]$ and $\mathcal{L}$-invariant $\mathcal{L} \in E$. We used the compatibility with respect to reduction mod p between the p-adic Local Langlands Correspondence and an Iwahori theoretic version of the mod p Local Langlands Correspondence. In particular, we showed the reduction $\bar{V}_{k,\mathcal{L}}$ varies through an alternating sequence of irreducible and reducible mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ [ChGh24, Theorem 1.1].

However, for lack of space and time, we had to make some intelligent guesses regarding the solutions to some matrix equations in [ChGh24]. These were flagged in 5 footnotes. The goal of this paper is to provide complete proofs of the statements in these footnotes. Let $r = k-2$, so that $r \in [1, p-1]$. In particular, we solve for the variables in the 5 matrix equations referred to in footnotes 10, 11 and 12 (for r odd) and in footnotes 16 and 17 (for r even) in [ChGh24]. This paper is therefore divided into 5 sections, one for each matrix equation.

We use Cramer's rule and so the proofs involve computing certain determinants whose entries contain binomial coefficients and sometimes partial harmonic sums. There are two parts to the computation:

- A column reduction argument to reduce these determinants to certain binomial identities involving partial harmonic sums.
- A verification of these identities using the `Sigma` [Sch07] or `fastZeil` [PaSc95] packages in Mathematica.[1]

This paper is essentially self-contained and may be of independent interest for people attempting to compute determinants involving binomial coefficients and partial harmonic sums.

Notation: For $n \geq 1$, let $H_n = \sum_{i=1}^{n} \frac{1}{i}$ be the n-th partial harmonic sum. Set $H_0 = 0$. Recall from [ChGh24] that v_-, and v_+, are the largest, respectively smallest, integers such that $v_- < \frac{r}{2} < v_+$ and for convenience we set $H_- = H_{v_-}$ and we set $H_+ = H_{v_+}$.

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[1] We remark that `fastZeil` provides a computer proof of the main recursive step in the proof of the identities which can then be verified by hand. So the proofs of the identities mentioned in the second bullet point are ultimately not based on any computer program.

1. Footnote 10

The first matrix equation we treat arises in footnote 10 in [ChGh24]. We wish to show that in the matrix equation

$$\begin{pmatrix} \binom{\frac{r+3}{2}}{0}\frac{r}{r} & 0 & 0 & \cdots & 0 \\ \binom{\frac{r+3}{2}}{1}\frac{r}{r-1} & \binom{\frac{r+1}{2}}{0}\frac{r-1}{(r-1)(r-2)} & 0 & \cdots & 0 \\ \binom{\frac{r+3}{2}}{2}\frac{r}{r-2} & \binom{\frac{r+1}{2}}{1}\frac{r-1}{(r-2)(r-3)} & \binom{\frac{r-1}{2}}{0}\frac{2!(r-2)}{(r-2)(r-3)(r-4)} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{\frac{r+3}{2}}{\frac{r-1}{2}}\frac{r}{\frac{r+1}{2}} & \binom{\frac{r+1}{2}}{\frac{r-3}{2}}\frac{r-1}{(\frac{r+1}{2})(\frac{r-1}{2})} & \binom{\frac{r-1}{2}}{\frac{r-5}{2}}\frac{2!(r-2)}{(\frac{r+1}{2})(\frac{r-1}{2})(\frac{r-3}{2})} & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_r \\ x_{r-1} \\ x_{r-2} \\ \vdots \\ x_{\frac{r+1}{2}} \end{pmatrix} = \begin{pmatrix} \binom{\frac{r+3}{2}}{0}\left(\mathcal{L}-H_{\frac{r+3}{2}}\right) \\ \binom{\frac{r+3}{2}}{1}\left(\mathcal{L}-H_{\frac{r+1}{2}}\right) \\ \binom{\frac{r+3}{2}}{2}\left(\mathcal{L}-H_{\frac{r-1}{2}}\right) \\ \vdots \\ \binom{\frac{r+3}{2}}{\frac{r-1}{2}}(\mathcal{L}-H_2) \end{pmatrix},$$

we have the solution $x_{\frac{r+1}{2}} = (-1)^{\frac{r-1}{2}}\dfrac{r+3}{4}\left(\mathcal{L}-H_{\frac{r-1}{2}}-H_{\frac{r+1}{2}}\right)$.

Let A be the coefficient matrix, X be the matrix of the indeterminates and write the matrix on the right as $B_1+B_2+B_3$, where

$$B_1 = \begin{pmatrix} \binom{\frac{r+3}{2}}{0}\mathcal{L} \\ \binom{\frac{r+3}{2}}{1}\mathcal{L} \\ \binom{\frac{r+3}{2}}{2}\mathcal{L} \\ \vdots \\ \binom{\frac{r+3}{2}}{\frac{r-1}{2}}\mathcal{L} \end{pmatrix}, B_2 = \begin{pmatrix} -\binom{\frac{r+3}{2}}{0}H_{\frac{r+3}{2}} \\ -\binom{\frac{r+3}{2}}{1}H_{\frac{r+3}{2}} \\ -\binom{\frac{r+3}{2}}{2}H_{\frac{r+3}{2}} \\ \vdots \\ -\binom{\frac{r+3}{2}}{\frac{r-1}{2}}H_{\frac{r+3}{2}} \end{pmatrix} \text{ and } B_3 = \begin{pmatrix} \binom{\frac{r+3}{2}}{0}(0) \\ \binom{\frac{r+3}{2}}{1}(1/\frac{r+3}{2}) \\ \binom{\frac{r+3}{2}}{2}(1/\frac{r+1}{2}+1/\frac{r+3}{2}) \\ \vdots \\ \binom{\frac{r+3}{2}}{\frac{r-1}{2}}(1/3+1/4+\cdots+1/\frac{r+3}{2}) \end{pmatrix}.$$

Let A_i be the matrix A with last column replaced by B_i for $i = 1, 2,$ and 3. We first claim that

$$\frac{\det A_1}{\det A} = (-1)^{\frac{r-1}{2}}\frac{r+3}{4}\mathcal{L} \quad \text{and} \quad \frac{\det A_2}{\det A} = (-1)^{\frac{r-1}{2}}\frac{r+3}{4}\left(-H_{\frac{r+3}{2}}\right).$$

Indeed, this follows immediately by showing that after making all but the last entry in the last column of the matrix

$$\begin{pmatrix} \binom{\frac{r+3}{2}}{0}\frac{r}{r} & 0 & 0 & \cdots & \binom{\frac{r+3}{2}}{0} \\ \binom{\frac{r+3}{2}}{1}\frac{r}{r-1} & \binom{\frac{r+1}{2}}{0}\frac{r-1}{(r-1)(r-2)} & 0 & \cdots & \binom{\frac{r+3}{2}}{1} \\ \binom{\frac{r+3}{2}}{2}\frac{r}{r-2} & \binom{\frac{r+1}{2}}{1}\frac{r-1}{(r-2)(r-3)} & \binom{\frac{r-1}{2}}{0}\frac{2!(r-2)}{(r-2)(r-3)(r-4)} & \cdots & \binom{\frac{r+3}{2}}{2} \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{\frac{r+3}{2}}{\frac{r-1}{2}}\frac{r}{\frac{r+1}{2}} & \binom{\frac{r+1}{2}}{\frac{r-3}{2}}\frac{r-1}{(\frac{r+1}{2})(\frac{r-1}{2})} & \binom{\frac{r-1}{2}}{\frac{r-5}{2}}\frac{2!(r-2)}{(\frac{r+1}{2})(\frac{r-1}{2})(\frac{r-3}{2})} & \cdots & \binom{\frac{r+3}{2}}{\frac{r-1}{2}} \end{pmatrix},$$

zero using column operations, the last entry becomes $(-1)^{\frac{r-1}{2}}\dfrac{r+3}{4}$.

Here are the column operations. The matrix above is a square matrix of size $\dfrac{r+1}{2}$. Number its columns using the labels 0 to $\dfrac{r-1}{2}$. First perform $C_{\frac{r-1}{2}} \to C_{\frac{r-1}{2}} - C_0$ to make the $(0, \frac{r-1}{2})^{\text{th}}$ entry 0. For $j \geq 1$, the $(j, \frac{r-1}{2})^{\text{th}}$ entry becomes

$$\binom{\frac{r+3}{2}}{j} - \binom{\frac{r+3}{2}}{j}\frac{r}{r-j} = -\binom{r+3}{2}\binom{\frac{r+1}{2}}{j-1}\frac{1}{r-j}.$$

So after this operation, the $(1, \frac{r-1}{2})^{\text{th}}$ entry is $-\binom{\frac{r+3}{2}}{}\frac{1}{r-1}$.

Now perform the operation $C_{\frac{r-1}{2}} \to C_{\frac{r-1}{2}} + \frac{r-2}{r-1}\binom{\frac{r+3}{2}}{} C_1$ to make the $(1, \frac{r-1}{2})^{\text{th}}$ entry 0. For $j \geq 2$, the $(j, \frac{r-1}{2})^{\text{th}}$ entry becomes

$$
\begin{aligned}
&-\left(\frac{r+3}{2}\right)\binom{\frac{r+1}{2}}{j-1}\frac{1}{r-j} + \frac{r-2}{(r-j)(r-j-1)}\left(\frac{r+3}{2}\right)\binom{\frac{r+1}{2}}{j-1} \\
&= \left(\frac{r+3}{2}\right)\left(\frac{r+1}{2}\right)\binom{\frac{r-1}{2}}{j-2}\frac{1}{(r-j)(r-j-1)}.
\end{aligned}
$$

Continuing this process, after making the first $\frac{r-1}{2}$ entries in the last column 0, for $j \geq \frac{r-1}{2}$, the $(j, \frac{r-1}{2})^{\text{th}}$ entry becomes

$$
(-1)^{\frac{r-1}{2}}\left(\frac{r+3}{2}\right)\left(\frac{r+1}{2}\right)\cdots(3)\binom{2}{j-(\frac{r-1}{2})}\frac{1}{(r-j)(r-j-1)\cdots(\frac{r+3}{2}-j)}.
$$

So the $\left(\frac{r-1}{2}, \frac{r-1}{2}\right)^{\text{th}}$ entry becomes $(-1)^{\frac{r-1}{2}}\frac{r+3}{4}$.

Next, we solve the equation $AX = B_3$ by solving for each fraction $1/(n+1)$ for $n = 2, 3, \ldots, \frac{r+1}{2}$ separately. We claim that the $1/(n+1)$ part of $x_{\frac{r+1}{2}}$ without $1/(n+1)$ is

$$
\left(\frac{r+3}{4}\right)\left[\sum_{j=1}^{n-1}(-1)^{j-1}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1}\right].
$$

This is obvious for $n = 2$ using Cramer's rule since only the last entry in the last column is non-zero. For $n = 3$, the last two entries are non-zero. So we perform the operation $C_{\frac{r-1}{2}} \to C_{\frac{r-1}{2}} - \binom{\frac{r+3}{2}}{\frac{r-3}{2}}\frac{(\frac{r+1}{2})(\frac{r-1}{2})}{(2)(1)}C_{\frac{r-3}{2}}$ to make the second-to-last entry 0. Therefore the last entry becomes

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{r+3}{2}\right)\left(\frac{r+1}{2}\right) - \frac{1}{6}\left(\frac{r+3}{2}\right)\left(\frac{r+1}{2}\right)^2\left(\frac{r-1}{2}\right)^2\frac{1}{2}3\left(\frac{r+3}{2}\right)\frac{1}{\left(\frac{r+1}{2}\right)\left(\frac{r-1}{2}\right)} \\
&= \binom{\frac{r+3}{2}}{2}\left[1 - \frac{1}{2}\left(\frac{r+3}{2}\right)\left(\frac{r-1}{2}\right)\right],
\end{aligned}
$$

verifying the claim in this case. The general case is based on an intelligent guess whose verification we leave to the reader.

In any case, to show that $x_{\frac{r+1}{2}} = (-1)^{\frac{r-1}{2}}\dfrac{r+3}{4}\left(\mathcal{L} - H_{\frac{r-1}{2}} - H_{\frac{r+1}{2}}\right)$, it suffices to prove the identity

$$
\sum_{n=2}^{\frac{r+1}{2}}\frac{1}{n+1}\sum_{j=1}^{n-1}(-1)^{j-1}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1} = (-1)^{\frac{r-1}{2}}\left(H_{\frac{r+3}{2}} - H_{\frac{r-1}{2}} - H_{\frac{r+1}{2}}\right). \tag{1.1}
$$

Switching the order of j and n, the double sum on the left is

$$
\begin{aligned}
\sum_{j=1}^{\frac{r-1}{2}}(-1)^{j-1}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1}\sum_{n=j+1}^{\frac{r+1}{2}}\frac{1}{n+1} &= \sum_{j=1}^{\frac{r-1}{2}}(-1)^{j-1}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1}(H_{\frac{r+3}{2}} - H_{j+1}) \\
= H_{\frac{r+3}{2}} \cdot \sum_{j=1}^{\frac{r-1}{2}}(-1)^{j-1}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1} &+ \sum_{j=1}^{\frac{r-1}{2}}(-1)^{j}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1}H_{j+1}.
\end{aligned} \tag{1.2}
$$

The first sum on the right of (1.2) is easy to compute. Rearranging the binomial coefficients using the identity $\binom{n}{m}\binom{m}{j} = \binom{n}{j}\binom{n-j}{m-j}$, it is just

$$\sum_{j=1}^{\frac{r-1}{2}} (-1)^{j-1} \binom{\frac{r-1}{2}+j}{j}\binom{\frac{r-1}{2}}{j-1} \quad = \quad (-1)^{\frac{r-1}{2}} \left(1 - \binom{r}{\frac{r+1}{2}}\right).$$

This can be seen by manipulating some identities in Gould [Gou72, volumes 4, 5]. However, alternatively, and more conceptually, we may use the Gauss' summation formula

$$_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix}; 1\right) = \frac{(c-b)_n}{(c)_n} \tag{1.3}$$

for the Gauss hypergeometric function $_2F_1\left(\begin{smallmatrix}a,b\\ c\end{smallmatrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$ noting that the series has finitely many terms when $a = -n$ is a negative integer since the rising factorial symbol $(a)_k = a\cdot(a+1)\cdots(a+k-1)$ eventually dies. This approach was shown to us by Peter Paule (in the context of Section 5. but we explain all the details in the present context and only recall some details there). Changing notation a bit we must evaluate

$$\begin{aligned}
\sum_{k=1}^{n}(-1)^{k-1}\binom{n+k}{k}\binom{n}{k-1} \quad &= \quad \sum_{k'=0}^{n-1}(-1)^{k'}\binom{n+k'+1}{k'+1}\binom{n}{k'} \\
&= \quad \sum_{k=0}^{\infty}(-1)^{k}\binom{n+k+1}{k+1}\binom{n}{k} - (-1)^n\binom{2n+1}{n+1} \\
&= \quad (n+1)\sum_{k=0}^{\infty}(-1)^{k}\binom{n+k+1}{k+1}\binom{n}{k}/(n+1) - (-1)^n\binom{2n+1}{n+1}
\end{aligned}$$

for $n = \frac{r-1}{2}$. Write $f(n,j) = (-1)^j\binom{n+j+1}{j+1}\binom{n}{j}/(n+1)$. Clearly $f(n,0) = 1$, and by telescoping

$$f(n,k) = \prod_{j=0}^{k-1}\frac{f(n,j+1)}{f(n,j)} = \prod_{j=0}^{k-1}\frac{(-n+j)(n+2+j)}{(j+2)}\cdot\frac{1}{j+1} = \frac{(-n)_k(n+2)_k}{(2)_k}\cdot\frac{1}{k!}.$$

Thus the last sum on the right above is just $_2F_1\left(\begin{smallmatrix}-n,n+2\\ 2\end{smallmatrix}; 1\right) = \frac{(-n)_n}{(2)_n}$, by (1.3). Thus the first term on the right above is

$$(n+1)\cdot\frac{(-n)_n}{(2)_n} = (n+1)\cdot\frac{(-n)\cdot(-n+1)\cdots(-n+n-1)}{2\cdot 3\cdots(2+n-1)} = (-1)^n$$

as desired.

The second sum on the right of (1.2) is trickier. Changing notation we may write it as

$$\sum_{k=1}^{n}(-1)^k\binom{n+k}{k}\binom{n}{k-1}H_{k+1}$$

for $n = \frac{r-1}{2}$. The embedded partial harmonic sum throws one off. However, this sum may be evaluated using the package `Sigma` in Mathematica (developed by Carsten Schneider). Remarkably, the sum satisfies a second order recurrence relation which has a solution that is an appropriate linear combination of a general solution and a particular solution, much as in the theory of 2^{nd} order ordinary differential equations. Here are the commands in Mathematica (in text mode):

```
In[1]  :=  << Sigma.m
           Sigma - A summation package by Carsten Schneider – ? RISC – V 2.89 (November 10, 2021)
In[2]  :=  mySum = SigmaSum[SigmaPower[-1,k] SigmaBinomial[n+k,k] SigmaBinomial[n,k-1] SigmaHNumber[k+1], {k,1,n}]
Out[2] =   SigmaSum[(1/(k+1) + SigmaHNumber[1, k]) SigmaBinomial[n, -1 + k] SigmaBinomial[k + n, k] SigmaPower[-1, k], {k, 1, n}]
In[3]  :=  rec = GenerateRecurrence[mySum][[1]]
Out[3] =   (1 + n) (5 + 2 n) SUM[n] + 2 (7 + 8 n + 2 n^2 ) SUM[1 + n] + (3 + n) (3 + 2 n) SUM[2 + n]
            == ((1 + 2 n) ((3 + 2 n) (944 + 1847 n + 1298 n^2 + 389 n^3 + 42 n^4 ) + (1 + n) (2 + n) (3 + n) (4 + n) (53 + 63 n + 18 n^2)
            SigmaHNumber[1, n]) SigmaBinomial[2 n, n] SigmaPower[-1, n]) / ((1 + n)^2 (2 + n) (3 + n) (4 + n)).
In[4]  :=  recSol = SolveRecurrence[rec, SUM[n]]
Out[4] =   {{0, SigmaPower[-1, n]}, {0, (-1/(n+1) - 2 SigmaHNumber[1, n]) SigmaPower[-1, n]},
            {1, ( ((1 + 2 n) (3 + 2 n)) / ((1 + n)^2 (2 + n)) + (1 + 2 n) SigmaHNumber[1, n] / (1 + n) )
            SigmaBinomial[2 n, n] SigmaPower[-1, n]}}
In[5]  :=  FindLinearCombination[recSol, mySum, n, 2]
Out[5] =   SigmaPower[-1, n]/(-1-n) - 2 SigmaHNumber[1, n] SigmaPower[-1, n]
            + ((1 + 2 n) (3 + 2 n)/ ((1 + n)^2(2 + n)) SigmaBinomial[2 n, n] SigmaPower[-1, n]
            + ((1 + 2 n)/(1+n)) SigmaHNumber[1, n] SigmaBinomial[2 n, n] SigmaPower[-1, n]
```

Thus within a few steps `Sigma` yields that the tricky sum above is

$$(-1)^n(-H_n - H_{n+1}) + (-1)^n\frac{(2n+1)(2n+3)}{(n+1)^2(n+2)}\binom{2n}{n} + (-1)^n\frac{(2n+1)}{(n+1)}\binom{2n}{n}H_n$$

for $n = \frac{r-1}{2}$. We remark that once the recurrence relation has been produced by `Sigma`, it can be checked that the sum above satisfies it (as an example we show how this can be done for a similar sum and recurrence relation in Section 5.). Since the purported answer above also satisfies the recurrence relation (a shorter by hand check), and both the sum and the answer above agree for small values of n, they must match for all values of n. This reasoning makes the evaluation of the tricky sum entirely theoretical!

In any case, plugging the evaluations of the easy sum and the tricky sum into (1.2), and simplifying noting $H_{\frac{r+3}{2}} = H_n + 1/(n+1) + 1/(n+2)$, we obtain the identity (1.1).

2. Footnote 11

To solve

$$\begin{pmatrix} \binom{\frac{r+1}{2}}{0}\frac{r}{r} & 0 & 0 & \cdots & 0 \\ \binom{\frac{r+1}{2}}{1}\frac{r}{r-1} & \binom{\frac{r-1}{2}}{0}\frac{r-1}{(r-1)(r-2)} & 0 & \cdots & 0 \\ \binom{\frac{r+1}{2}}{2}\frac{r}{r-2} & \binom{\frac{r-1}{2}}{1}\frac{r-1}{(r-2)(r-3)} & \binom{\frac{r-3}{2}}{0}\frac{2!(r-2)}{(r-2)(r-3)(r-4)} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{\frac{r+1}{2}}{\frac{r-1}{2}}\frac{r}{\frac{r+1}{2}} & \binom{\frac{r-1}{2}}{\frac{r-3}{2}}\frac{r-1}{(\frac{r+1}{2})(\frac{r-1}{2})} & \binom{\frac{r-3}{2}}{\frac{r-5}{2}}\frac{2!(r-2)}{(\frac{r+1}{2})(\frac{r-1}{2})(\frac{r-3}{2})} & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_r \\ x_{r-1} \\ x_{r-2} \\ \vdots \\ x_{\frac{r+1}{2}} \end{pmatrix} = \begin{pmatrix} \binom{\frac{r+1}{2}}{0}(\mathcal{L} - H_{\frac{r+1}{2}}) \\ \binom{\frac{r+1}{2}}{1}(\mathcal{L} - H_{\frac{r-1}{2}}) \\ \binom{\frac{r+1}{2}}{2}(\mathcal{L} - H_{\frac{r-3}{2}}) \\ \vdots \\ \binom{\frac{r+1}{2}}{\frac{r-1}{2}}(\mathcal{L} - H_1) \end{pmatrix}.$$

We have to prove that

$$x_{\frac{r+1}{2}} = (-1)^{\frac{r-1}{2}}(\mathcal{L} - H_- - H_+). \tag{2.4}$$

As in Section 1., we separate the matrix on the right as $B_1 + B_2 + B_3$, where

$$B_1 = \begin{pmatrix} \binom{\frac{r+1}{2}}{0}\mathcal{L} \\ \binom{\frac{r+1}{2}}{1}\mathcal{L} \\ \binom{\frac{r+1}{2}}{2}\mathcal{L} \\ \vdots \\ \binom{\frac{r+1}{2}}{\frac{r-1}{2}}\mathcal{L} \end{pmatrix}, B_2 = \begin{pmatrix} -\binom{\frac{r+1}{2}}{0}H_{\frac{r+1}{2}} \\ -\binom{\frac{r+1}{2}}{1}H_{\frac{r+1}{2}} \\ -\binom{\frac{r+1}{2}}{2}H_{\frac{r+1}{2}} \\ \vdots \\ -\binom{\frac{r+1}{2}}{\frac{r-1}{2}}H_{\frac{r+1}{2}} \end{pmatrix}, \text{ and } B_3 = \begin{pmatrix} \binom{\frac{r+1}{2}}{0}(0) \\ \binom{\frac{r+1}{2}}{1}(1/\frac{r+1}{2}) \\ \binom{\frac{r+1}{2}}{2}(1/\frac{r-1}{2} + 1/\frac{r+1}{2}) \\ \vdots \\ \binom{\frac{r+1}{2}}{\frac{r-1}{2}}(1/2 + 1/3 + \cdots + 1/\frac{r+1}{2}) \end{pmatrix}.$$

As before, we show that

$$\frac{\det A_1}{\det A} = (-1)^{\frac{r-1}{2}}\mathcal{L} \quad \text{and} \quad \frac{\det A_2}{\det A} = (-1)^{\frac{r-1}{2}}(-H_{\frac{r+1}{2}}).$$

Consider the following column operations on the matrix A_1 (with the common $\mathcal{L}$ term in every entry of the last column dropped and added in at the end). First perform $C_{\frac{r-1}{2}} \to C_{\frac{r-1}{2}} - C_0$ to make the $(0, \frac{r-1}{2})^{\text{th}}$ entry 0. For $j \geq 1$, the $(j, \frac{r-1}{2})^{\text{th}}$ entry becomes

$$\binom{\frac{r+1}{2}}{j} - \binom{\frac{r+1}{2}}{j}\frac{r}{r-j} = -\left(\frac{r+1}{2}\right)\frac{1}{r-j}\binom{\frac{r-1}{2}}{j-1}.$$

So the $(1, \frac{r-1}{2})^{\text{th}}$ entry is $-\left(\frac{r+1}{2}\right)\frac{1}{r-1}$.

Now perform $C_{\frac{r-1}{2}} \to C_{\frac{r-1}{2}} + (r-2)\left(\frac{r+1}{2}\right)\frac{1}{r-1}C_1$. For $j \geq 2$, the $(j, \frac{r-1}{2})^{\text{th}}$ entry becomes

$$\begin{aligned}
&-\left(\frac{r+1}{2}\right)\frac{1}{r-j}\binom{\frac{r-1}{2}}{j-1} + \left(\frac{r+1}{2}\right)\frac{r-2}{(r-j)(r-j-1)}\binom{\frac{r-1}{2}}{j-1} \\
&= \left(\frac{r+1}{2}\right)\left(\frac{r-1}{2}\right)\frac{1}{(r-j)(r-j-1)}\binom{\frac{r-3}{2}}{j-2}.
\end{aligned}$$

Continuing this process, after making the first $\frac{r-1}{2}$ entries 0, for $j \geq \frac{r-1}{2}$, the $(j, \frac{r-1}{2})^{\text{th}}$ entry becomes

$$(-1)^{\frac{r-1}{2}}\left(\frac{r+1}{2}\right)\left(\frac{r-1}{2}\right)\cdots(2)\frac{1}{(r-j)(r-j-1)\cdots(\frac{r+3}{2}-j)}\binom{1}{j-\frac{r-1}{2}}.$$

So the $\left(\frac{r-1}{2}, \frac{r-1}{2}\right)^{\text{th}}$ entry is just $(-1)^{\frac{r-1}{2}}$. Putting the $\mathcal{L}$ back in we get that the first ratio of determinants above is as claimed. The second ratio follows identically.

Next, we solve $AX = B_3$ by solving for $1/(n+1)$ for $n = 1, 2, \ldots, \frac{r-1}{2}$ separately. We claim that the solution to the $1/(n+1)$ part is

$$\sum_{j=1}^{n}(-1)^{j-1}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1}.$$

This is easy to check for $n = 1$ as before. For $n = 2$, we use column operations to reduce the last column to its last entry. We perform $C_{\frac{r-1}{2}} \to C_{\frac{r-1}{2}} - \binom{\frac{r+1}{2}}{\frac{r-3}{2}}\frac{\left(\frac{r+1}{2}\right)\left(\frac{r-1}{2}\right)}{(2)(1)}C_{\frac{r-3}{2}}$ to make the second-to-last entry in the last column 0. As a consequence, the last entry becomes

$$\frac{r+1}{2} - \frac{1}{2}\left(\frac{r+1}{2}\right)\left(\frac{r-1}{2}\right)\left(\frac{r+3}{2}\right) = \binom{\frac{r+1}{2}}{1} - 3\binom{\frac{r+3}{2}}{3},$$

verifying the claim when $n = 2$. The case of general n is an intelligent guess left to the reader as an exercise.

To show (2.4), we have to prove the identity

$$\sum_{n=1}^{\frac{r-1}{2}}\frac{1}{n+1}\sum_{j=1}^{n}(-1)^{j-1}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1} = (-1)^{\frac{r-1}{2}}(-H_{-}).$$

As before (cf. (1.2) above), the sum on the left can be broken into two sums and equals

$$H_{\frac{r+1}{2}} \cdot \sum_{j=1}^{\frac{r-1}{2}}(-1)^{j-1}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1} \quad + \quad \sum_{j=1}^{\frac{r-1}{2}}(-1)^{j}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1}H_j.$$

Changing notation (and binomial coefficients) a bit this equals

$$H_{\frac{r+1}{2}} \cdot \sum_{k=1}^{n} (-1)^{k-1} \binom{n+k}{k} \binom{n}{k-1} \quad + \quad \sum_{k=1}^{n} (-1)^{k} \binom{n+k}{k} \binom{n}{k-1} H_k$$

with $n = \frac{r-1}{2}$. The first (easy) sum was evaluated above. Using `Sigma` for the second, the above expression equals:

$$H_{n+1} \cdot (-1)^n \left(1 - \binom{2n+1}{n+1} \right) \quad + \quad (-1)^n (-H_n - H_{n+1}) + (-1)^n H_{n+1} \binom{2n+1}{n+1}$$

which clearly equals $(-1)^n(-H_n)$, as desired putting $n = \frac{r-1}{2}$.

3. Footnote 12

We show that in the matrix equation

$$\begin{pmatrix} \binom{\frac{r+1}{2}}{0}\frac{r}{r} & 0 & 0 & \cdots & 0 & 0 \\ \binom{\frac{r+1}{2}}{1}\frac{r}{r-1} & \binom{\frac{r-1}{2}}{0}\frac{r-1}{(r-1)(r-2)} & 0 & \cdots & 0 & 0 \\ \binom{\frac{r+1}{2}}{2}\frac{r}{r-2} & \binom{\frac{r-1}{2}}{1}\frac{r-1}{(r-2)(r-3)} & \binom{\frac{r-3}{2}}{0}\frac{2!(r-2)}{(r-2)(r-3)(r-4)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \binom{\frac{r+1}{2}}{\frac{r-1}{2}}\frac{r}{\frac{r+1}{2}} & \binom{\frac{r-1}{2}}{\frac{r-3}{2}}\frac{r-1}{(\frac{r+1}{2})(\frac{r-1}{2})} & \binom{\frac{r-3}{2}}{\frac{r-5}{2}}\frac{2!(r-2)}{(\frac{r+1}{2})(\frac{r-1}{2})(\frac{r-3}{2})} & \cdots & 1 & 0 \\ \binom{\frac{r+1}{2}}{\frac{r+1}{2}}\frac{r}{\frac{r-1}{2}} & \binom{\frac{r-1}{2}}{\frac{r-1}{2}}\frac{r-1}{(\frac{r-1}{2})(\frac{r-3}{2})} & \binom{\frac{r-3}{2}}{\frac{r-3}{2}}\frac{2!(r-2)}{(\frac{r-1}{2})(\frac{r-3}{2})(\frac{r-5}{2})} & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_r \\ x_{r-1} \\ x_{r-2} \\ \vdots \\ x_{\frac{r+1}{2}} \\ x_{\frac{r-1}{2}} \end{pmatrix} = \begin{pmatrix} \binom{\frac{r+1}{2}}{0}(\mathcal{L} - H_{\frac{r+1}{2}}) \\ \binom{\frac{r+1}{2}}{1}(\mathcal{L} - H_{\frac{r-1}{2}}) \\ \binom{\frac{r+1}{2}}{2}(\mathcal{L} - H_{\frac{r-3}{2}}) \\ \vdots \\ \binom{\frac{r+1}{2}}{\frac{r-1}{2}}(\mathcal{L} - H_1) \\ 0 \end{pmatrix},$$

we have

$$x_{\frac{r+1}{2}} = (-1)^{\frac{r-1}{2}} \left(\mathcal{L} - H_- - H_+ \right).$$

Since the coefficient matrix above is invertible, there is a unique solution. Moreover, since the matrix equation above contains exactly the same equations as the equations in the matrix equation in Section 2. (except for the last equation which may be used to solve for the last variable in terms of the previous ones), the formula for $x_{\frac{r+1}{2}}$ follows immediately from (2.4).

Next, we show that

$$x_{\frac{r-1}{2}} = -\mathcal{L} + (-1)^{\frac{r-1}{2}} \frac{2}{r+1} + (-1)^{\frac{r-1}{2}} \frac{r+1}{2} (\mathcal{L} - H_- - H_+).$$

This implies the check in footnote 12 since this expression reduces mod π to the one in that footnote, noting that $\nu = v_p(\mathcal{L} - H_- - H_+) \geq 0.5 > 0$ there.

Write the matrix on the right as $B_1 + B_2 + B_3 + B_4$, where

$$B_1 = \begin{pmatrix} \binom{\frac{r+1}{2}}{0}\mathcal{L} \\ \binom{\frac{r+1}{2}}{1}\mathcal{L} \\ \binom{\frac{r+1}{2}}{2}\mathcal{L} \\ \vdots \\ \binom{\frac{r+1}{2}}{\frac{r+1}{2}}\mathcal{L} \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -\mathcal{L} \end{pmatrix}, B_3 = \begin{pmatrix} \binom{\frac{r+1}{2}}{0}(-H_+) \\ \binom{\frac{r+1}{2}}{1}(-H_+) \\ \binom{\frac{r+1}{2}}{2}(-H_+) \\ \vdots \\ \binom{\frac{r+1}{2}}{\frac{r+1}{2}}(-H_+) \end{pmatrix} \text{ and } B_4 = \begin{pmatrix} \binom{\frac{r+1}{2}}{0}(0) \\ \binom{\frac{r+1}{2}}{1}(1/\frac{r+1}{2}) \\ \binom{\frac{r+1}{2}}{2}(1/\frac{r-1}{2} + 1/\frac{r+1}{2}) \\ \vdots \\ \binom{\frac{r+1}{2}}{\frac{r+1}{2}}(1 + 1/2 + \cdots + 1/\frac{r+1}{2}) \end{pmatrix}.$$

We solve the B_1 and B_3 parts of the matrix equation above together as usual. Consider the matrix obtained by replacing the last column in the coefficient matrix with B_1 but without the common $\mathcal{L}$

term. We perform the operation $C_{\frac{r+1}{2}} \to C_{\frac{r+1}{2}} - C_0$ to make the $(0, \frac{r+1}{2})^{\text{th}}$ entry 0. For $j \geq 1$, the $(j, \frac{r+1}{2})^{\text{th}}$ entry becomes

$$\binom{\frac{r+1}{2}}{j} - \binom{\frac{r+1}{2}}{j}\frac{r}{r-j} = \binom{\frac{r+1}{2}}{j}\left(\frac{-j}{r-j}\right) = -\left(\frac{r+1}{2}\right)\frac{1}{r-j}\binom{\frac{r-1}{2}}{j-1}.$$

So the $(1, \frac{r+1}{2})^{\text{th}}$ entry is $-\left(\frac{r+1}{2}\right)\frac{1}{r-1}$.

Next, we perform $C_{\frac{r+1}{2}} \to C_{\frac{r+1}{2}} + (r-2)\left(\frac{r+1}{2}\right)\frac{1}{r-1}C_1$ to make the $(1, \frac{r+1}{2})^{\text{th}}$ entry 0. For $j \geq 2$, the $(j, \frac{r+1}{2})^{\text{th}}$ entry becomes

$$\begin{aligned} &-\left(\frac{r+1}{2}\right)\frac{1}{r-j}\binom{\frac{r-1}{2}}{j-1} + (r-2)\left(\frac{r+1}{2}\right)\binom{\frac{r-1}{2}}{j-1}\frac{1}{(r-j)(r-j-1)} \\ &= \left(\frac{r+1}{2}\right)\left(\frac{r-1}{2}\right)\frac{1}{(r-j)(r-j-1)}\binom{\frac{r-3}{2}}{j-2}. \end{aligned}$$

Continuing as usual, and after making the first $\frac{r-1}{2}$ entries in the last column 0, for $j \geq \frac{r-1}{2}$, the $(j, \frac{r+1}{2})^{\text{th}}$ entry becomes

$$(-1)^{\frac{r-1}{2}}\left(\frac{r+1}{2}\right)\left(\frac{r-1}{2}\right)\cdots(2)\frac{1}{(r-j)(r-j-1)\cdots(\frac{r+3}{2}-j)}\binom{1}{j-\frac{r-1}{2}}.$$

The above column operations reduce the lower right 2 by 2 matrix to an upper triangular matrix with $(\frac{r+1}{2}, \frac{r+1}{2})^{\text{th}}$ entry equal to $(-1)^{\frac{r-1}{2}}\frac{r+1}{2}$. Note that the $(\frac{r-1}{2}, \frac{r-1}{2})^{\text{th}}$ entry remains unchanged and is 1 and the $(\frac{r-1}{2}, \frac{r+1}{2})^{\text{th}}$ entry is irrelevant when computing its determinant. By Cramer's rule (and putting $\mathcal{L}$ back in), we see that B_1 contributes $(-1)^{\frac{r-1}{2}}\frac{r+1}{2}\mathcal{L}$ to $x_{\frac{r-1}{2}}$.

Arguing similarly, we see that B_3 contributes $(-1)^{\frac{r-1}{2}}\frac{r+1}{2}(-H_+)$ to $x_{\frac{r-1}{2}}$. On the other hand, the B_2 part of the matrix equation clearly contributes just $-\mathcal{L}$ to $x_{\frac{r-1}{2}}$. So we have to show that B_4 contributes $(-1)^{\frac{r-1}{2}}\frac{2}{r+1} + (-1)^{\frac{r-1}{2}}\frac{r+1}{2}(-H_-)$ to $x_{\frac{r-1}{2}}$.

We solve $AX = B_4$ by solving as usual the $1/(n+1)$ parts separately and adding them up. The claim is that the $1/(n+1)$ part of $AX = B_4$ without $1/(n+1)$ is

$$1 + \left(\frac{r+1}{2}\right)\sum_{j=2}^{n}(-1)^{j-1}\frac{j-1}{j}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1}.$$

This can easily be seen for $n = 0, 1$ and we do not need to perform any column operations. However, for $n \geq 2$ we have to reduce the last column as follows. Assume $n = 2$. Note that the $(\frac{r-3}{2}, \frac{r+1}{2})^{\text{th}}$ entry without 1/3 is $\binom{\frac{r+1}{2}}{\frac{r-3}{2}}$. We have to kill this entry using the $(\frac{r-3}{2}, \frac{r-3}{2})^{\text{th}}$ entry, which is $\binom{2}{0}\frac{(\frac{r-3}{2})!(\frac{r+3}{2})}{(\frac{r+3}{2})(\frac{r+1}{2})\cdots(3)} = \frac{2}{(\frac{r+1}{2})(\frac{r-1}{2})}$.

We perform the operation $C_{\frac{r+1}{2}} \to C_{\frac{r+1}{2}} - (1/2)(\frac{r+1}{2})(\frac{r-1}{2})\frac{(\frac{r+1}{2})(\frac{r-1}{2})}{2}C_{\frac{r-3}{2}}$. This makes the matrix into a block lower triangular matrix whose lower right block is upper triangular with diagonal entries 1 and

$$1 - (1/2)\left(\frac{r+1}{2}\right)\left(\frac{r-1}{2}\right)\frac{(\frac{r+1}{2})(\frac{r-1}{2})}{2}\frac{(\frac{r+3}{2})}{(\frac{r-1}{2})} \quad = \quad 1 - \left(\frac{r+1}{2}\right)\frac{3}{2}\binom{\frac{r+3}{2}}{3},$$

which verifies the identity for $n = 2$. As usual, the verification of the claim for arbitrary n is left to the reader.

The identity that we have to prove is

$$\sum_{n=0}^{\frac{r-1}{2}} \frac{1}{n+1}\left(1+\binom{\frac{r+1}{2}}{}\sum_{j=2}^{n}(-1)^{j-1}\frac{j-1}{j}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1}\right) = (-1)^{\frac{r-1}{2}}\frac{2}{r+1} - (-1)^{\frac{r-1}{2}}\frac{r+1}{2}H_{-} \quad (3.5)$$

As before (cf. (1.2)) we switch the order of n and j (noting that $n=0,1$ do not contribute to the sum on j) and break the sum on the left into three parts:

$$H_{\frac{r+1}{2}} + \frac{r+1}{2}H_{\frac{r+1}{2}}\sum_{j=2}^{\frac{r-1}{2}}(-1)^{j-1}\frac{j-1}{j}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1} + \frac{r+1}{2}\sum_{j=2}^{\frac{r-1}{2}}(-1)^{j}\frac{j-1}{j}\binom{2j-1}{j}\binom{\frac{r-1}{2}+j}{2j-1}H_j.$$

Changing notation (and binomial coefficients) this may be rewritten as

$$H_{\frac{r+1}{2}} + \frac{r+1}{2}H_{\frac{r+1}{2}}\sum_{k=2}^{n}(-1)^{k-1}\frac{k-1}{k}\binom{n+k}{k}\binom{n}{k-1} + \frac{r+1}{2}\sum_{k=2}^{n}(-1)^{k}\frac{k-1}{k}\binom{n+k}{k}\binom{n}{k-1}H_k \quad (3.6)$$

with $n=\frac{r-1}{2}$. `Sigma` shows that the first sum above is

$$-\frac{1}{n+1}+(-1)^n-(-1)^n\frac{n(2n+1)}{(n+1)^2}\binom{2n}{n}$$

whereas the second is

$$-(-1)^n\frac{n}{(1+n)^2}-2(-1)^nH_n+(-1)^n\frac{n(2n+1)}{(n+1)^3}\binom{2n}{n}+(-1)^n\frac{n(2n+1)}{(n+1)^2}\binom{2n}{n}H_n.$$

Plugging these into (3.6), noting that $H_{n+1}=H_n+\frac{1}{n+1}$, and simplifying, (3.6) becomes

$$(-1)^n\frac{1}{n+1}-(-1)^n(n+1)H_n,$$

which gives the right hand side of (3.5) since $n=\frac{r-1}{2}$.

4. Footnote 16

We show that in

$$\begin{pmatrix} \binom{\frac{r+2}{2}}{0}\frac{r}{r} & 0 & 0 & \cdots & \binom{\frac{r+2}{2}}{0} \\ \binom{\frac{r+2}{2}}{1}\frac{r}{r-1} & \binom{r/2}{0}\frac{r-1}{(r-1)(r-2)} & 0 & \cdots & \binom{\frac{r+2}{2}}{1} \\ \binom{\frac{r+2}{2}}{2}\frac{r}{r-2} & \binom{r/2}{1}\frac{r-1}{(r-2)(r-3)} & \binom{\frac{r-2}{2}}{0}\frac{2!(r-2)}{(r-2)(r-3)(r-4)} & \cdots & \binom{\frac{r+2}{2}}{2} \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{\frac{r+2}{2}}{\frac{r-2}{2}}\frac{r}{\frac{r+2}{2}} & \binom{r/2}{\frac{r-4}{2}}\frac{r-1}{(\frac{r+2}{2})(r/2)} & \binom{\frac{r-2}{2}}{\frac{r-6}{2}}\frac{2!(r-2)}{(\frac{r+2}{2})(r/2)(\frac{r-2}{2})} & \cdots & \binom{\frac{r+2}{2}}{\frac{r-2}{2}} \end{pmatrix}\begin{pmatrix} x_r \\ x_{r-1} \\ x_{r-2} \\ \vdots \\ x_{\frac{r+2}{2}} \end{pmatrix}=\begin{pmatrix} \binom{\frac{r+2}{2}}{0}\left(\mathcal{L}-H_{\frac{r+2}{2}}\right) \\ \binom{\frac{r+2}{2}}{1}\left(\mathcal{L}-H_{r/2}\right) \\ \binom{\frac{r+2}{2}}{2}\left(\mathcal{L}-H_{\frac{r-2}{2}}\right) \\ \vdots \\ \binom{\frac{r+2}{2}}{\frac{r-2}{2}}(\mathcal{L}-H_2) \end{pmatrix},$$

we have $x_{\frac{r+2}{2}}=\mathcal{L}-H_{-}-H_{+}$. As usual, we separate the matrix on the right as B_1+B_2, where

$$B_1=\begin{pmatrix} \binom{\frac{r+2}{2}}{0}(\mathcal{L}-H_{+}) \\ \binom{\frac{r+2}{2}}{1}(\mathcal{L}-H_{+}) \\ \binom{\frac{r+2}{2}}{2}(\mathcal{L}-H_{+}) \\ \vdots \\ \binom{\frac{r+2}{2}}{\frac{r-2}{2}}(\mathcal{L}-H_{+}) \end{pmatrix} \text{ and } B_2=\begin{pmatrix} \binom{\frac{r+2}{2}}{0}(0) \\ \binom{\frac{r+2}{2}}{1}(1/\frac{r+2}{2}) \\ \binom{\frac{r+2}{2}}{2}(1/\frac{r}{2}+1/\frac{r+2}{2}) \\ \vdots \\ \binom{\frac{r+2}{2}}{\frac{r-2}{2}}(1/3+1/4+\cdots+1/\frac{r+2}{2}) \end{pmatrix}.$$

We first claim that the B_1 part of $x_{\frac{r+2}{2}}$ is $\mathcal{L} - H_+$. Indeed, this follows directly since the matrices A_1 and A differ only in the last column, and since the last column of A_1 is $(\mathcal{L} - H_+)$ times the last column of A.

Next, we show that the B_2 part of $x_{\frac{r+2}{2}}$ is $-H_-$. Firstly, we reduce A to a lower triangular matrix using the following column operations and see that the $(\frac{r-2}{2}, \frac{r-2}{2})^{\text{th}}$ entry is $(-1)^{\frac{r-2}{2}}$. We perform $C_{\frac{r-2}{2}} \to C_{\frac{r-2}{2}} - C_0$ to make the $(0, \frac{r-2}{2})^{\text{th}}$ entry 0. For $j \geq 1$, the $(j, \frac{r-2}{2})^{\text{th}}$ entry becomes

$$\binom{\frac{r+2}{2}}{j} - \binom{\frac{r+2}{2}}{j}\frac{r}{r-j} = -\binom{r+2}{2}\binom{\frac{r}{2}}{j-1}\frac{1}{r-j}.$$

Therefore the $(1, \frac{r-2}{2})^{\text{th}}$ entry becomes

$$-\binom{r+2}{2}\frac{1}{r-1}.$$

Now we perform $C_{\frac{r-2}{2}} \to C_{\frac{r-2}{2}} + (r-2)\binom{\frac{r+2}{2}}{}\frac{1}{r-1}C_1$ to make the $(1, \frac{r-2}{2})^{\text{th}}$ entry 0. For $j \geq 2$, the $(j, \frac{r-2}{2})^{\text{th}}$ entry becomes

$$\begin{aligned} &-\binom{r+2}{2}\binom{\frac{r}{2}}{j-1}\frac{1}{r-j} + \binom{r+2}{2}\binom{\frac{r}{2}}{j-1}\frac{r-2}{(r-j)(r-j-1)} \\ &= \binom{r+2}{2}\binom{r}{2}\frac{1}{(r-j)(r-j-1)}\binom{\frac{r-2}{2}}{j-2}. \end{aligned}$$

Continuing this and making the first $\frac{r-2}{2}$ entries 0, for $j \geq \frac{r-2}{2}$, the $(j, \frac{r-2}{2})^{\text{th}}$ entry becomes

$$(-1)^{\frac{r-2}{2}}\binom{r+2}{2}\binom{r}{2}\cdots(3)\frac{1}{(r-j)(r-j-1)\cdots(\frac{r+4}{2}-j)}\binom{2}{j-\frac{r-2}{2}}.$$

Therefore the $(\frac{r-2}{2}, \frac{r-2}{2})^{\text{th}}$ entry is $(-1)^{\frac{r-2}{2}}$.

Next, we solve the $\frac{1}{n+1}$ parts of the equation $AX = B_2$ separately. We claim that for $2 \leq n \leq r/2$ the $1/(n+1)$ part without $1/(n+1)$ is

$$(-1)^{\frac{r-2}{2}}\sum_{j=1}^{n-1}(-1)^{j-1}\binom{2j}{j-1}\binom{r/2+j}{2j}.$$

For $n = 2$, this is easy to see since the corresponding part of A_2 and the column reduced form of A found above differ only in the last column and so by Cramer's rule contributes $\frac{\binom{\frac{r+2}{2}}{\frac{r-2}{2}}}{(-1)^{\frac{r-2}{2}}}$. For $n = 3$, we perform $C_{\frac{r-2}{2}} \to C_{\frac{r-2}{2}} - \binom{\frac{r+2}{2}}{\frac{r-4}{2}}\frac{\binom{r+2}{2}\binom{r}{2}\binom{r-2}{2}}{6}C_{\frac{r-4}{2}}$ on the corresponding part of A_2. Therefore the $(\frac{r-2}{2}, \frac{r-2}{2})^{\text{th}}$ entry becomes

$$\frac{1}{2}\binom{r+2}{2}\binom{r}{2} - \frac{1}{6}\binom{r+2}{2}\binom{r}{2}\binom{r-2}{2}\binom{r+4}{2} \quad = \quad \binom{\frac{r+2}{2}}{2} - 4\binom{\frac{r+4}{2}}{4},$$

verifying the claim for $n = 3$. The verification of the general claim is left to the reader as usual.

To conclude, we have to prove the identity

$$(-1)^{\frac{r-2}{2}}\sum_{n=2}^{r/2}\frac{1}{n+1}\sum_{j=1}^{n-1}(-1)^{j-1}\binom{2j}{j-1}\binom{r/2+j}{2j} = -H_-.$$

Changing the order of n and j (cf. (1.2)), and changing notation (and binomial coefficients) slightly, the left hand side is

$$H_{n+1}(-1)^n \cdot \sum_{k=1}^{n-1}(-1)^k \binom{n+k}{k-1}\binom{n+1}{k+1} \quad + \quad (-1)^{n-1} \cdot \sum_{k=1}^{n-1}(-1)^k \binom{n+k}{k-1}\binom{n+1}{k+1} H_{k+1}$$

with $n = \frac{r}{2}$. As we shall see in the next section, the first sum is $(-1)^n - (-1)^n\binom{2n}{n-1}$ (replace $n+1$ by n in (5.8)), and the second sum equals $(-1)^n(H_{n-1}+H_{n+1}) - (-1)^n\binom{2n}{n-1}H_{n+1}$ (by (5.7)). So the expression above equals $-H_{n-1} = -H_-$.

5. Footnote 17

We show that in

$$\begin{pmatrix} \binom{r/2}{0}\frac{r}{r} & 0 & \cdots & \binom{r/2}{0} & 0 \\ \binom{r/2}{1}\frac{r}{r-1} & \binom{\frac{r-2}{2}}{0}\frac{r-1}{(r-1)(r-2)} & \cdots & \binom{r/2}{1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \binom{r/2}{r/2}\frac{r}{r/2} & \binom{\frac{r-2}{2}}{\frac{r-2}{2}}\frac{r-1}{(r/2)(\frac{r-2}{2})} & \cdots & \binom{r/2}{r/2} & 1 \end{pmatrix} \begin{pmatrix} x_r \\ x_{r-1} \\ \vdots \\ x_{\frac{r+2}{2}} \\ x_{r/2} \end{pmatrix} = \begin{pmatrix} -\frac{r+2}{2}\binom{r/2}{0}\left(H_{\frac{r+2}{2}}-1\right) \\ -\frac{r+2}{2}\binom{r/2}{1}\left(H_{r/2}-1\right) \\ \vdots \\ -\frac{r+2}{2}\binom{r/2}{\frac{r-2}{2}}\left(H_2-1\right) \\ -\frac{r+2}{2}\binom{r/2}{r/2}\left(H_1-1\right) \end{pmatrix},$$

we have

$$x_{\frac{r+2}{2}} = \frac{r+2}{2}(1-H_- - H_+) \text{ and } x_{r/2} = -(-1)^{r/2}\frac{1}{r/2}.$$

We separate the matrix on the right as $B_1 + B_2$, where

$$B_1 = \begin{pmatrix} \frac{r+2}{2}\binom{r/2}{0}(1-H_+) \\ \frac{r+2}{2}\binom{r/2}{1}(1-H_+) \\ \frac{r+2}{2}\binom{r/2}{2}(1-H_+) \\ \vdots \\ \frac{r+2}{2}\binom{r/2}{r/2}(1-H_+) \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} \frac{r+2}{2}\binom{r/2}{0}(0) \\ \frac{r+2}{2}\binom{r/2}{1}(1/\frac{r+2}{2}) \\ \frac{r+2}{2}\binom{r/2}{2}(1/\frac{r}{2}+1/\frac{r+2}{2}) \\ \vdots \\ \frac{r+2}{2}\binom{r/2}{r/2}(1/2+1/3+\cdots+1/\frac{r+2}{2}) \end{pmatrix}.$$

First, we solve for $x_{r/2}$. Let A_1 and A_2 be the matrices obtained by replacing the last column of A with B_1 and B_2, respectively. Note that the last two columns of A_1 are linearly dependent. Therefore B_1 does not contribute towards $x_{r/2}$. We now compute the determinant of A_2. Note that $\det A_2$ is $\frac{r+2}{2}$ times the determinant of

$$\begin{pmatrix} \binom{r/2}{0}\frac{r}{r} & 0 & \cdots & \binom{r/2}{0} & \binom{r/2}{0}(0) \\ \binom{r/2}{1}\frac{r}{r-1} & \binom{\frac{r-2}{2}}{0}\frac{r-1}{(r-1)(r-2)} & \cdots & \binom{r/2}{1} & \binom{r/2}{1}(1/\frac{r+2}{2}) \\ \vdots & \vdots & & \vdots & \vdots \\ \binom{r/2}{r/2}\frac{r}{r/2} & \binom{\frac{r-2}{2}}{\frac{r-2}{2}}\frac{r-1}{(r/2)(\frac{r-2}{2})} & \cdots & \binom{r/2}{r/2} & \binom{r/2}{r/2}(1/2+1/3+\cdots+1/\frac{r+2}{2}) \end{pmatrix}.$$

We first reduce the second-to-last column using the following column operations so that all but the last two entries are 0. We first perform $C_{\frac{r-2}{2}} \to C_{\frac{r-2}{2}} - C_0$ to make the $(0, \frac{r-2}{2})^{\text{th}}$ entry 0. For $j \geq 1$, the $(j, \frac{r-2}{2})^{\text{th}}$ entry becomes

$$\binom{r/2}{j} - \binom{r/2}{j}\frac{r}{r-j} = -\binom{r}{2}\binom{\frac{r-2}{2}}{j-1}\frac{1}{r-j}.$$

Therefore the $(1, \frac{r-2}{2})^{\text{th}}$ entry is

$$-\binom{r}{2}\frac{1}{r-1}.$$

Next, we perform $C_{\frac{r-2}{2}} \to C_{\frac{r-2}{2}} + (r-2)\binom{r}{2}\frac{1}{r-1}C_1$ to make the $(1, \frac{r-2}{2})^{\text{th}}$ entry 0. For $j \geq 2$, the $(j, \frac{r-2}{2})^{\text{th}}$ entry becomes

$$\begin{aligned} &-\binom{r}{2}\binom{\frac{r-2}{2}}{j-1}\frac{1}{r-j} + \binom{r}{2}\binom{\frac{r-2}{2}}{j-1}\frac{r-2}{(r-j)(r-j-1)} \\ &= \binom{r}{2}\binom{r-2}{2}\binom{\frac{r-4}{2}}{j-2}\frac{1}{(r-j)(r-j-1)}. \end{aligned}$$

Continuing this and making the first $\frac{r-2}{2}$ entries 0, for $j \geq \frac{r-2}{2}$, the $(j, \frac{r-2}{2})^{\text{th}}$ entry becomes

$$(-1)^{\frac{r-2}{2}}\binom{r}{2}\binom{r-2}{2}\cdots(2)\binom{1}{j-\frac{r-2}{2}}\frac{1}{(r-j)(r-j-1)\cdots(\frac{r+4}{2}-j)}.$$

Therefore the matrix becomes

$$\begin{pmatrix} \binom{r/2}{0}\frac{r}{r} & 0 & \cdots & 0 & \binom{r/2}{0}(0) \\ \binom{r/2}{1}\frac{r}{r-1} & \binom{\frac{r-2}{2}}{0}\frac{r-1}{(r-1)(r-2)} & \cdots & 0 & \binom{r/2}{1}(1/\frac{r+2}{2}) \\ \vdots & \vdots & & \vdots & \vdots \\ \binom{r/2}{\frac{r-2}{2}}\frac{r}{\frac{r+2}{2}} & \binom{\frac{r-2}{2}}{\frac{r-4}{2}}\frac{r-1}{(\frac{r+2}{2})(r/2)} & \cdots & (-1)^{\frac{r-2}{2}}\frac{2}{\frac{r+2}{2}} & \binom{r/2}{\frac{r-2}{2}}(1/3+1/4+\cdots+1/\frac{r+2}{2}) \\ \binom{r/2}{r/2}\frac{r}{r/2} & \binom{\frac{r-2}{2}}{\frac{r-2}{2}}\frac{r-1}{(r/2)(\frac{r-2}{2})} & \cdots & (-1)^{\frac{r-2}{2}} & \binom{r/2}{r/2}(1/2+1/3+\cdots+1/\frac{r+2}{2}) \end{pmatrix}.$$

Next, we claim that for $1 \leq n \leq r/2$, the $\dfrac{1}{n+1}$ part of $x_{r/2}$ in $AX = B_2$ without $1/(n+1)$ and up to the factor of $\frac{r+2}{2}$ is

$$(-1)^{n-1}n\binom{2n-1}{n-1}\frac{\binom{r/2+n}{r/2-n}}{\binom{r/2+1}{r/2-1}}.$$

This is easy to see for $n = 1, 2$ using Cramer's rule, noting only the bottom right 2×2 blocks are relevant. For $n = 3$, we kill the $(\frac{r-4}{2}, \frac{r}{2})^{\text{th}}$ entry above using the $(\frac{r-4}{2}, \frac{r-4}{2})^{\text{th}}$ entry, which is $\binom{2}{0}\frac{(\frac{r-4}{2})!(\frac{r+4}{2})}{(\frac{r+4}{2})(\frac{r+2}{2})\cdots(4)} = \frac{6}{(\frac{r+2}{2})(\frac{r}{2})(\frac{r-2}{2})}$. We perform $C_{r/2} \to C_{r/2} - \frac{(\frac{r+2}{2})(\frac{r}{2})(\frac{r-2}{2})}{6}\frac{(\frac{r}{2})(\frac{r-2}{2})}{2}C_{\frac{r-4}{2}}$. This reduces the 1/4 part of the matrix above to a block lower triangular matrix whose lower right block is

$$\begin{pmatrix} (-1)^{\frac{r-2}{2}}\frac{2}{\frac{r+2}{2}} & -(r/2)\frac{1}{12}(r^2+2r-20) \\ (-1)^{\frac{r-2}{2}} & 1-2\binom{\frac{r+4}{2}}{4} \end{pmatrix}.$$

This verifies the claim for $n = 3$ using Cramer's rule. As usual, the verification of the claim for general n is left to the reader.

Therefore, we have to prove the identity

$$\sum_{n=1}^{r/2}\binom{r+2}{2}(-1)^{n-1}\frac{n}{n+1}\binom{2n-1}{n-1}\frac{\binom{r/2+n}{r/2-n}}{\binom{r/2+1}{r/2-1}} = (-1)^{\frac{r+2}{2}}\frac{1}{r/2}.$$

Changing notation slightly, this reduces to proving

$$\sum_{k=1}^{n}(-1)^k\frac{2k}{k+1}\binom{2k-1}{k-1}\binom{n+k}{2k} = (-1)^n$$

for $n = r/2$. Noting that $2\binom{2k-1}{k-1} = \binom{2k}{k}$, the left hand side is

$$\sum_{k=1}^{n}(-1)^k\binom{2k}{k}\binom{n+k}{2k} - \sum_{k=1}^{n}(-1)^k\frac{1}{k+1}\binom{2k}{k}\binom{n+k}{2k}.$$

Including the $k = 0$ terms in both sums, the first sum is $(-1)^n$ and the second sum is 0 by [Gou72, vol 4, Equation 1.4] and [Gou72, vol 5, Equation 1.22] (taking $y = 1$), respectively.

Next we solve for $x_{\frac{r+2}{2}}$. Let now A_1', A_2' be the matrices obtained by replacing the second-to-last column in A with B_1, B_2, respectively. It is clear that A_1' and A only differ in the second-to-last column, and the second-to-last column in A_1' is $\frac{r+2}{2}(1-H_+)$ times that in A. This shows that the B_1 part of $x_{\frac{r+2}{2}}$ is $\frac{r+2}{2}(1-H_+)$. We claim that for $1 \le n \le r/2$, the $1/(n+1)$ part of $x_{\frac{r+2}{2}}$ in $AX = B_2$ without $(\frac{r+2}{2})(1/(n+1))$ is

$$(-1)^{\frac{r-2}{2}}\sum_{j=1}^{n-1}(-1)^{j-1}\frac{j}{j+1}\binom{r/2+j}{2j}\binom{2j}{j}.$$

This is clear for $n = 1, 2$. For $n = 3$, we kill the $(\frac{r-4}{2}, \frac{r-2}{2})^{\text{th}}$ entry of A_2' with the $(\frac{r-4}{2}, \frac{r-4}{2})^{\text{th}}$ entry, which is $\binom{2}{0}\frac{(\frac{r-4}{2})!(\frac{r+4}{2})}{(\frac{r+4}{2})(\frac{r+2}{2})\cdots(4)} = \frac{6}{(\frac{r+2}{2})(\frac{r}{2})(\frac{r-2}{2})}$. We perform $C_{\frac{r-2}{2}} \to C_{\frac{r-2}{2}} - \frac{(\frac{r+2}{2})(\frac{r}{2})(\frac{r-2}{2})}{6}\frac{(\frac{r}{2})(\frac{r-2}{2})}{2}C_{\frac{r-4}{2}}$. Therefore the $(\frac{r-2}{2}, \frac{r-2}{2})^{\text{th}}$ entry becomes

$$\frac{r}{2} - \frac{(\frac{r+2}{2})(\frac{r}{2})(\frac{r-2}{2})}{6}\frac{(\frac{r}{2})(\frac{r-2}{2})}{2}4\frac{\frac{r+4}{2}}{(\frac{r+2}{2})(\frac{r}{2})(\frac{r-2}{2})} = \frac{r}{2} - \frac{1}{3}\left(\frac{r+4}{2}\right)\left(\frac{r}{2}\right)\left(\frac{r-2}{2}\right).$$

Using Cramer's rule, we immediately get

$$(-1)^{\frac{r-2}{2}}\left(\binom{\frac{r+2}{2}}{2} - 4\binom{\frac{r+4}{2}}{4}\right),$$

which is the claim for $n = 3$. As usual, the claim for general n is left to the reader.

To find the B_2 part of $x_{\frac{r+2}{2}}$, we have to prove the identity

$$(-1)^{\frac{r-2}{2}}\left(\frac{r+2}{2}\right)\sum_{n=2}^{r/2}\frac{1}{n+1}\sum_{j=1}^{n-1}(-1)^{j-1}\frac{j}{j+1}\binom{r/2+j}{2j}\binom{2j}{j} = -\left(\frac{r+2}{2}\right)H_-.$$

Cancel $\frac{r+2}{2}$ from both sides and move the sign on the left to the right side. Changing the order of n and j (cf. (1.2)) and changing notation a bit we have to show that

$$H_{n+1}\cdot\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n+k}{k-1}\binom{n+1}{k+1} \quad + \quad \sum_{k=1}^{n-1}(-1)^k\binom{n+k}{k-1}\binom{n+1}{k+1}H_{k+1} = (-1)^n H_{n-1},$$

for $n = \frac{r}{2}$. The sum on the left is $-(-1)^n + (-1)^n\frac{n}{n+1}\binom{2n}{n}$. This follows from (5.8) below (replacing $n+1$ with n). Plugging this into the first sum and noticing that the second part of this is the $k = n$ term of the second sum, we are reduced to showing that

$$S(n) := \sum_{k=1}^{n}(-1)^k\binom{n+k}{k-1}\binom{n+1}{k+1}H_{k+1} = (-1)^n(H_{n-1} + H_{n+1}). \tag{5.7}$$

We can evaluate the 'tricky' sum $S(n)$ with `Sigma` but let us instead use `fastZeil`, since it also provides a theoretical proof (and it is perhaps good to give at least one example in this paper of how one can verify any given identity theoretically). We thank Peter Paule for explaining the entire argument below.

Let L be the operator which evaluates functions $f(x)$ at $x = 0$, i.e., $Lf(x) = f(0)$. Let D be the differentiation operator with respect to x, i.e., $Df(x) = f'(x)$. Clearly

$$LD\binom{x+k}{k} = H_k.$$

Using this, one sees that $S(n) = LDT(x,n)$ where

$$T(x,n) = \sum_{k=0}^{n}(-1)^k\binom{n+k}{k-1}\binom{n+1}{k+1}\binom{x+k+1}{k+1}.$$

The following commands using the package **fastZeil** in Mathematica produce a recurrence relation for $T(x,n)$:

```
In[1]  :=  << RISC`fastZeil`
           Fast Zeilberger Package version 3.61 written by Peter Paule, Markus Schorn and Axel Riese
           Copyright Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria
In[2]  :=  Zb[(-1)^k Binomial[n+k, k-1] Binomial[n+1, k+1] Binomial[x+k+1, k+1], {k,1,n}, n, 2]
Out[2]  =  n^2 (2 + n)^2 SUM[n] - (3 + 2n) (3 + 6 n + 2 n^2 + 4 x + 6 n x + 2 n^2x) SUM[1 + n] - (1 + n)^2 (3 + n)^2 SUM[2 + n] = 0
In[3]  :=  Prove[]
```

Here Out[2] means that:

$$n^2(n+2)^2T(x,n)-(3+2n)(3+6n+2n^2+(4+6n+2n^2)x)T(x,n+1)-(1+n)^2(3+n)^2T(x,n+2) = 0$$

for $n \geq 1$. Applying the operator LD we obtain

$$n^2(2+n)^2S(n)-(3+2n)(4+6n+2n^2)T(0,n+1)-(3+2n)(3+6n+2n^2)S(n+1)-(1+n)^2(3+n)^2S(n+2) = 0.$$

Now we may evaluate the 'easy' sum $T(0,n+1)$ with **Sigma** or by hand using ${}_2F_1$ as in the computation in footnote 10. We have:

$$\begin{aligned} T(0,n+1) &= \sum_{k=0}^{n+1}(-1)^k\binom{n+1+k}{k-1}\binom{n+2}{k+1} \\ &= -\frac{1}{2}(1+n)(2+n)\cdot {}_2F_1\left(\begin{matrix}-n, 3+n\\ 3\end{matrix}; 1\right) \\ &\overset{(1.3)}{=} -\frac{1}{2}(1+n)(2+n)\frac{(3-(3+n))_n}{(3)_n} \\ &= -\frac{1}{2}(1+n)(2+n)\frac{(-1)^n n!}{\frac{1}{2}(n+2)!} \\ &= (-1)^{n+1}. \end{aligned} \tag{5.8}$$

Plugging this into the recursion formula we obtain:

$$n^2(2+n)^2S(n)-(3+2n)(4+6n+2n^2)(-1)^{n+1}-(3+2n)(3+6n+2n^2)S(n+1)-(1+n)^2(3+n)^2S(n+2) = 0.$$

This gives a recurrence relation for the left hand side $S(n)$ of (5.7). Since an easy by hand check shows that the right hand side of (5.7) also satisfies the above recurrence relation and both sides of (5.7) agree for small values of n (the common values are $-3/2, 17/6, -43/12, \ldots$ for $n = 1, 2, 3, \ldots$), we see that (5.7) holds for all $n \geq 1$.

It remains to give a theoretical proof of the recurrence relation for $T(x,n)$ above. This is produced by the command in 'In[3]' above. The output is a 'Computer Theorem' with proof which we reproduce more or less identically below.

Theorem 5.1. (Computer Theorem) *Let*

$$F(k,n) = (-1)^k \binom{n+k}{k-1}\binom{n+1}{k+1}\binom{x+k+1}{k+1}$$

and

$$\mathrm{SUM}(n) = \sum_{k=1}^{n} F(k,n).$$

Then

$$n^2(2+n)^2\mathrm{SUM}(n) - (3+2n)(3+6n+2n^2+4x+6nx+2n^2x)\mathrm{SUM}(1+n) - (1+n)^2(3+n)^2\mathrm{SUM}(2+n) = 0.$$

Proof. Let Δ_k denote the forward difference operator in k and define

$$R(k,n) = \frac{2(k-1)(k+1)^2(n+1)(n+2)(n+3)}{(-k+n+1)(-k+n+2)}.$$

Then the theorem follows by summing the equation

$$\begin{aligned} n^2(2+n)^2F(k,n) - (3+2n)(3+6n+2n^2+4x+6nx+2n^2x)F(k,1+n) \\ -(1+n)^2(3+n)^2F(k,2+n) \quad = \quad \Delta_k(F(k,n)R(k,n)) \end{aligned}$$

over k from 1 to $n+2$ (since the right hand side telescopes to 0 since at the endpoints $F(n+3,n) = 0 = R(1,n)$). This equation is routinely verifiable by dividing both sides by $F(k,n)$ and checking the resulting rational equation holds.

We did indeed carry out the last check by hand, thereby removing all computer programs from the proof of the recurrence relation for $T(x,n)$ and hence the identity (5.7). Though, of course, without the computer's help in producing the auxiliary rational function $R(k,n)$ above, one would not even know where to start.

Acknowledgement. We are grateful to P. Paule for introducing us to the computer packages used in this paper. These packages were developed at the Research Institute of Symbolic Computation (RISC) at Johannes Kepler University in Linz, Austria.

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