On the Riesz means of $\delta_k(n)$

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Abstract. Let $k \geq 1$ be an integer. Let $\delta_k(n)$ denote the maximum divisor of n which is co-prime to k. We study the error term of the general m-th Riesz mean of the arithmetical function $\delta_k(n)$ for any positive integer $m \geq 1$, namely the error term $E_{m,k}(x)$ where

$$\frac{1}{m!}\sum_{n\leq x}\delta_k(n)\left(1-\frac{n}{x}\right)^m = M_{m,k}(x)+E_{m,k}(x).$$

We establish a non-trivial upper bound for $\left|E_{m,k}(x)\right|$, for any integer $m \geq 1$.

Keywords. Euler-totient function, Generating functions, Riemann zeta-function.

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1. Introduction

For any fixed positive integer k, we define

$$\delta_k(n) = \max\{d : d \mid n, \quad (d,k)=1\}. \tag{1.1}$$

Joshi and Vaidya [JV] proved that

$$\sum_{n\leq x}\delta_k(n) = \frac{k}{2\sigma(k)}x^2 + E_k(x), \tag{1.2}$$

with $E_k(x) = O(x)$ and $\sigma(k) = \sum_{d|k} d$, when k is a square free positive integer. They also proved that when $k=p$, a prime,

$$\underline{\lim}_{n\to\infty}\frac{E_p(x)}{x} = -\frac{p}{p+1}, \quad \text{and} \quad \overline{\lim}_{n\to\infty}\frac{E_p(x)}{x} = \frac{p}{p+1}.$$

It was proved by Maxsein and Herzog [MH] that for any square free positive integer k,

$$\underline{\lim}_{n\to\infty}\frac{E_k(x)}{x} \leq -\frac{k}{\sigma(k)}, \quad \text{and} \quad \overline{\lim}_{n\to\infty}\frac{E_k(x)}{x} \geq \frac{k}{\sigma(k)}.$$

Around the same time, Adhikari, Balasubramanian and Sankaranarayanan [ABS] proved the above results by a different method. While a tauberian theorem of Hardy-Littlewood and Karamata was used in [MH] to get the asymptotic formula for $\sum_{n\leq x}\gamma_k(n)$, where $\gamma_k(n)$ is defined by the relation $\delta_k(n) = \gamma_k * I(n)$ where $*$ is the Dirichlet convolution and I is the identity function, the method of [ABS] consists of averaging over arithmetical progressions.

For $k \geq 1$ and square free, Harzog and Maxsein [MH] had also observed that

$$\limsup_{x\to\infty}\frac{E_k(x)}{x} \leq \frac{1}{2}d(k),$$

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where $d(k)$ denotes the number of divisors of k. Later Adhikari and Balasubramanian [AB] improved this result of Maxsein and Herzog by showing that

$$\overline{\lim_{n\to\infty}} \frac{|E_k(x)|}{x} \leq \frac{1}{2}\left(1 - \frac{1}{p+1}\right) d(k),$$

where p denotes the smallest prime dividing k.

Writing

$$H_k(x) = \sum_{n\leq x} \frac{\delta_k(n)}{n} - \frac{kx}{\sigma(k)},$$

one observes (see [ABS]) that

$$\frac{E_k(x)}{x} = H_k(x) + O(1).$$

In [AS], more precise upper and lower bounds for the quantities $\underline{\lim}\, H_k(x)$ and $\overline{\lim}\, H_k(x)$ were established. The aim of this article is to study the error term of the general m-th Riesz mean related to the arithmetic function $\delta_k(n)$ for any positive integer $m \geq 1$ and $k \geq 1$. More precisely, we write

$$\frac{1}{m!} \sum_{n\leq x} \delta_k(n) \left(1 - \frac{n}{x}\right)^m = M_{m,k}(x) + E_{m,k}(x) \tag{1.3}$$

where $M_{m,k}(x)$ is the main term (exists) and $E_{m,k}(x)$ is the error term of the sum under investigation. We prove the following.

Theorem 1.1. *Let $x \geq x_0$ where x_0 is a sufficiently large positive number and let $c(\eta) = \frac{2}{1-2^{-\eta}}$ for any $\eta > 0$. For any integer $m \geq 1$ and for any integer $k \geq 1$, we have*

$$\frac{1}{m!} \sum_{n\leq x} \delta_k(n) \left(1 - \frac{n}{x}\right)^m = \frac{x^2}{(m+2)!} \prod_{p|k} \frac{p}{p+1} + E_{m,k}(x),$$

where

$$E_{1,k}(x) \ll kc(1/2)^{\omega(k)} x^{\frac{1}{2}} \log x,$$

and for $m \geq 2$, we have

$$E_{m,k}(x) \ll kc(\eta)^{\omega(k)} x^{\eta}$$

for any small fixed positive constant η and the implied constant is independent of m.

2. Notation

1. Throughout the paper, $s = \sigma + it$; the parameters T and x are sufficiently large real numbers and m is an integer ≥ 1.

2. η, ϵ always denote sufficiently small positive constants.

3. As usual $\zeta(s)$ denotes the Riemann zeta-function.

4. k is any square free positive integer.

3. Some Lemmas

Generating function for $\delta_k(n)$ is given by:

Lemma 3.1. *We have*

$$\sum_{n=1}^{\infty} \frac{\delta_k(n)}{n^s} = \zeta(s-1)G(s),$$

where

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_{p|k} \left(\frac{1 - \frac{p}{p^s}}{1 - \frac{1}{p^s}} \right) \ll k\ c(\eta)^{\omega(k)},$$

for $\sigma \geq \eta$ and

$$c(\eta) = \frac{2}{1 - 2^{-\eta}}.$$

Proof. We have (see [ABS, equation 2.2]),

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{\delta_k(n)}{n^s} &= \prod_p \left(1 + \frac{\delta_k(p)}{p^s} + \frac{\delta_k(p^2)}{p^{2s}} + \cdots \right) \\ &= \prod_{p|k} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \right) \prod_{p\nmid k} \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots \right) \\ &= \zeta(s-1) \prod_{p|k} \frac{1 - \frac{1}{p^{s-1}}}{1 - \frac{1}{p^s}} := \zeta(s-1)G(s), \end{aligned}$$

since

$$\delta_k(p^m) = \begin{cases} 1 & \text{if} \quad p \mid k \\ p^m & \text{if} \quad p \nmid k. \end{cases}$$

And for $\sigma \geq \eta$ (> 0), we observe that

$$|G(s)| = \prod_{p|k} \left| \frac{1 - \frac{1}{p^{s-1}}}{1 - \frac{1}{p^s}} \right| \leq \prod_{p|k} \frac{1 + p^{1-\eta}}{1 - \frac{1}{p^\eta}} \leq \prod_{p|k} \frac{2p}{1 - \frac{1}{2^\eta}} \leq kc(\eta)^{\omega(k)}.$$

Lemma 3.2. *Let m be an integer ≥ 1. Let c and y be any positive real numbers and $T \geq T_0$ where T_0 is sufficiently large. Then we have,*

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s(s+1)\cdots(s+m)} ds = \begin{cases} \frac{1}{m!} \left(1 - \frac{1}{y} \right)^m + O\left(\frac{4^m y^c}{T^m} \right) & \text{if} \quad y \geq 1, \\ O\left(\frac{1}{T^m} \right) & \text{if} \quad 0 < y \leq 1. \end{cases}$$

Proof. See [SS, Lemma 3.2] and also [In, p.31 Theorem B]).

Lemma 3.3. *The Riemann zeta-function $\zeta(s)$ is extended as a meromorphic function in the whole complex plane $\mathbb{C}$ with a simple pole at $s = 1$ and it satisfies a functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ where*

$$\chi(s) = \frac{\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)}{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)}.$$

Also, in any bounded vertical strip, using Stirling's formula, we have

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i\left(t+\frac{\pi}{4}\right)}\left(1+O\left(t^{-1}\right)\right)$$

as $|t| \to \infty$. *Thus, in any bounded vertical strip,*

$$|\chi(s)| \asymp t^{1/2-\sigma}\left(1+O\left(t^{-1}\right)\right)$$

as $|t| \to \infty$.

Proof. See [T, p.116] or [Iv, p.8-12].

Lemma 3.4. *We have for* $t \geq t_0$ *(sufficiently large),*

$$\zeta(\frac{1}{2}+it) \ll t^{1/6}(\log t)^{3/2}$$

and

$$\zeta(1+it) \ll \log t.$$

Proof. See [T, page 99, Theorem 5.5] and [T, page 49, Theorem 3.5]

4. Proof of theorem 1.1

From Lemma 3.2, with $c = 2 + \frac{1}{\log x}$ and writing $F(s) := \zeta(s-1)G(s)$, we have

$$\begin{aligned} S := \sum_{n\leq x} \delta_k(n)\left(1-\frac{n}{x}\right)^m &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^s}{s(s+1)\cdots(s+m)}\,ds \\ &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s(s+1)\cdots(s+m)}ds + O\left(\frac{4^m x^c \log x}{T^m}\right). \end{aligned} \tag{4.4}$$

Note that the tail portion error term in the above expression is actually

$$\ll \frac{4^m}{T^m}x^c\sum_{n\leq x}\frac{\delta_k(n)}{n^c} \ll \frac{4^m x^c \log x}{T^m},$$

since $\delta_k(n) \leq n$.

Case 1: Let $m = 1$. We move the line of integration in the above integral to $\Re s = \frac{1}{2}$. In the rectangular contour formed by the line segments joining the points $c-iT$, $c+iT$, $\frac{1}{2}+iT$, $\frac{1}{2}-iT$ and $c-iT$ in the anticlockwise order, we observe that $s=2$ is a simple pole of the integrand. Thus we get the main term $\frac{x^2}{(m+2)!}\prod_{p|k}\frac{p}{p+1}$ from the residue coming from the pole $s=2$.

We note that

$$\begin{aligned} &\frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s(s+1)}ds \\ &= \frac{1}{2\pi i}\left\{\int_{\frac{1}{2}+iT}^{c+iT}\cdots + \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\cdots + \int_{c-iT}^{\frac{1}{2}-iT}\cdots\right\} + \text{sum of the residues.} \end{aligned} \tag{4.5}$$

The left vertical line segment contributes the quantity:

$$
\begin{aligned}
Q_1 &:= \frac{1}{2\pi}\int_{-T}^{T} F(1/2+it)\frac{x^{1/2+it}dt}{(-1/2+it)(1/2+it)}\ dt\\
&= \frac{1}{2\pi}\left(\int\limits_{|t|\le t_0} + \int\limits_{t_0<|t|\le T}\right)\frac{x^{\frac{1}{2}+it}\zeta\left(-\frac{1}{2}+it\right)G\left(\frac{1}{2}+it\right)\ dt}{\left(\frac{1}{2}+it\right)\left(\frac{1}{2}+it\right))}\\
&\ll k\ c(1/2)^{\omega(k)}x^{1/2} + k\ c(1/2)^{\omega(k)}x^{1/2}\int\limits_{t_0<|t|\le T} t^{1/2-(-1/2)}\left|\zeta(3/2+it)G\left(\frac{1}{2}+it\right)\right|\frac{dt}{t^2}\\
&\ll k\ c(1/2)^{\omega(k)}x^{1/2} + k\ c(1/2)^{\omega(k)}x^{1/2}\int\limits_{t_0<t\le T}\frac{dt}{t}.\\
&\ll k\ c(1/2)^{\omega(k)}x^{1/2}\log T. \qquad (4.6)
\end{aligned}
$$

Now we will estimate the contributions coming from the upper horizontal line (estimation for the lower horizontal line is similar).
The horizontal lines in total contribute a quantity which is in absolute value

$$
\begin{aligned}
&\ll \int_{1/2}^{c}\left|\zeta(\sigma-1+iT)G(\sigma+iT)\frac{x^{\sigma+iT}}{(\sigma+iT)(\sigma+1+iT)}\right|d\sigma\\
&\ll \left(\int_{1/2}^{1}+\int_{1}^{3/2}+\int_{3/2}^{c}\right)|\zeta(\sigma-1+iT)G(\sigma+iT)|\frac{x^\sigma}{T^2}d\sigma\\
&\ll k\ c(1/2)^{\omega(k)}\left\{\left(\int_{1/2}^{1}+\int_{1}^{3/2}\right)T^{1/2-\sigma+1}|\zeta(2-\sigma+iT)|\frac{x^\sigma}{T^2}d\sigma\right.\\
&\qquad\left.+\int_{3/2}^{c}|\zeta(\sigma-1+iT)|\frac{x^\sigma}{T^2}d\sigma\right\}\text{(by Lemma 3.3)}\\
&\ll k\ c(1/2)^{\omega(k)}\left(\frac{x\log T}{T}+\frac{x^{3/2}\log T}{T^{3/2}}+\frac{x^2\log T}{T^{11/6}}\right)\text{(by Lemma 3.4).}
\end{aligned}
$$

Collecting all the estimates, and taking $T = x^{10}$ we get:

$$
\begin{aligned}
E_{1,k}(x) &\ll k\ c(1/2)^{\omega(k)}\left(x^{1/2}\log T+\frac{x^2}{T}+\frac{x\log T}{T}+\frac{x^{3/2}\log T}{T^{3/2}}+\frac{x^2\log T}{T^{11/6}}\right)\\
&\ll k\ c(1/2)^{\omega(k)}x^{1/2}\log x. \qquad (4.7)
\end{aligned}
$$

Case 2: Let $m \ge 2$. We move the line of integration to $\Re s = \eta\ (>0)$.
We note that

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{c-iT}^{c+iT}F(s)\frac{x^s}{s(s+1)\cdots(s+m)}ds\\
&= \frac{1}{2\pi i}\left\{\int_{\delta+iT}^{c+iT}\cdots+\int_{\delta-iT}^{\delta+iT}\cdots+\int_{c-iT}^{\delta-iT}\cdots\right\}+\text{sum of the residue.} \qquad (4.8)
\end{aligned}
$$

The left vertical line segment contributes the quantity:

$$\begin{aligned}
Q_m &:= \frac{1}{2\pi}\int_{-T}^{T} F(\eta+it)\frac{x^{\eta+it}dt}{(\eta+it)(\eta+1+it)\cdots(\eta+m+it)}\ dt \\
&= \frac{1}{2\pi}\left(\int\limits_{|t|\le t_0} + \int\limits_{t_0<|t|\le T}\right)\frac{x^{\eta+it}\zeta\left(\eta-1+it\right)G\left(\eta+it\right)\ dt}{(\eta+it)(\eta+1+it)\cdots(\eta+m+it)} \\
&\ll k\ c(\eta)^{\omega(k)}x^{\eta} + k\ c(\eta)^{\omega(k)}x^{\eta}\int\limits_{t_0<|t|\le T} t^{1/2-(\eta-1)}|\zeta(3/2-\eta+it)G(\eta+it)|\frac{dt}{t^{m+1}} \\
&\ll k\ c(\eta)^{\omega(k)}x^{\eta} + k\ c(\eta)^{\omega(k)}x^{\eta}\int\limits_{t_0<t\le T}\frac{t^{3/2-\eta}}{t^3}\ dt. \\
&\ll k\ c(\eta)^{\omega(k)}\ x^{\eta}. \qquad (4.9)
\end{aligned}$$

Now we will estimate the contributions coming from the upper horizontal line (estimation for the lower horizontal line is similar).
The horizontal lines in total contribute a quantity which is in absolute value

$$\begin{aligned}
&\ll \int_{\eta}^{c}\left|\zeta(\sigma-1+iT)G(\sigma+iT)\frac{x^{\sigma+iT}}{(\sigma+iT)(\sigma+1+iT)\cdots(\sigma+m+iT)}\right| d\sigma \\
&\ll c(\eta)^{\omega(k)}k\left(\int_{\eta}^{1}+\int_{1}^{3/2}+\int_{3/2}^{c}\right)|\zeta(\sigma-1+iT)|\frac{x^{\sigma}}{T^{k+1}} \\
&\ll k\ c(\eta)^{\omega(k)}\left\{\left(\int_{\eta}^{1/2}+\int_{1/2}^{1}+\int_{1}^{3/2}\right)T^{1/2-\sigma+1}|\zeta(2-\sigma+iT)|\frac{x^{\sigma}}{T^{m+1}}d\sigma\right. \\
&\qquad\qquad \left. + \int_{3/2}^{c}|\zeta(\sigma-1+iT)|\frac{x^{\sigma}}{T^{m+1}}d\sigma\right\} \\
&\ll k\ c(\eta)^{\omega(k)}\left(\frac{x^{1/2}}{T^{m-1/2+\eta}}+\frac{x\log T}{T^m}+\frac{x^{3/2}(\log T)^{3/2}}{T^{m+5/6}}+\frac{x^2(\log T)^{3/2}}{T^{m+5/6}}\right)
\end{aligned}$$

Collecting all the estimates, and taking $T = x^{10}$, for $m \ge 2$ we get:

$$E_{m,k}(x) \ll k\ c(\eta)^{\omega(k)}x^{\eta}. \qquad (4.10)$$

This proves Theorem 1.1.

Remark 4.1. *For $m \ge 2$ we may try to move the line of integration slightly left of vertical line 0. On the line $\Re s = 0$, the function $G(s)$ has simple poles at the points $s(\ell,p) = \frac{2\pi i\ell}{\log p}$ $\forall \ell \in \mathbb{Z}$ and for each prime $p \mid k$. let $p_1, p_2, \cdots p_{r_k}$ be the primes dividing k. The total contribution from the simple poles at the points $s(\ell,p) = \frac{2\pi i\ell}{\log p_j}$ for $1 \le j \le r_k$ is given by:*

$$M = \sum_{j=1}^{r_k}\sum_{|\ell|\le\frac{T\log p_j}{2\pi}}\zeta\left(\frac{2\pi i\ell}{\log p_j}-1\right)\prod_{p_i\ne p_j}\left(1-\frac{p_i}{p_j^{\frac{2\pi i\ell}{\log p_j}}}\right)\frac{x^{\frac{2\pi i\ell}{\log p_j}}}{\frac{2\pi i\ell}{\log p_j}\left(\frac{2\pi i\ell}{\log p_j}+1\right)\cdots\left(\frac{2\pi i\ell}{\log p_j}+m\right)}.$$

If one establishes that $M = o\left(x^{\eta}\right)$, then this will improve the error term. This seems to be really difficult.

Remark 4.2. *From the Theorem 1.1 observe that*

$$E_{1,k}(x) \ll_\epsilon x^{1/2+10\epsilon}$$

uniformly for $3 \leq k \ll x^\epsilon$ *since* $\omega(k) \ll \frac{\log}{\log\log k}$ *for* $k \geq 3$. *Also* $E_{m,k}(x) \ll x^{c_1\eta}$ *uniformly for* $3 \leq k \ll x^\epsilon$, *where* c_1 *is effective positive constant.*

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