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Contributions of Ramachandra to the Theory of the Riemann Zeta-Function

A. Sankaranarayanan

To the memory of K. Ramachandra (18-08-1933 to 18-01-2011)

Abstract. This is a survey article covering certain important mathematical contributions of K. Ramachandra to the theory of the Riemann zeta-function and their impact on current research.

Keywords. Riemann zeta-function, Omega theorems, mean-value theorems, fractional moments.

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1. Introduction

Kanakanahalli Ramachandra was one of the world renowned mathematicians whose active research period spanned second half of the 20th century and early 21st century. He published almost one hundred and seventy five research articles in reputed mathematics journals. His main research interest was in Number Theory and it spread across its various branches like Algebraic Number Theory, Transcendental Number Theory, Analytic Number Theory (in particular the theory of the Riemann zeta-function), Additive Number Theory, Elementary Number Theory to name a few. He contributed several important results in each area of this broad spectrum, and that makes him one of a rare kind on the vast firmament of Number Theory.

When I started working under his guidance in 1985, he was at his peak making several important contributions to the theory of the Riemann zeta-function. This was the main reason why I also got interested in this area. I am sure that the mathematical community will agree that it is a difficult task to write a survey article covering all of his work. Even writing a survey exposition of his work in the Riemann zeta-function theory was a daunting task for me since he worked in this area roughly during 1970-2010, for almost four decades and published around seventy five research papers. However, in this article, I have tried my level best to write a survey with some comments on nineteen selected papers of Ramachandra in this area (of course I like them very much because they are like shining diamonds of a crown) and also comment their impact on the subsequent contributions by other leading players in the field. I also present some brief ideas for most of the items I mention in the sequel.

2. Notations and Preliminaries

Notations: 1. Throughout the article, $s = \sigma + it, s_0 = \sigma_0 + it_0$; the parameters T and x are sufficiently large real numbers.
2. $\delta,\ \epsilon$ with or without suffixes always denote sufficiently small positive constants.
3. $A, B, C, \ldots, a, b, c, \ldots$ with or without suffixes denote absolute constants unless specified otherwise and they need not be the same at each occurrence.

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The Riemann zeta-function $\zeta(s)$ is defined as:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \text{ for } \sigma > 1, \tag{2.1}$$

and it is continued analytically to the whole complex plane with a simple pole at $s = 1$. It is well known that it satisfies a functional equation whose symmetric form can be written as

$$\pi^{-s/2}\Gamma\left(s/2\right)\zeta(s) = \pi^{1-s/2}\Gamma\left(1 - s/2\right)\zeta\left(1 - s/2\right). \tag{2.2}$$

One observes that $\zeta(s)$ vanishes at $s = -2, -4, -6, \ldots$ and these are called the trivial zeros of $\zeta(s)$. It is well known that $\zeta(s)$ has non-trivial complex zeros.
Riemann Hypothesis (RH): It asserts that all the non-trivial complex zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$.
Riemann Hypothesis (Stronger form (SRH)): It asserts that all the non-trivial complex zeros of $\zeta(s)$ are simple and they lie on the critical line $\sigma = 1/2$.
To know more about this function, the readers are advised to refer [Ing95], [Iv03] and [Ti86].

3. Omega Theorems

A theorem of Titchmarsh (see Theorem 8.12 of [Ti86] or [Ti28]) asserts that for σ a fixed number in the range $\frac{1}{2} \leq \sigma < 1$, the inequality

$$|\zeta(\sigma + it)| > \exp\left(\log^{\alpha} t\right) \tag{3.1}$$

is satisfied for some indefinitely large values of t provided that

$$\alpha < 1 - \sigma. \tag{3.2}$$

Ramachandra was fascinated by this result of Titchmarsh and in [Ra74] and established a stronger version of the above result of Titchmarsh namely that, for $T \geq T_0(\sigma, \epsilon, c)$, the inequality

$$\max_{T \leq t \leq T+H} |\zeta(\sigma + it)| > \exp\left((\log H)^{1-\sigma-\varepsilon}\right) \tag{3.3}$$

holds for any H satisfying $(\log T)^c \leq H \leq T$. The beauty of this result is that every horizontal block of width H will contain a value t for which (3.3) is satisfied and thus it provides the information locally in short intervals. This result is considered to be an initial push and an improvement on the result of Titchmarsh. Ramachandra also proved that: For $\frac{1}{2} < \sigma < 1$, $\varepsilon > 0$ (both σ and ε are fixed), the inequality

$$\min_{T \leq t \leq 2T} |\zeta(\sigma + it)| < \exp\left(-(\log T)^{1-\sigma-\varepsilon}\right) \tag{3.4}$$

holds.

1. Levinson's Method: Levinson developed a method (see [Le72]) to study Omega theorems for $\zeta(s)$. Let $\frac{1}{2} \leq \sigma \leq 1$. The main idea here is to consider the following line integral (for example when $\sigma = 1$ where it yields the best result), (with a and b are certain real positive constants which might depend on the parameter k (where k is an integer ≥ 2 and the quantity n)

$$J = -i\, b^{-1}\, \pi^{-\frac{1}{2}} \int\limits_{\sigma=1+a} \zeta^k(s)\, n^s\, \exp\left(\frac{(s-1-a)^2}{b^2}\right)\, ds \tag{3.5}$$

and observe that $J > d_k(n) > 0$. Let $m(p) =: \left[\frac{k}{p-1}\right]$.

Then choosing n (depending on k) (namely $n = \prod_{p \le k} p^{m(p)}$) so that (as k increases), we have

$$\left(\frac{d_k(n)}{n}\right)^{\frac{1}{k}} = e^{\gamma}\ \log k + O(1), \tag{3.6}$$

where γ is the Euler's constant. With the choice of $a = \frac{k}{\log n}$ and $b = e^k$, by splitting the line integral suitably into three parts and estimating them, he could establish that (with $T = 2ke^k$)

$$J \le 2n \left(\max_{1\le|t|\le T} |\zeta(1+it)|\right)^k + 6e^{-\gamma k}\ d_k(n). \tag{3.7}$$

The lower and upper bounds for J indeed imply that

$$\left(1 - 6e^{-\gamma k}\right)^{\frac{1}{k}}\ \left(\frac{d_k(n)}{n}\right)^{\frac{1}{k}} \le 2^{\frac{1}{k}}\ M_T^*$$

and hence, we get

$$M_T^* \ge e^{\gamma} \log\log T + O(1), \tag{3.8}$$

where $M_T^* =: \max_{1\le|t|\le T} |\zeta(1+it)|$. This is an improvement of a result of Littlewood, namely

$$\overline{\lim_{T\to\infty}} \frac{M_T^*}{\log\log T} \ge e^{\gamma}. \tag{3.9}$$

By a similar argument, he also obtained that if $\frac{1}{2} \le \sigma < 1$, then the lower bound estimate

$$\max_{1\le|t|\le T} |\zeta(\sigma+it)| \ge \exp\left(\frac{C(\log T)^{1-\sigma}}{\log\log T}\right), \tag{3.10}$$

holds. This is an improvement of a result of Titchmarsh. Levinson could also prove that there exists arbitrarily large t for which the inequality

$$\frac{1}{|\zeta(1+it)|} \ge \frac{6e^{\gamma}}{\pi^2} (\log\log t - \log\log\log t) + O(1) \tag{3.11}$$

holds and thus improving a result of Littlewood.

It should be mentioned that Granville and Soundararajan (see [GrSou06], a paper dedicated to Ramachandra) have improved upon Levinson's results (3.8) and (3.11). For instance, they proved that the set of points $t \in [0, T]$ with

$$|\zeta(1+it)| \ge e^{\gamma} (\log\log T + \log\log\log T - \log\log\log\log T - \log A + O(1)) \tag{3.8a}$$

has measure at least $T^{1-1/A}$, uniformly for $A \ge 10$. They have also conjectured (more precise than Littlewood) that

$$\max_{t\in[T,2T]} |\zeta(1+it)| \ge e^{\gamma} (\log\log T + \log\log\log T + \log C)$$

for some constant C. Similar results for large values of $\frac{1}{|\zeta(1+it)|}$ with out $\log\log\log t$ term in (3.11) are also proved in [GrSou06].

2. Montgomery's Method: When Montgomery was in IAS, Princeton, he was influenced by the above results of Titchmarsh, Levinson and Ramachandra. He developed a method (see [Mo77]) to study the omega theorems for $\zeta(s)$.

Let $0 \le \theta < 2\pi$, $\alpha > 0$ and $\frac{1}{2} \le \sigma_0 < 1$ be constants and let $s = \sigma + it$, $s_0 = \sigma_0 + it_0$. Let x satisfy $10 \le x \ll (\log t_0)(\log\log t_0)$.

Let us define the kernel function,

$$K(s,\alpha,\theta,x) =: \left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 \left(2+x^s e^{i\theta}+x^{-s}e^{-i\theta}\right). \tag{3.12}$$

Here, one starts with the following line integral,

$$\begin{aligned} I &=: \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\left(e^{-i\theta}\log\zeta\left(s+s_0\right)\right)\, K(s,\alpha,\theta,x)\, ds \\ &= \sum_{\left|\log\left(\frac{p}{x}\right)\right|\le 2\alpha} p^{-s_0}\left(2\alpha-\left|\log\left(\frac{p}{x}\right)\right|\right)+O\left((\log x)^2\right). \end{aligned} \tag{3.13}$$

We observe that the kernel function $K(s,\alpha,\theta,x)$ is real and positive when $\sigma=0$. He combined this idea with the box-principle to establish a much better Ω bound (see [Mo77]) namely: for any real θ, there are arbitrarily large values t_0 such that

$$\Re\left\{e^{i\theta}\log\zeta(\sigma_0+it_0)\right\} \ge \frac{1}{20}\,\left(\sigma-\frac{1}{2}\right)^{-1}\,\frac{(\log t_0)^{1-\sigma}}{(\log\log t_0)^{\sigma}} \tag{3.14}$$

holds for any σ_0 in the range $\frac{1}{2}<\sigma_0<1$. Here $\log\zeta(s)$ is defined by continuous variation along lines parallel to the real axis using the Dirichlet series

$$\log\zeta(s) = \sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^s} \quad (\sigma>1)$$

where $\Lambda_1(n)=\Lambda(n)/\log n$ with $\Lambda(n)$ being the usual Von-Mangoldt function defined on primes and its powers. In particular under the assumption of RH, Montgomery showed that

$$\zeta\left(\frac{1}{2}+it_0\right) = \Omega\left(\exp\left(\frac{1}{20}\left(\frac{\log t_0}{\log\log t_0}\right)^{1/2}\right)\right), \tag{3.15}$$

and

$$\arg\zeta\left(\frac{1}{2}+it_0\right) = \Omega\left(\left(\frac{\log t_0}{\log\log t_0}\right)^{1/2}\right). \tag{3.16}$$

3. Balasubramanian-Ramachandra Method: Ramachandra introduced the notion of "Titchmarsh Series" which we describe here in the following:

Definition: Let $C\le H\le T$ where C is a certain positive constant. A series of the form $f(s)=\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n^s}$ is called a " Titchmarsh Series " if it satisfies the following conditions:

1. $\{a_n\}$ is a sequence of complex numbers satisfying $a_1=1$ and $|a_n|\le(nH)^{A_1}$ where A_1 is a positive constant.
2. For $n=1,2,3,\ldots$, we have $\frac{1}{A_1}\le\lambda_{n+1}-\lambda_n\le A_1$.
3. The Titchmarsh series $f(s)=\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n^s}$ is continuable analytically in $\{\sigma\ge 0, T\le t\le T+H\}$ and there

   $$\max|f(s)| \le \exp\exp\left\{\frac{H}{100A_1}\right\}$$

   holds.
4. $f(s)$ is continuous in $\{\sigma\ge 0, T\le t\le T+H\}$.

We state below a weaker version of a general Theorem.

Theorem 3.1. *(A general theorem) (see [Ra81]). For a "Titchmarsh Series" $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$ we have*

$$\frac{1}{H}\int_T^{T+H} |f(it)|^2 dt > C_{A_1} \sum_{\lambda_n \le X} |a_n|^2 \left(1 - \frac{\log \lambda_n}{\log H} + \frac{1}{\log\log H}\right),$$

where $X = 2 + D_{A_1}H$ and C_{A_1}, D_{A_1} are positive constants depending on A_1 provided $H \ge C(>0)$ a certain constant.

For an improved version of this general Theorem (Balasubramanian-Ramachandra method) we refer to [BaRa86], [BaRa90] and [BaRa92].

Applying this general theorem A to the $2k$-th moments of the $\zeta(s)$, in [BaRa77], Ramachandra and Balasubramanian established successfully that

$$\max_{T \le t \le T+H} \left|\zeta\left(\frac{1}{2} + it\right)\right| \ge \exp\left(c_1 \left(\frac{\log H}{\log\log H}\right)^{1/2}\right) \tag{3.17}$$

for $100(\log\log T) \le H \le T$ and $T \ge T_0(\delta)$. The importance of this result is that it is unconditional (No assumption of RH) and the method works for short intervals of width H as stated above. Later Balasubramanian [Ba86] optimized c_1 in (3.17) and showed that one can take $c_1 = 0.530\cdots$ in (3.17).

Inspired by the result of Montgomery [Mo77], Ramachandra and Sankaranarayanan [RaSa91a] obtained a more precise form of omega theorems, namely: For any real θ with $0 \le \theta < 2\pi$, (θ and σ fixed), there are arbitrarily large values t such that

$$\Re\left\{e^{i\theta} \log\zeta(\sigma + it)\right\} \ge \frac{c}{1-\sigma}\,\frac{(\log t)^{1-\sigma}}{(\log\log t)^{\sigma}} \tag{3.18}$$

where $c = c(\sigma)$ can be taken to be any positive constant < 0.17. The feature of this result is that when σ approaches 1 from the left, the implied constant goes to ∞.

Levinson's method yields better result on the line $\sigma = 1$. An advantage of Montgomery's method is that it gives a better Ω-result for $|\zeta(\sigma + it)|$ (for $\frac{1}{2} < \sigma < 1$) than the result of Levinson and Ramachandra. However, Ramachandra's method works for short intervals and you do not need the assumption of RH. It also works for L series and so on.

4. Soundararajan's Resonance Method: In 2008, Soundararajan [Sou08] introduced a new approach called "Resonance Method". The idea here is to find a Dirichlet polynomial $R(t) = \sum_{n \le N} a(n) n^{-it}$ which 'resonates' with $\zeta\left(\frac{1}{2} + it\right)$ and picks out its large values. More precisely, one has to study the smoothed moments

$$M_1 := \int_{-\infty}^{\infty} |R(t)|^2\, \Phi\left(\frac{t}{T}\right)\, dt \quad \text{and} \quad M_2 := \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2} + it\right)\, |R(t)|^2\, \Phi\left(\frac{t}{T}\right)\, dt. \tag{3.19}$$

Here Φ denotes a smooth, non-negative function, compactly supported in $[1, 2]$ with $\Phi(y) \le 1$ for all y and $\Phi(y) = 1$ for $5/4 \le y \le 7/4$. It is easy to see that

$$\max_{T \le t \le 2T} \left|\zeta\left(\frac{1}{2} + it\right)\right| \ge \frac{M_2}{M_1}. \tag{3.20}$$

For $N \le T^{1-\varepsilon}$, one has to evaluate M_1 and M_2. These are quadratic forms in the unknown coefficients $a(n)$ and the problem now is to maximize this ratio of these quadratic forms. Thus he established that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \geq \exp\left((1+o(1))\left(\frac{\log T}{\log\log T}\right)^{1/2}\right), \tag{3.21}$$

holds for some t with $T \leq t \leq 2T$ if T is sufficiently large.

5. Bondarenko-Seip GCD-Resonance Method: Using a modified version of Soundararajan's resonance method together with ideas of Hilberdink (see [Hil09]), Aistleitner (see [Ai16]) has proved that for fixed $\alpha \in (1/2, 1)$, we have

$$\max_{0\leq t\leq T} |\zeta(\alpha+it)| \geq \exp\left(\frac{c_\alpha(\log T)^{1-\alpha}}{(\log\log T)^\alpha}\right)$$

for all sufficiently large T, where we can choose $c_\alpha = 0.18(2\alpha-1)^{1-\alpha}$. This may be compared with the result of Montgomery (see [Mo77], who established almost the same result with a smaller c_α). The arguments of [Ai16] allows us to obtain lower bounds for the measure of those $t \in [0, T]$ for which $\zeta(\alpha+it)$ is of the order mentioned above.

Recently, Andriy Bondarenko and Kristian Seip [BoSe16] studied a good lower bound for sums involving $\gcd(n_k, n_l)$ where $\{n_j\}$ is a sequence of positive integers, more precisely they showed that there exists a sequence of positive numbers $\{c_j\}$ $(j \geq 1)$ and a positive integer sequence $\{n_j\}$ $j \geq 1$ such that (for every N sufficiently large), the inequality

$$\sum_{k,l=1}^{N} c_k c_l \frac{\gcd(n_k, n_l)}{\sqrt{n_k n_l}} \geq \left(\sum_{j=1}^{N} c_j^2\right) \exp\left(\gamma\left(\frac{\log N \log\log\log N}{\log\log N}\right)^{1/2}\right) \tag{3.22}$$

holds for any given constant γ satisfying $0 < \gamma < 1$.

They clubbed this idea along with the resonance method of Soundararajan and proved that for any given β with $0 < \beta < 1$, the inequality

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \geq \exp\left(c\left(\frac{\log T \log\log\log T}{\log\log T}\right)^{1/2}\right), \tag{3.23}$$

holds for some t with $T^\beta \leq t \leq T$ if T is sufficiently large, where c is any positive number satisfying

$$c < \sqrt{\min\left(\frac{1}{2}, 1-\beta\right)}. \tag{3.24}$$

6. On the Hurwitz zeta-functions: In [RaSa89] and [RaSa91b], Ramachandra and Sankaranarayanan proved Omega theorems for the Hurwitz zeta-functions. For instance, in [RaSa91b] we proved:

Let $0 \leq \beta < 1$, $\frac{1}{2} \leq \sigma_0 < 1$, $0 \leq \theta < 2\pi$. Let y_0 be the positive solution of the equation $e^{y_0} = 2y_0 + 1$, let l be an integer constant satisfying $l \geq 6$, $c_2 = 2y_0/(2y_0+1)^2$, $0 < c_1 < c_2$. Then the inequalities

$$\Re\left(e^{-i\theta}\zeta(\sigma_0 + it_0, \beta)\right) \geq \frac{1}{1-\sigma_0}\, c_0 c_1\, (\log t_0)^{1-\sigma_0} \tag{3.25}$$

and

$$\Re\left(e^{-i\theta}\zeta(1 + it_1, \beta)\right) \geq \left(\frac{1}{2}\cos^2(\theta/2) - \varepsilon_1\right)\, (\log\log t_1) \tag{3.26}$$

hold at least for one t_0 and one t_1 with $\frac{1}{2}T^\varepsilon \leq t_0, t_1 \leq \frac{3}{2}T$.

The proofs of the above results make use of the Montgomery's method but with a different kernel function

$$K_1(s,\alpha,\theta,x) =: \left(\frac{e^{i\theta+\alpha s} - e^{-i\theta-\alpha s} - e^{i\theta} + e^{-i\theta}}{s}\right)^2 \tag{3.27}$$

when $\sigma_0 = 1$. Note that on the line 1, it yields the expected Omega result for the Hurwitz zeta-function. In [RaSa89], they applied the method of Balasubramanian and Ramachandra to study the Omega theorems when β is "well approximated".

4. Mean-Value Theorems

1. Mean fourth power: In [Ra75a], Ramachandra gives an elegant method to prove estimates for discrete mean fourth power of $|\zeta(1/2+it)|$ and $|L(1/2+it,\chi)|$. More precisely, he established that: If $3 \le t_1 < t_2 < \cdots < t_R \le T$, $(R \ge 2)$ and $t_{j+1} - t_j \ge 1$, then

$$\sum_{r=1}^{R} |\zeta\left(1/2+it_r\right)|^4 \ll T(\log T)^{50}. \tag{4.1}$$

Let χ be a character $\mod q$ (q fixed), $T \ge 3$, $-T \le t_{\chi,1} < t_{\chi,2} < \cdots < t_{\chi,R_\chi} \le T$, $(R_\chi \ge 2)$ and $t_{\chi,j+1} - t_{\chi,j} \ge 1$. If with each χ we associate such points $t_{\chi,j}$, then

$$\sum_{\chi \mathrm{mod}\ q}^{\star} \sum_{r=1}^{R_\chi} |L\left(1/2+it_{\chi,r},\chi\right)|^4 \ll qT(\log(qT))^{50}. \tag{4.2}$$

Here $\star$ denotes the sum over primitive characters mod q.

Previously such estimates were deduced via the approximate functional equation (see theorem 10.1 of [Mo71]). However in the case of Dirichlet L-functions the proof is tedious. Ramachandra's method here just uses the functional equation and contour integration to obtain an expression for $L^2(s,\chi)$ in the form

$$\sum_{n=1}^{\infty} \tau(n)\chi(n)e^{-n/N}n^{-s} + I_1 + I_2 + \text{ small error} \tag{4.3}$$

where I_1 and I_2 are certain contour integrals involving the segments $\sum_{n=1}^{N}$ and $\sum_{n=N+1}^{\infty}$ of the Dirichlet series $L^2(s,\chi)$. This idea had been applied on several occasions by many researchers later.

Ingham (see [Ing28]) used the approximate functional equation (the version of Hardy-Littlewood [HaLi29]) and proved the asymptotic formula:

$$\int_0^T |\zeta(1/2+it)|^4 \, dt = \frac{1}{2}\pi^{-2}T\log^4 T + O\left(T\log^3 T\right). \tag{4.4}$$

In [Ra75b], Ramachandra dispensed again the approximate functional equation and established the very same asymptotic formula (4.4) of Ingham just by using the ideas developed by him in [Ra75a], namely a smoothed expression for ζ^2 and using a mean value estimate of Montgomery and Vaughan [MoVa74].

2. Estimates of the integral $I_k(T)$:
Let k be a complex constant and let

$$I_k(T) = \int_0^T |\zeta(1/2+it)|^{2k} \, dt. \tag{4.5}$$

$(\zeta(s))^k$ is the analytic continuation (from $\sigma \geq 2$) along lines (parallel to the σ-axis) not containing zeros or poles of $\zeta(s)$. Also, let

$$(\zeta(s))^k = \sum_{n=1}^{\infty} d_k(n) n^{-s} \ (\sigma \geq 2) \tag{4.6}$$

and

$$C_k = \left(\Gamma\left(|k^2|+1\right)\right)^{-1} \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{|k^2|} \sum_{m=0}^{\infty} \left|d_k\left(p^m\right)\right|^2 p^{-m} \right\}. \tag{4.7}$$

An important and well-studied problem in analytic number theory is determining the moments of the Riemann zeta function $\zeta(s)$ on the critical line. The only values of k where the main term of $I_k(T)$ is known (other than $k = 0$) are k=1 (due to Hardy and Littlewood) and k=2 (due to Ingham). There are conjectures for positive k. It is believed that

$$I_k(T) \sim C_k \ T(\log T)^{k^2} \tag{4.8}$$

for some $C_k > 0$. Several approaches to this conjecture are there ranging from random matrix theory [CoFKR05], [KaSar99], [KeSn00], multiple Dirichlet series [DiGH03]. For some previous history related to the work on $I_k(T)$ (for real $k \geq 0$), the readers are referred to [CoGh84].

Lower bound estimate: In 1984, J. B. Conrey and A. Ghosh [CoGh84] showed that for all real $k \geq 0$,

$$I_k(T) \geq (C_k + o(1))\, T(\log T)^{k^2} \tag{4.9}$$

on RH with an explicitly given value of C_k. Balasubramanian and Ramachandra [BaRa90] removed RH in (4.9) when k is a positive integer. Soundararajan [Sou95] improved (4.9) that one can replace C_k by $2C_k$ for any integer $k \geq 2$. He also gave some refinements of (4.9) under Lindelöf Hypothesis (LH).

It should be mentioned here that Balasubramanian and Ramachandra [BaRa90] have shown that for all complex constants k and for $D_k \log\log T \leq H \leq T$, where D_k is a certain positive constant, we have

$$\frac{1}{H} \int_T^{T+H} |\zeta(1/2+it)|^{2k} \ dt \geq C_k (\log H)^{|k^2|} \left(1 + O\left((\log H)^{-1} + H^{-1} \log\log T\right)\right) \tag{4.10}$$

on RH. This follows from their third main theorem of [BaRa90].

Further Lower and Upper bound estimates: Ramachandra was the first to prove (see [Ra80b]) that:

$$I_{1/2}(T) \ll T(\log T)^{1/4} \tag{4.11}$$

and in fact more, namely:

$$\frac{1}{H} \int_T^{T+H} |\zeta(1/2+it)| \ dt \ll (\log T)^{1/4} \tag{4.12}$$

for $H = T^{\lambda}$ where $\lambda \in (1/2, 1]$. Together with the Omega-estimate mentioned before, this settles the open problem on the average order of $|\zeta(1/2+it)|$. He also unconditionally established (see [Ra78] and [Ra80a]):

$$\frac{1}{H}\int_T^{T+H} |\zeta(1/2+it)|^{2k}\ dt \gg (\log H)^{k^2}$$
$$\text{(when } 2k>0 \text{ integer and } T \geq H \geq (\log T)^\delta > 100, \delta > 0), \tag{4.13}$$

and

$$\frac{1}{H}\int_T^{T+H} |\zeta(1/2+it)|^{2k}\ dt \gg (\log H)^{k^2}\ (\log\log H)^{-c_k}$$
$$\text{(when } 2k>0 \text{ irrational and } T \geq H \geq (\log T)^\delta > 100, \delta > 0). \tag{4.14}$$

On the assumption of RH, Ramachandra established that

$$\frac{1}{H}\int_T^{T+H} |\zeta(1/2+it)|^{2k}\ dt \gg (\log H)^{k^2}$$
$$\text{(when } 2k>0 \text{ irrational and } T \geq H \geq (\log T)^\delta > 100, \delta > 0). \tag{4.15}$$

From the works of Heath-Brown (see [Hea81] and [Hea93]), it is now known unconditionally that:

$$I_k(T) \gg_k T(\log T)^{k^2} \text{ for all rational } k, \tag{4.16}$$

$$I_k(T) \ll_k T(\log T)^{k^2} \text{ for all } k = \frac{1}{n}\ (n = 2,3,4,\cdots), \tag{4.17}$$

and on the assumption of RH, we have

$$I_k(T) \gg T(\log T)^{k^2}\ (k>0 \text{ irrational})\ ;\ I_k(T) \ll T(\log T)^{k^2}\ (2 \geq k > 0). \tag{4.18}$$

The main difference between the methods of Ramachandra and Heath-Brown is that the proofs of Ramachandra are self-contained but Heath-Brown uses certain convexity theorems of Gabriel, which not only helps to simplify the Ramachandra's proofs but has extra advantages, namely (i) $2k$ rational instead of $2k$ integer can be dealt with while dealing with lower bounds, and (ii) $k = 1/n, n = 2,3,4,\ldots$ can be dealt with instead of $k = 1/2$. The Heath-Brown's idea of using Gabriel's convexity theorems is very useful.

In [RuSou05], Rudnik and Soundararajan developed a new method of obtaining lower bounds for rational moments of L-functions varying in certain families. In [RadSou13], Radziwill and Soundararajan extended these ideas to get the result even in the irrational case. More precisely, their lower bound result for the Riemann zeta-function is that: unconditionally we have

$$I_k(T) \geq e^{-30k^4} T(\log T)^{k^2} \tag{4.18a}$$

for any real $k > 1$ and for all large T.

Let $1 \leq V \leq \log T$ and let

$$M_T(V) := \left\{ t \in [0,T] : \left|\zeta(\frac{1}{2}+it)\right| \right\}.$$

Jutila [Ju83] studied the measure $\mu(M_T(V))$ of the set $M_T(V)$ and showed that

$$\mu(M_T(V)) \ll T\exp\left(-\frac{\log^2 V}{\log\log T}\left(1+O\left(\frac{\log V}{\log\log T}\right)\right)\right) \tag{4.19}$$

and also that

$$\mu(M_T(V)) \ll T \exp\left(-c\frac{\log^2 V}{\log\log T}\right) \tag{4.20}$$

where c is a positive constant. As a consequence of this, he deduced two interesting consequences, namely: if $\omega(T)$ is any positive function such that $\lim_{T\to\infty} \omega(T) = \infty$, then

$$|\zeta(1/2+it)| \leq \exp\left(\omega(T)(\log\log T)^{1/2}\right) \tag{4.21}$$

for the numbers $t \in [0, T]$ which do not belong to an exceptional set of measure $o(T)$ and there exist positive constants a_1, a_2, a_3 such that the inequalities

$$\exp\left(a_1(\log\log T)^{1/2}\right) \leq |\zeta(1/2+it)| \leq \exp\left(a_2(\log\log T)^{1/2}\right) \tag{4.22}$$

hold for a subset of measure at least $a_3 T$ of the interval $[0, T]$.

Thus, we observe that in support of these conjectures (4.8), we have upper and lower bounds for $I_k(T)$. For the lower bound, we have $I_k(T) \gg_k T(\log T)^{k^2}$ unconditionally if k is a positive rational and under the Riemann Hypothesis (RH) for k positive. In the other direction, we have $I_k(T) \ll_k T(\log T)^{k^2}$ for $0 \leq k \leq 2$ under RH. Soundararajan [Sou09] improved the upper bound by showing that the RH implies

$$I_k(T) \ll_{k,\varepsilon} T(\log T)^{k^2+\varepsilon}. \tag{4.23}$$

One may compare this bound with a consequence of the Lindelöf Hypothesis namely

$$I_k(T) \ll_{k,\varepsilon} T^{1+\varepsilon}.$$

The result (4.23) follows from estimates on the measure of large values of $|\zeta(1/2+it)|$. Letting

$$M_T^*(V) := \{t \in [T, 2T] : \log|\zeta(1/2+it)| \geq V\}, \tag{4.24}$$

Selberg's log-normal law leads to measures of the size of $M_T^*(V)$ for V of size $\sqrt{\log\log T}$ (see [Sel44], [Sel46] and [Sel92]). This is extended under RH. The proofs on upper bounds of $M_T^*(V)$ for various ranges of V extend Selberg's technique, The key observation is that if one is only interested with an upper bound for $|\log\zeta(1/2+it)|$, then the zeros of $\zeta(s)$ very close to the line $\frac{1}{2}$ will serve the purpose. In the same article [Sou09], Soundararajan's main theorem yields two more interesting results pertaining to the large values of $\zeta(s)$ on the critical line and a growth estimate of $\zeta(1/2+it)$ under RH, namely:

$$\text{meas}\{t \in [0, T] : |\zeta(1/2+it)| \geq (\log T)^k\} = T(\log T)^{-k^2+o(1)} \tag{4.25}$$

and under the assumption of RH

$$|\zeta(1/2+it)| \leq \exp\left(\frac{3\log t}{8\log\log t}\right). \tag{4.26}$$

In [Iv09b], under RH, Ivić established a more refined estimate analogous to (4.24) over short intervals. More precisely, he proved that: If $H = T^\theta$ where $0 < \theta < 1$ is a fixed number and k is any fixed positive number, then under RH the estimate

$$\int_{T-H}^{T+H} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll H(\log T)^{k^2\left(1+O\left(\frac{1}{\log\log\log T}\right)\right)} \tag{4.27}$$

holds. This conditional upper bound (4.27) of Ivić is very tight in the sense that when $k \in \mathbb{N}$, it differs from the unconditional lower bound (4.10) of Balasubramanian and Ramachandra by only a factor of $(\log T)^{O\left(\frac{k}{\log\log\log T}\right)}$. Method of proof here depends on the measure of the set:

$$M^{**}_{T,H}(V) := \{t \in [T, T+H] : \log|\zeta(1/2+it)| \geq V\}, \tag{4.28}$$

under RH for V of size $\sqrt{\log\log T}$. Recently, Harper (see [Har13]) has established sharp conditional upper bound estimate for $I_k(T)$, namely: Assuming RH, we have

$$\int_T^{2T} |\zeta(1/2+it)|^{2k}\, dt \ll_k T(\log T)^{k^2} \tag{4.23a}$$

for any fixed $k > 0$ and for all large T. The main difference is that while Soundararajan bounded $\log|\zeta(1/2+it)|$ by a single Dirichlet polynomial and investigated how often it attains large values, Harper bounds $\log|\zeta(1/2+it)|$ by a sum of many Dirichlet polynomials and investigate their combined behaviour of all of them. He also works directly with moments throughout rather than passing through estimates for large values.

Mean-value theorems for error terms of the integral in (4.5) for $k = 1, 2$ have been studied by many authors. The readers may refer to [IvMo94], [IvMo95] and [Ju84]. For mean-square estimates of smoothed version of moments of $\zeta(s)$ (for $k = 1, 2$) over short intervals, the readers may refer to [Iv09a].

5. Growth estimate of $\zeta(s)$

1. Growth estimates on vertical lines: A consequence of RH (see Theorem 14.14 (A) of [Ti86]) is that:

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left(\frac{A\log t}{\log\log t}\right) \tag{5.1}$$

for some absolute constant A. This is due to Littlewood and the proof is really hard. However, Selberg's proof (see [Sel46] or [Ti86]) of this result is some what simpler than Littlewood's one and understandable. Ramachandra and Sankaranarayanan [RaSa93] proved (5.1) with an economical positive constant A (adapting the method of Selberg). More precisely, they showed that (on the assumption of RH) for $t \geq t_0$,

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left(\frac{0.4666\ldots\cdot\log t}{\log\log t}\right) \tag{5.2}$$

and

$$\left|\arg\zeta\left(\frac{1}{2}+it\right)\right| \leq \frac{3.515\ldots\cdot\log t}{\log\log t}. \tag{5.3}$$

The result in (5.2) may be compared with (4.26) of Soundararajan. For further improvements on (5.2), the readers may refer to [CaCha11], [ChaSou11].

Again a consequence of RH is the Lindelöf hypothesis (LH) which asserts that: For every $\sigma \geq \frac{1}{2}$ and for every $\varepsilon > 0$,

$$\zeta(\sigma+it) \ll (|t|+10)^{\kappa+\varepsilon} \tag{5.4}$$

holds with $\kappa = 0$. The important problem here is to find the least positive κ so that the inequality (5.4) holds. Though it is well known that $\kappa = 1/4$ from the functional equation of $\zeta(s)$ (this is so called the convexity bound in the literature and considered to be a trivial bound), the very first non-trivial value $\kappa = 1/6$ is due to Hardy [Ti86]. The best value of $\kappa = 53/342$ $(< 1/6)$ is due to Bourgain [Bou16]. There are several intermediate values of κ due to Walfisz, Titchmarsh, Phillips, Rankin, Titchmarsh, Min, Haneke, Chen, Kolesnik, Ivić, Huxley. The references of their articles may be found in [Ti86] or [Iv03].

2. Growth estimates on certain horizontal lines: On the assumption of RH, Littlewood (see Theorem 14.16 of [Ti86]) proved that:

$$\min_{T\le t\le T+1}\ \max_{1/2\le\sigma\le 2}\ |\zeta(\sigma+it)|^{-1} < \exp\left(\frac{C_1\log T}{\log\log T}\right). \tag{5.5}$$

Without any hypothesis, the upper bound in (5.5) is T^C (see Theorem 9.7 of [Ti86]). Indeed it means that under RH, every unit interval of type $[T,T+1]$ will contain a point t^* such that

$$\max_{1/2\le\sigma\le 2}\ |\zeta\left(\sigma+it^*\right)|^{-1} < \exp\left(\frac{C_2\log T}{\log\log T}\right). \tag{5.6}$$

So, it is very natural to ask if we compromise on the length of this t-interval, whether one can improve the Littlewood's result (5.5) either qualitatively or quantitatively. In [RaSa91c] Ramachandra and Sankaranarayanan established the following: Let $1\ge\delta>0$ be a constant and let $H=T^\delta$. Then

$$\min_{T\le t\le T+H}\ \max_{1/2\le\sigma\le 2}\ |\zeta(\sigma+it)|^{-1} \le \exp(C_3(\log\log T)^2) \tag{5.7}$$

holds with $C=C(\delta)$ and $T\ge T_0(\delta)$ provided we take $\delta=1/3$. The inequality (5.7) is true with $0<\delta<1/3$ if we assume RH. It is plain that the result (5.7) is unconditional for $\delta=1/3$ and it is much better on both the counts qualitatively and quantitatively provided $\delta\ge 1/3$. The proof stems upon the zero-density estimate of $\zeta(s)$ over short intervals H with $T\ge H\ge T^{1/3}$ and an upper bound estimate for the quantity

$$F=\sum_J\ \max_{\sigma\ge 1/2+2\nu,s\in J}\ |\log\zeta(s)|^{2k} \tag{5.8}$$

for suitable k and ν (being functions of T). The sum is taken over all zero-free horizontal slabs (disjoint!) of width $(\log T)^D$ for some suitable positive constant D. This is one of the most celebrated results in the Riemann zeta-function theory. As bi-products, we obtain: For $T^{1/3}\le H\le T$, every interval of type $[T,T+H]$ will contain at least two points t_1^* and t_2^* such that

$$\max_{1/2\le\sigma\le 2}\ |\zeta\left(\sigma+it_1^*\right)|^{-1} < \exp\left(C_4(\log\log T)^2\right), \tag{5.9}$$

and

$$\max_{1/2\le\sigma\le 2}\ |\zeta\left(\sigma+it_2^*\right)| < \exp\left(C_5(\log\log T)^2\right). \tag{5.10}$$

These results (5.9) and (5.10) have some important applications for which the readers are referred to [GaSa06], [SaSau13] and [LaSa15].

6. The upper bound of $|\zeta(1+it)|$

From the best known zero-free for $\zeta(s)$ (see p.135 of [Ti86]), we observe that (for $t\ge t_0$ with t_0 sufficiently large)

$$\zeta(1+it)\ll(\log t)^{2/3}\ ;\ \ \frac{1}{\zeta(1+it)}\ll(\log t)^{2/3}(\log\log t)^{1/3}, \tag{6.1}$$

and on the assumption of Riemann Hypothesis, we have (see Theorem 14.9 of [Ti86] due to Littlewood)

$$|\zeta(1+it)|\le 2e^\gamma(1+o(1))\log\log t\ ;\ \ \frac{1}{|\zeta(1+it)|}\le\frac{12e^\gamma}{\pi^2}(1+o(1))\log\log t. \tag{6.2}$$

It should be mentioned that it is not known whether the quantities

$$\overline{\lim_{t\to\infty}} \frac{|\zeta(1+it)|}{\log\log t}\ ; \quad \overline{\lim_{t\to\infty}} \frac{1/|\zeta(1+it)|}{\log\log t} \tag{6.3}$$

are bounded (unconditionally!). In this connection, the following result of Ramachandra ([Ra87]) is of great interest. He proved: Let $T \geq 1000$ and put $X = \exp\left(\frac{\log\log T}{\log\log\log T}\right)$. Consider any set of disjoint open intervals I each of length $\frac{1}{X}$ all contained in the interval $T \leq t \leq T + e^X$. Let ε be any positive constant not exceeding 1. Then with the exception of K intervals I (where K depends only on ε) we have

$$|\log\zeta(1+it)| \leq \varepsilon\ \log\log T. \tag{6.4}$$

The result in (6.4) may be compared with (6.1). The proof of (6.4) depends on the inequalities

$$\left|a + \sum_{\alpha} p^{-i\alpha}\right|^2 \geq 0 \quad \text{and} \quad \left|ai + \sum_{\alpha} p^{-i\alpha}\right|^2 \geq 0, \tag{6.5}$$

where a is any real number and α runs over a finite set of distinct real numbers.

7. On some special problems

1. On a problem of Srinivasa Ramanujan: In [Ram16], Ramanujan records (without proof) many curious asymptotic formulae. One of them is

$$d^2(1) + d^2(2) + \cdots + d^2(n) = An(\log n)^3 + Bn(\log n)^2 + Cn\log n + Dn + O\left(n^{\frac{3}{5}+\epsilon}\right). \tag{7.1}$$

Also he records (without proof) the result that on the assumption of Riemann hypothesis, the error term in (7.1) can be improved to $O\left(n^{\frac{1}{2}+\epsilon}\right)$. In view of a method due to H.L. Montgomery and R.C. Vaughan (see [MoVa79]), it is very likely that the error term is $O\left(n^{\frac{1}{2}}\right)$. We propose this as a conjecture. Unconditionally, the error term related to $d^2(j)$ is known to be $O\left(n^{\frac{1}{2}+\varepsilon}\right)$ for any positive constant ε (see for example the equation (14.30) of [Iv03] and also [ChGoo83]). In a general setting with generating function of the arithmetic function satisfying a functional equation of the Riemann zeta-type (with multiple gamma factors), O and Ω estimates for the error terms of the summatory function up to x of the arithmetic function under consideration have been studied by K. Chandrasekharan and R. Narasimhan (see [ChNa62]).

Let

$$E(x) = \sum_{n\leq x} d^2(n) - xP_3(\log x) \tag{7.2}$$

where $P_3(y)$ is a polynomial in y of degree 3. From a general theorem of M. Kühleitner and W.G. Nowak (see for example (5.4) of [KuNo94]), it follows that

$$E(x) = \Omega\left(x^{\frac{3}{8}}\right).$$

In [RaSa03], Ramachandra and Sankaranarayanan unconditionally established that:

$$E(x) \ll x^{1/2}(\log x)^5(\log\log x). \tag{7.3}$$

Naturally the proof employs the Perron's formula with the generating function of $d^2(n)$, namely the Dirichlet series $\frac{\zeta^4(s)}{\zeta(2s)}$. Thus the fourth power moment of Riemann zeta-function plays an effective role provided we handle the denominator factor $\zeta(2s)$ some what carefully. The crucial part of the proof is to show that the number of bad points (where $\left|\zeta(s)^{-1}\right|$ assumes large values of certain order on the line 1) is considerably small so that the contributions coming from them can be ignored ultimately. This theme of ideas had been used later on several applications for which the readers are referred to [KuNo04] and [GaLNo06]. Professor K. Chandrasekharan was very happy about this result (7.3) on seeing the Annual Working Report of TIFR. He wrote an appreciation letter to the then Dean, School of Mathematics, TIFR. This article is something special which was dedicated to the fiftieth birthday of Professor R. Balasubramanian.

For an improvement of (7.3), the readers are referred to [JiSa14]. As such it looks to be a challenging problem to reduce the exponent 5 of the factor $\log x$ in (7.3).

2. On a problem of Ivić: Let $\varepsilon > 0$ and denote by γ the imaginary parts of the nontrivial zeros of the Riemann zeta-function $\zeta(s)$. For sufficiently large T, Ivić [Iv01] proved that

$$\sum_{T<\gamma\leq 2T} |\zeta(1/2+i\gamma)|^2 \ll_\varepsilon T(\log T)^2(\log\log T)^{3/2+\varepsilon}, \tag{7.4}$$

where the implicit constant depends only on ε. In [Ra00], Ramachandra improved this result on both the sides of the inequality (7.4) by:

(i) replacing $|\zeta(1/2+i\gamma)|^2$ by a bigger quantity $M(\gamma) := \max|\zeta(s)|^2$, where the maximum is taken over all $s = \sigma + it$ in the rectangle

$$\{1/2 - A/\log T \leq \sigma \leq 2,\ |t-\gamma| \leq B(\log\log T)/\log T\}$$

with some fixed positive constants A, B
and

(ii) replacing the upper bound in (7.4) by a smaller quantity $T(\log T)^2 \log\log T$.
The method of proof differs completely from Ivić's approach. Ramachandra also discusses bounds for higher moments, namely unconditional lower bound and conditional upper bound for the discrete sum $\sum\limits_{\gamma\in I}(M(\gamma))^k$ for k any positive constant where I is any sub-interval of the interval $[T, 2T]$ of length $H \geq C\frac{\log\log T}{\log T}$. As a rare gesture, this paper was dedicated to Sankaranarayanan on his fortieth birthday.

8. Concluding Remarks

It is very clear that some results of Professor Ramachandra, as yet unimproved, are likely to stay as such for many years. He was highly dedicated and uncompromising on quality. He developed a strong Number Theory group in TIFR, Mumbai. He and his students have been successful in removing the assumption of RH in certain situations. He was a good teacher and an excellent researcher of high quality. He was a great mentor and guide, and he seeded and developed an interest in Number theory to many, the list includes T.N. Shorey, R. Balasubramanian, K. Srinivas and myself. We were really very fortunate and are proud to say that Ramachandra was our teacher. All of us are doing very well mathematically and this adds more to say that Ramachandra had been very successful throughout his research life. When I was participating in the conference dinner (Professor Paul Turan's Centenary International Conference on Number Theory, Analysis, Combinatorics held jointly by Alfréd Rényi Institute of Mathematics and Eötvös Loránd University in August 2011) at Budapest, Hungary, Professor Alan Baker enquired with me about the well being of Professor Ramachandra. When I informed him that Ramachandra expired in January 2011, I even recall now that tears rolled down from his eyes on hearing Ramachandra's sad demise. International mathematical community respected Professor Ramachandra a lot. I am sure that his mathematical contributions will certainly attract younger generations in the years to come and he will, forever, live in our memories.

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School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai - 400 005, INDIA.
e-mail: sank@math.tifr.res.in