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On coprimality of consecutive elements in certain sequences

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Dedicated to Kalyan Chakraborty and Srinivas Kotyada
on the occasion of their 60th anniversaries

Abstract. The study of finding blocks of primes in certain arithmetic sequences is one of the classical problems in number theory. It is also very interesting to study blocks of consecutive elements in such sequences that are pairwise coprime. In this context, we show that if f is a twice continuously differentiable real-valued function on $[1, \infty)$ such that $f''(x) \to 0$ as $x \to \infty$, then there exist arbitrarily long blocks of pairwise coprime consecutive elements in the sequence $(\lfloor f(n) \rfloor)_n$ if and only if f' is unbounded. Among other, this result extends the qualitative part of a recent result by the first author, Drmota and Müllner.

We also prove that, under the same conditions, there exists a subset $\mathcal{A} \subseteq \mathbb{N}$ having upper Banach density one such that for any two distinct integers $m, n \in \mathcal{A}$, the integers $\lfloor f(m) \rfloor$ and $\lfloor f(n) \rfloor$ are pairwise coprime.

Further, we show that there exist arbitrarily long blocks of consecutive elements in the sequence $(\lfloor f(n) \rfloor)_n$ such that no two of them are coprime.

Keywords. Segal-Piatetski-Shapiro sequences, Regular sequences, Pairwise coprime, Banach density

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1. Introduction and Statements of Results

The Segal-Piatetski-Shapiro sequences are sequences of the form $(\lfloor n^c \rfloor)_n$ for a fixed c in $(1, \infty) \setminus \mathbb{N}$. In [Pia53], Piatetski-Shapiro proved that there exist infinitely many primes in the sequence $(\lfloor n^c \rfloor)_n$ if $c \in \left(1, \frac{12}{11}\right)$. In this article, we are interested in finding blocks of consecutive elements in similar sequences, which are pairwise coprime. In a recent work [DDM23], the first author, Drmota and Müllner showed that if $c \in (1, 2)$ and $0 < \alpha < \min\left(c - 1, 1 - \frac{c}{2}\right)$, then there exist infinitely many positive integers n such that for any positive integer $H \le \alpha \log n$, all the elements in the sequence $\{\lfloor n^c \rfloor, \lfloor (n+1)^c \rfloor, \cdots, \lfloor (n+H)^c \rfloor\}$ are pairwise coprime. We refer the reader to [BaSh23, BeRi17, DelDes02, ErLo59, LaMo55, Wat53] for related literature.

In this article, we characterize the twice continuously differentiable functions f such that f'' tends to 0 at infinity for which there exist arbitrarily long sequences of consecutive pairwise coprime integers in the sequence $(\lfloor f(n) \rfloor)_n$.

Theorem 1. *Let f be a twice continuously differentiable real valued function on $[1, \infty)$ such that*

$$f''(x) \to 0 \quad \text{as } x \to \infty \tag{1.1}$$

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Then, for any positive integer H, there exist infinitely many positive integers n such that all the elements in the sequence $\{\lfloor f(n)\rfloor, \lfloor f(n+1)\rfloor, \cdots, \lfloor f(n+H)\rfloor\}$ are distinct and pairwise coprime if and only if

$$f' \text{ is not bounded.} \tag{1.2}$$

Remark 1.1. *In general, pairs of coprime integers are distinct, except for the values ± 1. The phrasing of Theorem 1 avoids obvious uninteresting counterexamples with functions with bounded derivative containing blocks with many ± 1's.*

Remark 1.2. *For c in $(1, 2)$, the function $f(x) = x^c$ satisfies the conditions (1.1) and (1.2) of Theorem 1. This implies the existence of arbitrarily long blocks of consecutive elements in the sequence $(\lfloor n^c\rfloor)_n$ which are pairwise coprime, thus recovering the qualitative part of the main result of [DDM23].*

Remark 1.3. *We note that the first condition in Theorem 1 cannot be replaced by the weaker condition $f'(x) = o(x)$. For $n \geq 2$, let a_n be an even integer in $[n^{3/2} - 2, n^{3/2}]$. Observe that $a_n < a_{n+1}$ and $a_{n+1} - a_n = (3/2 + o(1))\, n^{1/2}$. One can construct a twice continuously differentiable real valued function g on $[1, \infty)$ such that $g(n) = a_n$ and $x^{1/2} < g'(x) < 2x^{1/2}$ for sufficiently large x. Then we have $g'(x) = o(x)$ and $\limsup_{x\to\infty} g'(x) = \infty$, but all the $g(n)$ are distinct and even for all $n \geq 2$.*

In order to state the next result, let us recall the following notion of density. For a subset $\mathcal{S}$ of natural numbers, the upper Banach density (or upper uniform density, see [GTT10, Rib93] for more details) of $\mathcal{S}$ is defined by

$$\lim_{T\to\infty} \limsup_{x\to\infty} \frac{\#\left(\mathcal{S} \cap (x, x+T]\right)}{T}.$$

In this set-up, we prove the following result.

Theorem 2. *Let f satisfy conditions (1.1) and (1.2). Then there exists a subset $\mathcal{A}$ of natural numbers with upper Banach density equal to 1 such that for any distinct pair of integers m and n contained in $\mathcal{A}$, the integers $\lfloor f(m)\rfloor$ and $\lfloor f(n)\rfloor$ are pairwise distinct and coprime.*

In the opposite direction to Theorem 1, it is also possible to find arbitrarily long blocks of elements in the sequence $(\lfloor f(n)\rfloor)_n$ such that no two elements are pairwise coprime. In fact, we prove the following theorem.

Theorem 3. *Let f satisfy conditions (1.1) and (1.2). Then for any positive integer H, there exist infinitely many positive integers n such that all the elements in the sequence $\{\lfloor f(n)\rfloor, \lfloor f(n+1)\rfloor, \cdots, \lfloor f(n+H)\rfloor\}$ are even.*

2. Preliminary results

The following proposition is at the heart of our proof of Theorem 1.

Proposition 4. *Let $H \geq 2$ be a positive integer and $\Pi_H = \prod_{p \leq H} p$ be the product of all primes less than or equal to H. Let $f \in \mathcal{C}^2([1,\infty))$ be a twice continuously differentiable real valued function on $[1,\infty)$ and n be a positive integer satisfying*

$$\{f(n)\} \leq \tfrac{1}{3}, \quad \tfrac{1}{9H} \leq \{f'(n)\} \leq \tfrac{1}{3H} \quad \text{and} \quad |f''(x)| \leq \tfrac{1}{10H^2} \quad \text{for} \quad x \in [n, n+H], \tag{2.3}$$

$$\lfloor f'(n) \rfloor \neq 0 \quad \text{and} \quad \lfloor f'(n) \rfloor \equiv 0 \pmod{\Pi_H}, \tag{2.4}$$

$$\gcd\left(\lfloor f(n) \rfloor, \lfloor f'(n) \rfloor\right) = 1. \tag{2.5}$$

Then the integers in the set $\{\lfloor f(n+h) \rfloor : h \in [H/2, H] \cap \mathbb{N}\}$ are pairwise distinct and coprime.

Proof. By Taylor's theorem, for any integer $h \in [H/2, H]$, there exists $\theta_h \in [0,1]$ such that

$$\begin{aligned} f(n+h) &= f(n) + hf'(n) + \frac{h^2}{2} f''(n+\theta_h h) \\ &= \lfloor f(n) \rfloor + h\lfloor f'(n) \rfloor + \{f(n)\} + h\{f'(n)\} + \frac{h^2}{2} f''(n+\theta_h h). \end{aligned}$$

Note that

$$0 \leq f(n+h) - \lfloor f(n) \rfloor - h\lfloor f'(n) \rfloor \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{20} < 1.$$

This implies that for any integer $h \in [H/2, H]$, we have

$$\lfloor f(n+h) \rfloor = \lfloor f(n) \rfloor + h\lfloor f'(n) \rfloor. \tag{2.6}$$

Notice that (2.6) and (2.4) imply that the $\lfloor f(n+h) \rfloor$'s are pairwise distinct. Observe that for any prime $p \leq H$, we have $p \nmid \lfloor f(n+h) \rfloor$ for any $H/2 \leq h \leq H$. This is because, from (2.4), we have $p \mid \lfloor f'(n) \rfloor$ and from (2.5), $p \nmid \lfloor f(n) \rfloor$ and thus we get $p \nmid \lfloor f(n+h) \rfloor$ by (2.6). Suppose that there exists a prime $p > H$ such that $p \mid \gcd\left(\lfloor f(n+h) \rfloor, \lfloor f(n+k) \rfloor\right)$ for some pair of integers h and k with $H/2 \leq k < h \leq H$. Then $p \mid \lfloor f(n+h) \rfloor - \lfloor f(n+k) \rfloor$. From (2.6), we have

$$\lfloor f(n+h) \rfloor - \lfloor f(n+k) \rfloor = (h-k)\lfloor f'(n) \rfloor.$$

Hence $p \mid \lfloor f'(n) \rfloor$. From (2.6), we deduce that $p \mid \lfloor f(n) \rfloor$, which is a contradiction to (2.5). Thus $\lfloor f(n+h) \rfloor$ and $\lfloor f(n+k) \rfloor$ are coprime for any two distinct integers $h, k \in [H/2, H]$.

The following "folklore" lemma gives an equivalence criterion for $\lfloor x \rfloor$ to be even.

Lemma 5. *Let x be a real number. We have*

$$\lfloor x \rfloor \equiv 0 \pmod{2} \iff \left\{\frac{x}{2}\right\} \in \left[0, \frac{1}{2}\right).$$

The following lemma was surely known, at least by Erdős and Selfridge (cf. [ErSe71]). We however give a simple proof of it.

Lemma 6. *Let N be an integer larger than 1 and $\ell(N)$ be the minimum of $a_N - a_1$ over all sets of integers $a_1 < \ldots < a_N$ which are pairwise coprime. Then, one has*

$$\lim_{N\to\infty} \frac{\ell(N)}{N} = \infty. \tag{2.7}$$

Proof. Let $N \geq 2$ be given and $a_1 < \ldots < a_N$ be a set of pairwise coprime integers for which $a_N - a_1 = \ell(N)$. Denoting by p_i the i-th positive prime, we let k be such that $P_k = p_1 \cdots p_k \leq N < p_1 \cdots p_{k+1}$. We have $k = O(\log N) = o(\ell(N))$. For any i between 1 and k, there is at most one element a_n which is divisible by p_i; thus, aside from at most k elements, all the other elements a_n are coprime to P_k; we thus have

$$\begin{aligned} N &\leq k + \left\lfloor \frac{\ell(N)}{P_k} + 1 \right\rfloor \varphi(P_k) \\ &\leq k + 2\ell(N) \prod_{i \leq k} \left(1 - \frac{1}{p_i}\right) = o(\ell(N)). \end{aligned}$$

3. Proof of Theorem 1

3.A. The case when f' is bounded

Let us assume that a real differentiable function f is such that for all H there exists a number $n(= n(H))$ such that the numbers $\lfloor f(n+1) \rfloor, \ldots, \lfloor f(n+H) \rfloor$ are pairwise distinct and coprime. We can write them in increasing order, i.e. find a permutation σ in $\mathfrak{S}(H)$ such that

$$a_1 = \lfloor f(n + \sigma(1)) \rfloor < \ldots < a_H = \lfloor f(n + \sigma(H)) \rfloor$$

are pairwise coprime. By Lemma 6, we have, as H tends to infinity,

$$\frac{\lfloor f(n + \sigma(H)) \rfloor - \lfloor f(n + \sigma(1)) \rfloor}{H} \to +\infty,$$

which is not compatible with f' bounded.

3.B. The case when f' is not bounded

If f' is not bounded we have either

$$\limsup_{x \to \infty} f'(x) = +\infty \tag{3.8}$$

or

$$\liminf_{x \to \infty} f'(x) = -\infty. \tag{3.9}$$

Both cases can be treated in a similar way and we shall restrict ourselves to the case when f' satisfies (3.8).

Let f satisfy (1.1) and (3.8) and let $H \geq 2$ be an integer. Also let $x_0 > 1$ be a real number such that

$$|f''(x)| \leq \frac{1}{(100H\Pi_H)^3} \tag{3.10}$$

for $x \geq x_0$. Let $q > H$ be a prime number which is sufficiently large. Then there exists a positive integer $m = m(q) > x_0 + 1$ such that

$$\lfloor f'(m-1) \rfloor < \lfloor f'(m) \rfloor = q\Pi_H. \tag{3.11}$$

By the mean value theorem and (3.10), we have for any $x \geq x_0$,

$$|f'(x+1) \;-\; f'(x)| \;\leq\; \frac{1}{(100H\Pi_H)^3} \tag{3.12}$$

which implies in conjunction with the inequality in (3.11) that

$$\{f'(m)\} \;<\; \frac{1}{100H}. \tag{3.13}$$

From (3.12), (3.13) and the fact that $\limsup_{x\to\infty} f'(x) = \infty$, there exists an integer $n_0 \geq m$ such that

$$\lfloor f'(n_0)\rfloor = \lfloor f'(m)\rfloor = q\Pi_H \quad \text{and} \quad \{f'(n_0)\} \;\in\; \left(\frac{1}{6H}, \frac{1}{5H}\right). \tag{3.14}$$

Let $K = 15H\Pi_H$. By Taylor's formula, for any integer $k \in [0, K]$, we have

$$f(n_0+k) \;=\; f(n_0) \;+\; kf'(n_0) \;+\; \frac{k^2}{2}f''(n_0+\theta_k k)$$

for some $\theta_k \in [0,1]$. Let $\epsilon_k = \frac{k^2}{2}f''(n_0+\theta_k k)$. From (3.10), we have

$$|\epsilon_k| \;<\; \frac{1}{20H}. \tag{3.15}$$

We can write

$$f(n_0+k) \;=\; f(n_0) \;+\; kq\Pi_H \;+\; k\{f'(n_0)\} \;+\; \epsilon_k.$$

Let us consider the sequence $(a_k)_{0\leq k\leq K}$, where

$$a_k \;=\; f(n_0+k) - kq\Pi_H \;=\; f(n_0) \;+\; k\{f'(n_0)\} \;+\; \epsilon_k.$$

From (3.14) and (3.15), we have

$$a_K \;-\; a_0 \;=\; K\{f'(n_0)\} \;+\; \epsilon_K \;>\; 15H\Pi_H\cdot\frac{1}{6H} \;-\; \frac{1}{20H} \;>\; 2\Pi_H.$$

For any $k \in [0, K-1]$, we get

$$0 \;<\; a_{k+1} \;-\; a_k \;=\; \{f'(n_0)\} \;+\; \epsilon_{k+1} \;-\; \epsilon_k \;<\; \frac{1}{5H} \;+\; \frac{1}{20H} \;+\; \frac{1}{20H} \;<\; \frac{1}{3H}. \tag{3.16}$$

Thus, there exists an integer $k_0 \in [0, K]$ such that

$$\lfloor a_{k_0}\rfloor \;\equiv\; 1 \pmod{\Pi_H}, \quad \lfloor a_{k_0}\rfloor \;\not\equiv\; 0 \pmod{q} \quad \text{and} \quad \{a_{k_0}\} \;<\; \frac{1}{3}.$$

To see this, let $b \in [a_0, a_K)$ be the smallest integer such that

$$b \;\equiv\; 1 \pmod{\Pi_H} \quad \text{and} \quad b \;\not\equiv\; 0 \pmod{q}.$$

Such an integer exists, since $a_K - a_0 > 2\Pi_H$. Let $k_0 \in [0, K]$ be the smallest integer such that $a_{k_0} \geq b$. Then from (3.16), we have $\lfloor a_{k_0}\rfloor = b$ and $\{a_{k_0}\} < \frac{1}{3H}$. We set $n = n_0 + k_0$. It is easy to see that

$$\begin{aligned} &\lfloor f'(n)\rfloor \;=\; q\Pi_H, \quad \gcd(\lfloor f(n)\rfloor, \, q\Pi_H) \;=\; 1, \\ &\{f(n)\} \;\leq\; \frac{1}{3}, \quad \frac{1}{9H} \;\leq\; \{f'(n)\} \;\leq\; \frac{1}{3H}, \quad |f''(n)| \;\leq\; \frac{1}{10H^2}. \end{aligned} \tag{3.17}$$

Hence there exist infinitely many pairs positive integers q, $n = n(q)$ satisfying (3.17). Now Theorem 1 follows from Proposition 4. □

4. Proof of Theorems 2 and 3

4.A. Proof of Theorem 2

Let f satisfy (1.1) and (3.8). Let $H_1 \geq 2$ be a natural number. Then from Theorem 1, there exists $n_1 \in \mathbb{N}$ such that the integers $\lfloor f(n_1) \rfloor, \lfloor f(n_1+1) \rfloor, \cdots, \lfloor f(n_1+H_1) \rfloor$ are pairwise coprime. Let H_2 be a natural number strictly greater than $H_1 + \max_{0 \leq h \leq H_1} |f(n_1+h)|$. Now proceeding as in the proof of Theorem 1, we can find a prime $q_2 > H_2$ and a natural number $n_2 > n_1 + H_1$ such that

$$\lfloor f'(n_2) \rfloor \;=\; q_2 \Pi_{H_2}, \quad \gcd\left(\lfloor f(n_2) \rfloor, q_2 \Pi_{H_2}\right) \;=\; 1 \quad \text{and} \quad \lfloor f(n_2+h) \rfloor \;=\; \lfloor f(n_2) \rfloor + h \lfloor f'(n_2) \rfloor$$

for $H_2/2 \leq h \leq H_2$. Then the integers in the set $\{\lfloor f(n_2+h) \rfloor : h \in [H_2/2, H_2] \cap \mathbb{N}\}$ are pairwise coprime. Also, if p is a prime which divides $\lfloor f(n_1+h) \rfloor$ for some $0 \leq h \leq H_1$, then p divides Π_{H_2}, since $H_2 > H_1 + \max_{0 \leq h \leq H_1} |f(n_1+h)|$. Hence p does not divide $\lfloor f(n_2+h) \rfloor$ for any integer $h \in [H_2/2, H_2]$. By induction, there exist integers $H_r > H_{r-1} + \max\{|f(n_i+h_j)| : h_j \in [H_i/2, H_i] \cap \mathbb{N}, 1 \leq i \leq r-1\}$ and $n_r > n_{r-1} + H_{r-1}$ such that the integers in the set $\{\lfloor f(n_i+h_j) \rfloor : h_j \in [H_i/2, H_i] \cap \mathbb{N},\ 1 \leq i \leq r\}$ are pairwise coprime. Let

$$\mathcal{A} \;=\; \{n_i + h_j \;:\; h_j \in [H_i/2, H_i] \cap \mathbb{N},\ i \in \mathbb{N}\}.$$

Then clearly the upper Banach density of $\mathcal{A}$ is equal to 1. This completes the proof of Theorem 2. ☐

4.B. Proof of Theorem 3

Arguing as in the proof of Theorem 1 (cf. (3.10) and (3.14) for the function $f/2$), for given H, we can find infinitely many positive integers n such that

$$\left\{\frac{f'(n)}{2}\right\} \;\in\; \left(\frac{1}{6H}, \frac{1}{5H}\right) \quad \text{and} \quad |f''(x)| \;\leq\; \frac{1}{1000H^3}$$

for $x \geq n$. The Taylor expansion of f leads to

$$\frac{f(n+h)}{2} \;=\; \frac{f(n)}{2} \;+\; h\frac{f'(n)}{2} \;+\; \epsilon_h$$

with $|\epsilon_h| < \frac{1}{50H}$ for $0 \leq h \leq 10H$. We have

$$\left\{\frac{f(n+h)}{2}\right\} \;=\; \left\{\left\{\frac{f(n)}{2}\right\} \;+\; h\left\{\frac{f'(n)}{2}\right\} \;+\; \epsilon_h\right\}.$$

Set $\xi_h = h\left\{\frac{f'(n)}{2}\right\} + \epsilon_h$. Then we have

$$\xi_{10H} \;>\; 1 \quad \text{and} \quad 0 \;<\; \xi_{h+1} \;-\; \xi_h \;<\; \frac{1}{4H}$$

for any integer $h \in [0, 10H)$. Thus one can find at least H consecutive values of $h \in [0, 10H)$ such that $\left\{\frac{f(n+h)}{2}\right\} \in \left[0,\ \frac{1}{2}\right)$, which, thanks to Lemma 5, proves Theorem 3. ☐

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