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ON THE BARBAN-DAVENPORT-HALBERSTAM THEOREM: X

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Before resuming our researches on the theme of the third [1] and ninth [3] articles of this series (denoted, respectively, by III and IX in what follows) concerning general theorems of Barban-Davenport-Halberstam type, we should remark that the class of sequences treated in the former and initial paper was influenced by our prior appreciation of the way in which the properties of the prime numbers entered into known proofs of the Barban-Davenport-Halberstam theorem itself. All in fact that seemed to be needed in this respect was tantamount to just the prime number theorem for arithmetical progressions, the statement of which for present purposes is best divided into two parts. The first is the existence of an asymptotic formula of the form

$$E(x;a,k) = \theta(x;a,k) - f(a,k)x = \sum_{\substack{p\le x\\ p\equiv a,\ \mathrm{mod}\ k}} \log\, p - f(a,k)x = O(x\log^{-A} x) \tag{1}$$

for any positive constant A, without which one would not naturally consider the moments

$$G(x,Q) = \sum_{k\le Q}\ \sum_{0<a\le k} E^2(x;a,k)$$

and without which the estimate for $G(x,Q)$ supplied by the Barban- Davenport-Halberstam theorem would be false. The second part is that

$$f(a,k) = \begin{cases} 1/\phi(k), if\ (a,k)=1, \\ 0, otherwise, \end{cases}$$

a feature expressing the equi-distribution of the primes among residue classes, mod k, for given values of k and *(a,k)* that is exploited directly and indirectly in several parts of the proof. In reflection of these features the sequences of integers s within the purview of III and the intermediate paper[2] were subject to the Criterion U to the effect that

$$E(x;a,k) = S(x;a,k) - g\{k,(a,k)\}x = \sum_{\substack{s\le x\\ s\equiv a,\mathrm{mod}\ k}} 1 - g\{k,(a,k)\}x = O(x\log^{-A} x)$$

for any positive constant A, the conclusion being that

$$G(x,Q) = \sum_{k\le Q}\ \sum_{0<a\le k} E^2(x;a,k) = O(Qx) + O(x^2\log^{-A} x). \tag{2}$$

In the present article we begin by showing that Criterion U unnecessarily restricts the class of sequences that enjoy the feature of being the subject of a theorem of Barban-Davenport-Halberstam type, since we shall demonstrate that we can altogether dispense with any requirement about the equi-distribution of the sequence s in various arithmetical progressions to a given modulus k. Thus, now only assuming that

$$S(x;a,k) = \sum_{\substack{s\leq x \\ s\equiv a, \bmod k}} 1 = f(a,k)x + O(x\log^{-A} x) \tag{3}$$

and writing

$$E(x;a,k) = S(x;a,k) - f(a,k)x, \tag{4}$$

we establish (2) in its extended sense and thus shew there is a strict equivalence between it and (3) that was previously absent. Of central importance in the new treatment is the entrance of inequalities of large sieve type in more contexts than in our [2], on which our work is partially based.

We then proceed to the subject of generalized Barban-Montgomery theorems, one version of which was produced in IX for sequences (of positive density) satisfying Criterion U and a condition S to the effect that $g(k,0)$ was the product $C\psi(k)$ of a positive constant C and a multiplicative function $\psi(k)$. Here, initiating our analysis by using the new result on the first part, we again shew that some earlier imposed restraints are unnecessary and prove the asymptotic formulae

$$G(x,Q) = [D_1 + o(1)]\, Qx + O(x^2\log^{-A} x) \quad (Q \leq x; o(1) \to 0 \ as \ x/Q \to \infty), \tag{5}$$

$$G(x,x) = D_2x^2 + O(x^2\log^{-A} x) \tag{6}$$

of Barban-Montgomery type subject only to the fundamental condition (3) above. Much of the method in IX is retained, although there is an important departure from past practice that entails yet another application of a large sieve inequality. Yet the new result does not altogether imply the superannuation of its predecessor, since the condition under which the latter was established led to a determination of the constant D_1 whose significance was indicated at the end of IX.

It might be asked why the trouble is taken to enunciate the Barban-Davenport-Halberstam type result when under the stated conditions it is contained in the Barban-Montgomery result. In response, we would say, first that the former constitutes an important stage in the proof of the latter and, secondly, that it is desirable to regard these theorems as being in different genres, it not necessarily being the case in general that two such results share a common domain of validity.

Throughout A denotes a positive absolute constant on which and on the sequence the constants implied by the O-notation at most depend. The letter s not only denotes a member of the given sequence but also a complex variable $\sigma + it$, it being clear from the context which meaning is intended.

The statement of the large sieve inequality required for the first part is due to Montgomery [4], who also recorded it in his book [5]. Translated into the language of this paper and specialized for the sequence of numbers s, this asserts that

$$\sum_{l \le Q_1} \frac{1}{l} \sum_{0<a\le l} \left[\sum_{d|l} \mu\left[\frac{l}{d}\right] dS(x;a,d)\right]^2 = O\left[(Q_1^2 + x)\sum_{s\le x} 1\right]$$

$$= O\left[Q_1^2 x\right] + O(x^2) \tag{7}$$

in contrast to a later quoted result in which the Möbius function occurs at a different place in the summation.

We then take up the story at a point corresponding to the beginning of § 4, I I I in [2], having noted the bounds

$$S(x;a,k) = O(x/k)\ (k \le x),\ f(a,k) = O(1/k) \tag{8}$$

that spring from the same obvious source as did Lemma 2 in article III. If, as before, we write

$$G(x,Q) = \sum_{k\le Q} \sum_{0<a\le k} E^2(x;a,k) = \sum_{k\le Q} H(x,k)$$

and assume throughout that $Q \le x$, then (4) implies that

$$H(x,k) = \sum_{0<a\le k} \{S(x;a,k) - f(a,k)x\}^2$$

$$= \sum_{0<a\le k} S^2(x;a,k) - 2x \sum_{0<a\le k} f(a,k)S(x;a,k) + x^2 \sum_{0<a\le k} f^2(a,k)$$

$$= Y_x(k) - 2x\Psi_x(k) + x^2 M(k), \text{say}, \tag{9}$$

in emulation of the initial stage of the derivation of (163) in [2]. Yet this is as far as we can go without taking a serious detour, since $H(x,k)$ is no longer essentially a variance and since therefore $\Psi_x(k)$ becomes a covariance that cannot necessarily be easily estimated.

The analysis of $\Psi_x(k)$ falls into two parts, in the first of which we obtain

$$\Psi_k(k) = \sum_{0<a\le k} f(a,k)\left\{f(a,k)x + O\left[\frac{x}{\log^{2A+4} x}\right]\right\}$$

$$= x\sum_{0<a\le k} f^2(a,k) + O\left[\frac{x}{k\log^{2A+4} x}\sum_{0<a\le k} 1\right]$$

$$= xM(k) + O(x\log^{-2A-4} x) \tag{10}$$

by (3), (8), and (9). But having a weak remainder term for larger values of k, this estimate must be complemented by a result on the sum

$$\sum_{k \le Q} \Psi_x(k), \tag{11}$$

which is the subject of the second part.

To estimate (11) let us first extend the definition of $f(a,k)$ in the obvious way by making it a periodic function in a, modulo k. Secondly, set

$$f(a,k) = \frac{1}{k} \sum_{l|k} w(a,l) \tag{12}$$

so that

$$w(a,l) = \sum_{d|l} \mu \left[\frac{l}{d}\right] df(a,d) \tag{13}$$

by one of the Möbius formulae, wherefore $w(a,l)$ is periodic, mod ℓ, and

$$w(a,l) = O\left[\sum_{d|l} 1\right] = O\{d(l)\}$$

by (8). Next, since

$$\Psi_x(k) = \sum_{0<a\le k} f(a,k) \sum_{\substack{s \le x \\ s \equiv a, \text{ mod k}}} 1 = \sum_{s \le x} f(s,k)$$

on account of (9), we have

$$\begin{aligned}\sum_{k \le Q} \Psi_x(k) &= \sum_{s \le x} \sum_{k \le Q} f(s,k) = \sum_{s \le x} \sum_{lm \le Q} \frac{w(s,l)}{lm} \\ &= \sum_{lm \le Q} \frac{1}{lm} \sum_{s \le x} w(s,l) \\ &= \sum_{lm \le Q} \frac{W(x,l)}{lm}, \text{ say,}\end{aligned} \tag{14}$$

wherein

$$W(x,l) = \sum_{0<a\le l} w(a,l) \sum_{\substack{s \le x \\ s \equiv a, \text{ mod } \ell}} 1 = \sum_{0<a\le l} w(a,l) S(x;a,l). \tag{15}$$

The sum $W(x,l)$ is another example of an entity whose estimation must fall into two portions. On the one hand, if $l \le \xi_1 = \log^{4A+10} x$, then we shall apply the equation

$$W(x,l) = x \sum_{0<a\le l} w(a,l) f(a,l) + O\left[\frac{x}{\log^{10A+26} x} \sum_{0<a<l} |w(a,l)|\right],$$

$$= xN(l) + O\left[\frac{xld(l)}{\log^{10A+26} x}\right], \text{say} \tag{16}$$

that stems from (15) and (3), having noted in particular that

$$N(l) = O\left[\frac{1}{l}\sum_{0<a\leq l} |w(a,l)|\right].$$

But, on the other hand, in the contrary instance we merely use (8) to obtain the trivial estimate

$$W(x,l) = O\left[\frac{x}{l}\sum_{0<a\leq l} |w(a,l)|\right] = xN(l) + O\left[\frac{x}{l}\sum_{0<a\leq l} |w(a,l)|\right],$$

which, when inserted with (16) into (14), yields

$$\sum_{k\leq Q} \Psi_x(k) = x\sum_{lm\leq Q} \frac{N(l)}{lm} + O\left[\frac{x}{\log^{10A+25} x}\sum_{l\leq \xi_1} ld(l)\right]$$

$$+O\left[x\left\{\sum_{m\leq Q}\frac{1}{m}\right\}\left\{\sum_{\xi_1<l\leq Q}\frac{1}{l^2}\sum_{0<a\leq l}|w(a,l)|\right\}\right]$$

$$= x\sum_{lm\leq Q}\frac{N(l)}{lm} + O\left[\frac{x\xi_1^2\log\xi_1}{\log^{10A+25} x}\right] + O\left[x\log x\sum_{\xi_1<l\leq Q}\frac{1}{l^2}\sum_{0<a\leq l}|w(a,l)|\right]$$

$$= x\sum_{k\leq Q} M_1(k) + O\left[\frac{x}{\log^{2A+4} x}\right] + O\left[x\log x\sum_{\xi_1<l\leq Q}\frac{1}{l^2}\sum_{0<a\leq l}|w(a,l)|\right] \tag{17}$$

where

$$M_1(k) = \frac{1}{k}\sum_{l|k} N(l) \tag{18}$$

and where it is helpful in what follows to note that the last double sum occurring in (17) is zero for given Q as $x \to \infty$.

The above estimate would actually fully serve our purposes in the proof of the first theorem as soon as the final remainder term therein was suitably measured. But a more polished and tidy product is derived by immediately comparing (17) with (10) as $x \to \infty$ to deduce that

$$M_1(k) = M(k), \tag{19}$$

as identification that anyway becomes imperative when we embark on the demonstration of the second theorem. Proceeding then to the final remainder term in (18), we need to bring the large sieve inequality (7) into play by substituting (3) into it and deducing that

$$\sum_{l\leq Q_1}\frac{1}{l}\sum_{0<a\leq l}\left\{\sum_{d|l}\mu\left[\frac{l}{d}\right]d\left[f(a,d)x + O\left[\frac{x}{\log^A x}\right]\right]\right\}^2 = O\left[Q_1^2 x\right] + O(x^2),$$

from which, by dividing by x^2 and letting $x \to \infty$, we conclude that

$$\sum_{l \leq Q_1} \frac{1}{l} \sum_{0<a\leq l} w^2(a,l) = O(1) \text{ and } \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sum_{0<a\leq \ell} \mathrm{w}^2(\mathrm{a},\mathrm{l}) \text{ converges.}$$

Hence, by the Cauchy-Schwarz inequality,

$$\sum_{l>\xi_1} \frac{1}{l^2} \sum_{0<a\leq l} |w(a,l)| \leq \left[\sum_{l>\xi_1} \frac{1}{l^3} \sum_{0<a\leq l} 1\right]^{\frac{1}{2}} \left[\sum_{l=1}^{\infty} \frac{1}{l} \sum_{0<a\leq l} w^2(a,l)\right]^{\frac{1}{2}}$$

$$= O\left\{\left[\sum_{l<\xi_1} \frac{1}{l^2}\right]^{\frac{1}{2}}\right\} = O\left[\frac{1}{\xi_1^{\frac{1}{2}}}\right]$$

and so we finally see from (18) and (19) that

$$\sum_{k\leq Q} \Psi_x(k) = x \sum_{k\leq Q} M(k) + O\left[\frac{x}{\log^{2A+4} x}\right], \tag{20}$$

which result is valid for all Q and contains the preliminary (10).

Temporarily retreating to (9) to sum up what has so far been achieved, we obtain a counterpart of (163) in [2] in the form of the equation

$$H(x,k) = \sum_{0<a\leq k} S^2(x;a,k) - x^2 M(k) + Z_x(k) \tag{21}$$

$$= Y_x(k) - x^2 M(k) + Z_x(k), \text{say}, \tag{22}$$

where

$$\sum_{k\leq Q} Z_x(k) = O\left[\frac{x^2}{\log^{2A+4} x}\right] \text{ and } Z_x(k) = O\left[\frac{x^2}{\log^{2A+4} x}\right]. \tag{23}$$

This will be used in company with the bound

$$H(x,k) = O\left[\frac{kx^2}{\log^{3A+5} x}\right] \tag{24}$$

that is immediate from (9) and (3).

There are at least two ways of advancing from (22) to the first theorem, one of which entails the appearance of two different versions of the large sieve inequality that are distinct from (7). We therefore retrace as closely as possible the final part of [2], which only needed one large sieve inequality and permits us to present the treatment with brevity. First, letting $y \geq x$ be an auxiliary variable, we have

$$Y_y(k) = \frac{1}{k} \sum_{q|k} \sum_{\substack{0<p\leq q \\ (p,q)=1}} \left|\sum_{s\leq y} e^{2\pi i p s/q}\right|^2 = \frac{1}{k} \sum_{q|k} V_y(q)$$

as in equation (164) of [2]. Secondly, if

$$M(k) = \frac{1}{k}\sum_{q|k} v(l)$$

(so that actually $v(q) = N(q)$, the replacement notation having been previously avoided lest it were felt that (19) was being assumed without proof) and $\xi = \log^{A+1} x$, then

$$H(y,k) = \frac{1}{k}\sum_{\substack{q|k \\ q\le\xi}} \left\{V_y(q) - y^2 v(q)\right\} + \frac{1}{k}\sum_{\substack{q|k \\ q>\xi}} V_y(q) - \frac{y^2}{k}\sum_{\substack{q|k \\ q>\xi}} v(q) + Z_y(k) \tag{25}$$

by (22). Next, as this, (24), and (23) imply that

$$\sum_{q|k}\left\{V_y(q) - y^2 v(q)\right\} = O\left[\frac{k^2y^2}{\log^{3A+5} x}\right] + O\left[\frac{ky^2}{\log^{2A+4} x}\right] = O\left[\frac{y^2}{\log^{A+3} x}\right]$$

for $k \le \xi$, it follows that

$$V_y(q) - y^2 v(q) = O\left[\frac{d(q)y^2}{\log^{A+3} x}\right]$$

for $q \le \xi$ and hence that

$$\sum_{\substack{q|k \\ q\le\xi}}\left\{V_y(q) - y^2 v(q)\right\} = O\left[\frac{d_3(k)y^2}{\log^{A+3} x}\right]$$

always. Hence (25) becomes

$$H(y,k) = \frac{1}{k}\sum_{\substack{q|k \\ q\ge\xi}} V_y(q) - \frac{y^2 R(k,\xi)}{k} + O\left[\frac{d_3(k)y^2}{k\log^{A+3} x}\right] + Z_y(k)$$

in the notation of [2], which equation when summed over k yields

$$G(y,Q) = O\left[\sum_{\xi<q\le Q}\frac{1}{q}\log\frac{2Q}{q}V_y(q)\right] - y^2\sum_{k\le Q}\frac{R(k,\xi)}{k} + O\left[\frac{y^2}{\log^A x}\right]$$

$$= O(Qy) + O\left[\frac{y^2}{\log^A x}\right] - y^2\sum_{k\le Q}\frac{R(k,\xi)}{k}$$

when the method of [2] is followed (the temporary restriction that $Q > \xi$ imposed in [2] is not really essential provided that empty sums be properly interpreted). From this, as in [2], we get

$$G(y,Q) = O(Qy) + O(y^2\log^{-A} x)$$

and then, setting $y = x$, obtain our

THEOREM 1. *For any sequence satisfying condition* (3),*we have*

$$\sum_{k\le Q}\ \sum_{0<a\le k}\{S(x;a,k)-f(a,k)x\}^2=O(Qx)+O(x^2\log^{-A}x)$$

for $Q\le x$ *and any positive constant* A.

The earlier part of the treatment of the second theorem is almost identical in method and notation to that of its analogue in IX, it therefore being enough merely to indicate the minor alterations needed. In the case $Q>x\log^{-A}x$ that it suffices to consider in view of Theorem 1, we substitute our equation (21) for IX (9), using our present $M(k)$ in (9) above to denote what was previously written as $C^2M(k)$. Then, summing over k by means of (24) and retaining IX (10) with the new definition of $M(k)$, we replace IX(11) by

$$G(x;Q_1,Q_2)=\Gamma(x;Q_1,Q_2)-x^2\{T(Q_2)-T(Q_1)\}+O(x^2\log^{-A}x),$$

to which must be appended IX (12) with

$$C=f(0,1). \tag{26}$$

Afterwards $f(b,l)$ must substitute for $g(l,\delta)$, whence we modify IX (17) by using the new $M(l)$ and omitting C.

No longer do we treat the generating function for the sums $T^*(x/Q)$ and $T(Q)$ by the sieve method of IX but instead bring in the bound

$$\sum_{n\le Q_1}\ \sum_{l|n}\mu\left[\frac{n}{l}\right]l\sum_{0<a\le l}S^2(x;a,l)=O(Q_1^2x)+O(x^2),$$

which is a corollary of the large sieve inequality

$$\sum_{q\le Q_1}\ \sum_{\substack{0<p\le q\\(p,q)=1}}\left|\sum_{s\le x}e^{2\pi isp/q}\right|^2=O(Q_1^2x)+O(x^2)$$

and

$$\sum_{\substack{0<p\le q\\(p,q)=1}}\left|\sum_{s\le x}e^{2\pi isp/q}\right|^2=\sum_{d|q}\mu\left[\frac{q}{d}\right]\sum_{0<h\le d}\left|\sum_{s\le x}e^{2\pi ish/d}\right|^2$$

$$=\sum_{d|q}\mu\left[\frac{q}{d}\right]d\sum_{o<a\le d}S^2(x;a,d). \tag{27}$$

In this we substitute formula (3) for $S(x;a,d)$ and, letting $x\to\infty$ as in the application of (7), infer that

$$\sum_{n\le Q_1}na_n=O(1),$$

where

$$na_n=\sum_{l|n}\mu\left[\frac{n}{l}\right]lM(l),$$

or, what is the same,

$$M(l) = \frac{1}{l}\sum_{n|l} na_n.$$

Hence

$$\sum_{l=1}^{\infty} na_n$$

is absolutely convergent because (27) means that a_n is non-negative. Consequently, if $F(s)$ be the Dirichlet's series

$$\sum_{n=1}^{\infty} \frac{M(l)}{l^s},$$

then

$$F(s) = \zeta(s+1)\Phi(s),$$

wherein

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is absolutely convergent for $\sigma \geq -1$ and regular for $\sigma > -1$.

We have reached the same point in our work as at IX (28) and have set the scene for a treatment of $T^*(x/Q)$ and $T(Q)$ that is identical to that of IX save for the substitution of $\Phi(s)$ for $c^2\Phi(s)$. Combining as before the consequential estimates with the earlier equations, we infer the truth of

THEOREM 2. *For any sequence satisfying condition* (3), *we have*

$$\sum_{k\leq Q}\ \sum_{0<a\leq k} \{S(x;a,k) - f(a,k)x\}^2 = [D_1 + o(1)]\, Qx + O\left[x^2 \log^{-A} x\right],$$

for $Q \leq x$ and any given positive constant A, where D_1 is a non-negative constant depending only on the sequence and where $o(1) \to 0$ as $x/Q \to \infty$. Also

$$\sum_{k\leq x}\ \sum_{0<a\leq k} \{S(x;a,k) - f(a,k)x\}^2 = D_2 x^2 + O\left[x^2 \log^{-A} x\right]$$

where D_2 is a non-negative constant depending only on the sequence.

We note that

$$D_1 = f(0,1) - \sum_{n=1}^{\infty} na_n,$$

a determination that ceases to be easy to interpret when Criteria U and S are no longer in place. However, we should at least observe that $f(0,1)$ is zero when and only when the sequence has zero density in which situation $f(a,k)$ and therefore D_1 are also zero. In this case we would need to adopt appropriate alternatives to supposition (3) in order to seek a theorem of Barban-Montgomery type in which the explicit term would be of a lower order of magnitude than Qx unless weights were attached to the sequence.

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