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Sieve functions in arithmetic bands

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Abstract. An arithmetic function f is a sieve function of range Q, if its Eratosthenes transform $g = f * \mu$ is supported in $[1, Q] \cap \mathbb{N}$, where $g(q) \ll_\varepsilon q^\varepsilon$, $\forall \varepsilon > 0$. Here, we study the distribution of f over the so-called short arithmetic bands $\bigcup_{1 \le a \le H} \{n \in (N, 2N] : n \equiv a \,(\mathrm{mod}\, q)\}$, with $H = o(N)$, and give applications to both the correlations and to the so-called weighted Selberg integrals of f, on which we have concentrated our recent research.

Keywords. mean squares, arithmetic progressions, short intervals.

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Contents

1. Introduction and statement of the results

If an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is the convolution product of g and the constantly 1 function, i.e.

$$f(n) = (g * \mathbf{1})(n) = \sum_{d|n} g(d),$$

we say, with Wintner [Win43], that $g = f * \mu$ is the *Eratosthenes transform* of f (where μ is the well-known Möbius function). We call f a *sieve function* of *range* Q, if its Eratosthenes transform g is *essentially bounded*, namely $g(d) \ll_\varepsilon d^\varepsilon$, $\forall \varepsilon > 0$, and vanishes outside $[1, Q]$ for some $Q \in \mathbb{N}$, that is to say,

$$f(n) = \sum_{\substack{d|n \\ d \le Q}} g(d).$$

As usual, $\ll$ is Vinogradov's notation, synonymous to Landau's O-notation. In particular, $\ll_\varepsilon$ means that the implicit constant might depend on an arbitrarily small and positive real number ε, which might change at each occurrence. We also write $g_Q \overset{def}{=} g \cdot \mathbf{1}_{[1,Q]}$ to mean that g vanishes outside $[1, Q]$ (hereafter, $\mathbf{1}_B$ denotes the indicator function of the set $B \cap \mathbb{Z}$). Moreover, note that $f = g * \mathbf{1}$ is essentially bounded if and only if so is g.

Sieve functions are ubiquitous in analytic number theory. For example, the truncated divisor sum Λ_R, exploited by Goldston in [Gol92], is a linear combination of sieve functions of range R (see Sect. 5.). Compare also [Cop10b] for more examples of sieve functions. However, the reader is cautioned that by a sieve function some authors simply mean any sieve-related function that often arises within the theory of sieve methods (see [DiaHal08]).

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The first author has intensively investigated symmetry properties of sieve functions in short intervals through the study of their *correlations* and the associated *Selberg integrals* ([Cop10a], [Cop10b] and [CopLap15]). Here we wish to relate such a study to the distribution of a sieve function in modular arithmetic *short bands*. More precisely, for given positive integers q, N, H we search for non-trivial bounds on the *total (balanced) value* of f in *arithmetic bands* modulo q defined as

$$T_f(q,N,H) \overset{def}{=} \sum_{a\le H} \sum_{\substack{n\sim N \\ n\equiv a \,(\mathrm{mod}\ q)}} f(n) - \frac{H}{q}\sum_{n\sim N} f(n),$$

where $n \sim N$ means that $n \in (N, 2N]\cap\mathbb{N}$ (hereafter, we omit $a \ge 1$ in sums like $\sum_{a\le H}$). In particular, given any $N, H \in \mathbb{N}$, we prove that (see the remark after Theorem 1.1) for every real sieve function f of range $Q \ll N$ and every $q \ll N$ one has

$$T_f(q,N,H) \ll_\varepsilon N^\varepsilon(N/q + q + Q). \tag{1..1}$$

It transpires from our method that similar bounds can be immediately established for *weighted* versions of the above problem, namely

$$T_{w,f}(q,N,H) \overset{def}{=} \sum_{0\le|a|\le H} w(a) \sum_{\substack{n\sim N \\ n\equiv a \,(\mathrm{mod}\ q)}} f(n) - \frac{1}{q}\sum_{0\le|h|\le H} w(h) \sum_{n\sim N} f(n),$$

whenever $w : \mathbb{R} \to \mathbb{R}$ is a piecewise-constant *weight*. Indeed, it is plain that $T_f(q,N,H) = T_{u,f}(q,N,H)$ involves the *unit step* weight

$$u(h) \overset{def}{=} \begin{cases} 1 \text{ if } h > 0 \\ 0 \text{ otherwise.} \end{cases}$$

However, we give more general conditions on w to treat $T_{w,f}(q,N,H)$. First, let us set

$$w_H(h) \overset{def}{=} w\cdot \mathbf{1}_{[-H,H]}(h) = \begin{cases} w(h) & \text{if } h \in [-H,H]\cap\mathbb{Z} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{L}^1_\ell(\widehat{w_H}) \overset{def}{=} \frac{1}{\ell}\sum_{\substack{j<\ell \\ (j,\ell)=1}} \left|\widehat{w_H}\Big(\frac{j}{\ell}\Big)\right|, \quad \text{where} \quad \widehat{w_H}(\beta) \overset{def}{=} \sum_{0\le|h|\le H} w(h)e(h\beta),$$

(hereafter, $e(\alpha) \overset{def}{=} e^{2\pi i\alpha}$ $\forall\alpha\in\mathbb{R}$, and $(j,\ell) \overset{def}{=}$ g.c.d.(j,ℓ), as usual in number theory). Thus, we can write

$$\sum_a w_H(a) \sum_{\substack{n\sim N \\ n\equiv a \,(\mathrm{mod}\ q)}} f(n) = \frac{\widehat{w_H}(0)}{q}\sum_{n\sim N} f(n) + T_{w,f}(q,N,H)$$

and state our first result.

Theorem 1.1. *Let q, N, H, Q be positive integers such that $q \ll N$ and $Q \ll N$, as $N \to \infty$. For every sieve function $f : \mathbb{N} \to \mathbb{R}$ of range Q and every weight $w : \mathbb{R} \to \mathbb{R}$ one has*

$$T_{w,f}(q,N,H) \ll_\varepsilon N^\varepsilon\Big(\frac{N}{q} + q + Q\Big) \max_{\substack{\ell>1 \\ \ell|q}} \mathcal{L}^1_\ell(\widehat{w_H}).$$

Remark 1.2. By taking $w = u$ and recalling $\|r\| \overset{def}{=} \min_{n\in\mathbb{Z}}|r-n|$, $\forall r\in\mathbb{R}$, we have $\forall\ell > 1$, (see [Da00], Ch.25),

$$\mathcal{L}^1_\ell(\widehat{u_H}) = \frac{1}{\ell}\sum_{\substack{j<\ell \\ (j,\ell)=1}} \left|\sum_{h\le H} e\Big(h\frac{j}{\ell}\Big)\right| \ll \frac{1}{\ell}\sum_{\substack{j<\ell \\ (j,\ell)=1}} \frac{1}{\|j/\ell\|} \ll \sum_{j\le\ell/2}\frac{1}{j} \ll \log\ell.$$

Therefore, (1..1) follows immediately from Theorem 1.1.

Another remarkable instance concerns the *correlation* of w_H given by

$$W_H(a) \overset{def}{=} \sum_{\substack{h_1 \\ h_2-h_1=a}} \sum_{h_2} w_H(h_1) w_H(h_2) = \sum_{\substack{0\le|h|\le H \\ 0\le|h-a|\le H}} w(h)w(h-a).$$

Note that W_H vanishes outside $[-2H, 2H]$. Moreover, uniformly in $\beta \in [0,1]$, one has

$$\begin{aligned} \widehat{W_H}(\beta) &= \sum_{0\le|h|\le 2H} W_H(h)e(h\beta) = \sum_h \sum_{m-n=h} w_H(m)w_H(n)e(h\beta) \\ &= \Big|\sum_r w_H(r)e(r\beta)\Big|^2 = |\widehat{w_H}(\beta)|^2. \end{aligned}$$

Besides revealing that not all the weights are correlations of other weights, this yields

$$\widehat{W_H}(0) = \widehat{w_H}(0)^2 \ll H^2,$$

when w_H is uniformly bounded as $H \to \infty$. Moreover, if w_H also satisfies the inequality

$$\mathcal{L}^2_\ell(\widehat{w_H}) \overset{def}{=} \frac{1}{\ell^2} \sum_{j<\ell} \left|\widehat{w_H}\left(\frac{j}{\ell}\right)\right|^2 \ll \frac{H}{\ell}, \ \forall \ell \ge 1, \tag{1..2}$$

then

$$\mathcal{L}^1_\ell(\widehat{W_H}) = \frac{1}{\ell} \sum_{\substack{j<\ell \\ (j,\ell)=1}} \widehat{W_H}\left(\frac{j}{\ell}\right) \le \ell \mathcal{L}^2_\ell(\widehat{w_H}) \ll H, \ \forall \ell \ge 1.$$

$\diamond$

(Hereafter, $\diamond$ indicates the end of a remark).

According to [CopLap16], a uniformly bounded weight w_H (as $H \to \infty$) is said to be *good*, if it satisfies (1..2). Thus, the following result is immediately established in a completely analogous way to the proof of Theorem 1.1.

Corollary 1.3. *Let q, N, H, Q be positive integers such that $q \ll N$ and $Q \ll N$, as $N \to \infty$. For every sieve function $f : \mathbb{N} \to \mathbb{R}$ of range Q and every good weight $w : \mathbb{R} \to \mathbb{R}$ one has*

$$\sum_a W_H(a) \sum_{\substack{n\sim N \\ n\equiv a \ (\mathrm{mod}\ q)}} f(n) = \frac{\widehat{W_H}(0)}{q} \sum_{n\sim N} f(n) + O_\varepsilon\Big(N^\varepsilon H\Big(\frac{N}{q} + q + Q\Big)\Big),$$

where W_H is the correlation of w_H.

Remark 1.4. Though analogous definitions can be easily formulated for a complex weight w (with the only exception of W_H, whose definition has to be modified by taking the complex conjugate of $w_H(h_1)$), here we stick to real weights and real sieve functions for simplicity. $\diamond$

Remark 1.5. From [CopLap16] (see Propositions 2 and 3 there) it turns out that, beyond the unit step function u defined above, other remarkable examples of good weights are the *sign* function and the *Cesaro* weight, respectively defined as

$$\mathrm{sgn}(h) \overset{def}{=} \begin{cases} 0 & \text{if } h = 0 \\ h/|h| & \text{otherwise,} \end{cases} \qquad C_H(h) \overset{def}{=} \begin{cases} 1 - |h|/H & \text{if } |h| \le H \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$C_H(h) = \frac{1}{H} \sum_{t \le H-|h|} 1 = \frac{1}{H} \sum\sum_{\substack{m,n \le H \\ m-n=h}} 1,$$

then HC_H is the correlation of u_H, and consequently $\widehat{C_H}(0) = \widehat{u_H}(0)^2/H = H$. We conclude that Corollary 1.3 is non-trivial for $w_H = u_H$, yielding

$$\sum_a C_H(a) \sum_{\substack{n \sim N \\ n \equiv a \,(\mathrm{mod}\, q)}} f(n) = \frac{H}{q} \sum_{n \sim N} f(n) + O_\varepsilon\Big(N^\varepsilon\Big(\frac{N}{q} + q + Q\Big)\Big).$$

◇

Remark 1.6. The main terms in the formulæ furnished by Theorem 1.1 and Corollary 1.3 can be explicitly related to the Eratosthenes transform of $f = g_Q * \mathbf{1}$, with $Q \ll N$. Indeed,

$$\begin{aligned}
\sum_{\substack{n \sim N \\ n \equiv a \,(\mathrm{mod}\, q)}} f(n) &= \sum_{\substack{n \sim N \\ n \equiv a \,(\mathrm{mod}\, q)}} \sum_{d|n} g_Q(d) = \sum_{d \le Q} g(d) \sum_{\substack{n \sim N \\ n \equiv a \,(\mathrm{mod}\, q) \\ n \equiv 0 \,(\mathrm{mod}\, d)}} 1 \\
&= \sum_{\substack{d \le Q \\ (d,q)|a}} g(d) \sum_{\substack{n \sim N/d \\ nd \equiv a \,(\mathrm{mod}\, q)}} 1 = \sum_{\substack{d \le Q \\ (d,q)|a}} g(d) \left(\frac{N}{dq}(d,q) + O(1)\right) \\
&= \frac{N}{q} \sum_{\substack{d \le Q \\ (d,q)|a}} \frac{g(d)}{d}(d,q) + O_\varepsilon\left(Q^{1+\varepsilon}\right).
\end{aligned}$$

In particular, for the *long intervals* we get the formula

$$\sum_{n \sim N} f(n) = R_1(f)N + O_\varepsilon\left(Q^{1+\varepsilon}\right), \tag{1..3}$$

where the so-called first *Ramanujan coefficient* $R_1(f)$ is the mean value of f (see Sect. 2.):

$$\begin{aligned}
R_1(f) &\overset{def}{=} \sum_{d \le Q} \frac{g(d)}{d} \\
&= \lim_{x \to \infty} \Big(\sum_{d \le Q} \frac{g(d)}{d} + \frac{1}{x} \sum_{d \le Q} O\left(|g(d)|\right)\Big) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n).
\end{aligned}$$

On the other side, by taking F as the Dirichlet series generating f, one has

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n) = \operatorname*{Res}_{s=1} F(s) \frac{x^{s-1}}{s}.$$

Since $f = g_Q * \mathbf{1}$ is a sieve function, then F can be expressed in terms of the Riemann zeta function ζ and the Dirichlet polynomial generating its Eratosthenes transform, namely

$$F(s) \overset{def}{=} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \sum_{d \le Q} \frac{g(d)}{d^s}.$$

Note that the zeta function forces F to have a simple pole at $s = 1$, provided the g series does not vanish at $s = 1$. Thus, if $f = g_Q * \mathbf{1}$ is gauged by a weight w in the *short interval* $[x - H, x + H]$

(i.e. $H = o(N)$, as $N \to \infty$), then it is natural to take the expected *mean value* of $w_H(n-x)f(n)$ for $N < x \le 2N$ to be (compare [CopLap14])

$$\widehat{w_H}(0)R_1(f) = \sum_a w_H(a) \sum_{d \le Q} \frac{g(d)}{d} \quad \text{(that is independent of } x\text{)}.$$

Indeed, a basic tool for the study of the distribution of the sieve function f in short intervals is its *weighted Selberg integral*

$$J_{w,f}(N,H) \overset{def}{=} \sum_{x \sim N} \Big| \sum_n w_H(n-x)f(n) - \widehat{w_H}(0)R_1(f) \Big|^2,$$

whose non-trivial bounds might lead to results on the distribution of f in *almost all* short intervals $[x-H, x+H]$, i.e. with $o(N)$ possible exceptions $x \in (N, 2N] \cap \mathbb{N}$. Observe that the trivial bound for $J_{w,f}(N,H)$ is $N^{1+\varepsilon}H^2$, because f is essentially bounded. In [CopLap14] and [CopLap16] we have investigated and exploited the link between $J_{w,f}(N,H)$ and the *correlation*

$$\mathcal{C}_f(a) \overset{def}{=} \sum_{n \sim N} f(n)f(n-a),$$

in order to get non-trivial bounds under suitable conditions on f and a good weight w. ◇

As a consequence of Theorem 1.1, we obtain a further result on such a link with a slight generalization. Let us define the correlation of real arithmetic functions f_1 and f_2 as

$$\mathcal{C}_{f_1,f_2}(a) \overset{def}{=} \sum_{n \sim N} f_1(n)f_2(n-a).$$

In such a context, we might refer to $\mathcal{C}_f = \mathcal{C}_{f,f}$ as the *autocorrelation* of f. Since the *shift* a is confined to $a \ll H$, the conditions $n \sim N$ and $H = o(N)$ clearly yield $\max(n, n-a) \le 2N + |a| \le 3N$. Moreover, if f_1 and f_2 are essentially bounded, then trivially $\mathcal{C}_{f_1,f_2}(0) \ll N^{1+\varepsilon}$, and for any $a \ll H$ one has

$$\mathcal{C}_{f_1,f_2}(a) = \sum_{\substack{n_1 \sim N \\ n_2 - n_1 = a}} \sum_{n_2 \sim N} f_1(n_1)f_2(n_2) + O_\varepsilon(N^\varepsilon H)$$

(to be compared to the previous definition of the correlation of a weight).
Correspondingly, the *mixed* weighted Selberg integral associated to the pair (f_1, f_2) is (compare [Cop14])

$$J_{w,(f_1,f_2)}(N,H) \overset{def}{=} \sum_{x \sim N} \prod_{j=1,2} \Big(\sum_n w_H(n-x)f_j(n) - \widehat{w_H}(0)R_1(f_j) \Big).$$

By assuming that $f_2 = g_2 * \mathbf{1}$ is a sieve function of range Q_2 it is readily seen that (see also the proof of Lemma 3.3 below)

$$\mathcal{C}_{f_1,f_2}(a) = \sum_{q \le Q_2} g_2(q) \sum_{\substack{n \sim N \\ n \equiv a\,(q)}} f_1(n).$$

Moreover, Lemma 3.3 also shows that if f_1 and f_2 are sieve functions, then $J_{w,(f_1,f_2)}(N,H)$ can be expressed in terms of arithmetic bands of f_1 or f_2. Such favorable circumstances allow us to apply Theorem 1.1 in order to obtain both a so-called *first generation* formula (consistently with the terminology of [CopLap14]) for the correlation of the sieve functions f_1 and f_2 (more precisely, here we use (1..1) that is a consequence of Theorem 1.1 as it is showed in Remark 1.2) and an estimate of $J_{w,(f_1,f_2)}(N,H)$ once f_1 and f_2 are gauged by a good weight w.

Corollary 1.7. *Let N, H, Q_1, Q_2 be positive integers with $Q_1 \le Q_2 \ll N$, as $N \to \infty$. For any real and essentially bounded arithmetic functions g_1 and g_2 supported in $[1, Q_1]$ and $[1, Q_2]$, respectively, one has*

$$\sum_{a \le H} \mathcal{C}_{f_1,f_2}(a) = R_1(f_1)R_1(f_2)NH + O_\varepsilon\big(N^\varepsilon(N + Q_2^2 + Q_1 H)\big),$$

where $f_j = g_j * \mathbf{1}$ for $j = 1, 2$. Furthermore, if $H = o(N)$, as $N \to \infty$, and $w : \mathbb{R} \to \mathbb{R}$ is a good weight, then

$$J_{w,(f_1,f_2)}(N, H) \ll_\varepsilon N^\varepsilon\,(NH + Q_2 H^2 + Q_2^2 H + H^3).$$

Remark 1.8. For every real sieve function f of range $Q \ll N$, this corollary gives

$$\sum_{a \le H} \mathcal{C}_f(a) = R_1^2(f)NH + O_\varepsilon\big(N^\varepsilon(N + Q^2 + QH)\big),$$

$$J_{w,f}(N, H) \ll_\varepsilon N^\varepsilon\,(NH + QH^2 + Q^2 H + H^3).$$

We stress that such a bound for the weighted Selberg integral has been already established in Theorem 3 of [CopLap16]. In Sect. 4. we propose a much simpler proof through the new approach of the *arithmetic bands* formulæ provided by Theorem 1.1.

Furthermore, from such an approach we find an important relation between weighted Selberg integrals and the *total (weighted) content* of a sieve function f of range $Q \ll N$, namely (see Lemma 3.3 and the proof of Corollary 1.7)

$$J_{w,f}(N, H) \ll_\varepsilon N^\varepsilon \sum_{q \le Q} |T_{W,f}(q, N, H)| + N^\varepsilon H^2(Q + H), \tag{1..4}$$

where for the correlation of w_H we set

$$T_{W,f}(q, N, H) \overset{def}{=} \sum_a W_H(a) \sum_{\substack{n \sim N \\ n \equiv a \,(\mathrm{mod}\ q)}} f(n) - \frac{\widehat{W_H}(0)}{q} \sum_{n \sim N} f(n).$$

◇

Beyond Corollary 1.3, more generally, given an essentially bounded f, a non-trivial bound like

$$\sum_{q \le Q} |T_{W,f}(q, N, H)| \ll N^{1-\delta} H^2, \text{ for some real } \delta > 0,$$

might yield a non-trivial bound of the same type for $J_{w,f}(N, H)$ (but not necessarily with the same *gain* N^δ) by means of (1..4). Analogous considerations hold for mixed weighted Selberg integrals. Rather surprisingly, in spite of the fact that the presence of absolute values in the total content seems to prevent it from further possible cancellation, the next theorem makes it clear that there are non-trivial bounds for (weighted) Selberg integrals, involving a sieve function f of range $Q \ll N^{1-\delta}$ for some $\delta > 0$, if and only if there are non-trivial results on the distribution of f in short arithmetic bands.

Theorem 1.9. *Let $f : \mathbb{N} \to \mathbb{R}$ be a sieve function of range $Q \ll N^{1-\delta}$, for some $\delta > 0$, and let $w : \mathbb{R} \to \mathbb{R}$ be such that w_H is uniformly bounded for any $H \ll N^{1-\delta}$, as $N \to \infty$.*

I) The following three assertions are equivalent:

i) a non-trivial bound holds for $\sum_{q \le Q} |T_{W,f}(q, N, H)|$

ii) a non-trivial bound holds for $J_{w,f}(N,H)$

iii) a non-trivial bound holds for $J_{w,(f,f_1)}(N,H)$*, where* f_1 *is any sieve function of range* Q*.*

II) If $N^{\delta/2} \ll H \ll N^{1-\delta}$*, as* $N \to \infty$*, then the following assertions are equivalent:*

iv) a non-trivial bound holds for $\sum_{q \le Q} |T_f(q,N,H)|$

v) a non-trivial bound holds for the Selberg integral

$$J_f(N,H) \stackrel{def}{=} \sum_{x \sim N} \Big| \sum_{x<n\le x+H} f(n) - R_1(f)H \Big|^2.$$

Note that in $iv)$ a non-trivial bound is meant to be of the type $N^{1-\delta}H$ for some $\delta > 0$.

After a short section on some further notation and basic formulæ, in Sect. 3. we give the necessary lemmata for our theorems and for Corollary 1.7, whose proofs constitute the fourth section, whereas we omit the proof of Corollary 1.3, it being completely analogous to the proof of Theorem 1.1. In Sect. 4. we specialize the results of the present article to the aforementioned function Λ_R. The last section is devoted to a comparison between classical results in arithmetic progressions and ours in arithmetic bands.

2. Further notation and standard properties

As already mentioned, we omit $a \ge 1$ in sums like $\sum_{a \le X}$. For the same sake of brevity, at times we write $n \equiv a\,(q)$ in place of $n \equiv a \pmod q$. Thus, the well-known *orthogonality of additive characters*,

$$e_q(r) \stackrel{def}{=} e(r/q) = e^{2\pi i r/q},\ (q \in \mathbb{N},\ r \in \mathbb{Z}),$$

can be written as

$$\frac{1}{q}\sum_{j\,(q)} e_q(j(n-m)) = \frac{1}{q}\sum_{j \le q} e_q(j(n-m)) = \begin{cases} 1 \text{ if } n \equiv m \pmod q \\ 0 \text{ otherwise} \end{cases},$$

since the sum is over a complete set of residue classes $j \pmod q$.
We write $\sum_{j(q)}^{*}$ to mean that the sum is over a complete set of reduced residue classes (mod q), i.e. the set $\mathbb{Z}_q^*$ of $1 \le j \le q$ such that $(j,q)=1$. In particular, the *Ramanujan sum* is written as

$$c_q(n) \stackrel{def}{=} \sum_{j(q)}^{*} e_q(jn).$$

Without further references, we will appeal to the well-known inequality (see [Da00], Ch.25)

$$\sum_{V_1 < v \le V_2} e(v\alpha) \ll \min\Big(V_2 - V_1, \frac{1}{\|\alpha\|}\Big).$$

Recalling that $\mathbf{1}(n) \stackrel{def}{=} 1,\ \forall n \in \mathbb{N}$, we set

$$\mathbf{1}_{d|n} \stackrel{def}{=} \begin{cases} 1 \text{ if } d|n \\ 0 \text{ otherwise.} \end{cases}$$

Consequently, the aforementioned orthogonality of characters becomes

$$\mathbf{1}_{d|n} = \frac{1}{d}\sum_{j'(d)} e_d(j'n) = \frac{1}{d}\sum_{\ell|d}\sum_{\substack{j'(d)\\(j',d)=d/\ell}} e_d(j'n) = \frac{1}{d}\sum_{\ell|d} c_\ell(n).$$

Therefore, one has the following *Ramanujan expansion* of a sieve function $f = g_Q * \mathbf{1}$:

$$\begin{aligned} f(n) &= \sum_{d|n} g_Q(d) = \sum_{d\le Q} g(d)\mathbf{1}_{d|n} = \sum_{d\le Q}\frac{g(d)}{d}\sum_{\ell|d} c_\ell(n) \\ &= \sum_{\ell\le Q}\sum_{\substack{d\le Q\\ d\equiv 0\,(\ell)}}\frac{g(d)}{d}c_\ell(n) = \sum_{\ell\le Q} R_\ell(f)c_\ell(n), \end{aligned}$$

where we have introduced the so-called $\ell-$th *Ramanujan coefficient* of f, i.e.

$$R_\ell(f) \stackrel{def}{=} \sum_{d\equiv 0\,(\ell)} \frac{g_Q(d)}{d}.$$

The hypothesis that g is essentially bounded yields the bound

$$R_\ell(f) \ll \frac{1}{\ell}\sum_{m\le\frac{Q}{\ell}}\frac{|g(\ell m)|}{m} \ll_\varepsilon \frac{Q^\varepsilon}{\ell}\sum_{m\le\frac{Q}{\ell}}\frac{1}{m} \ll_\varepsilon \frac{Q^\varepsilon}{\ell}. \tag{2..1}$$

We refer the reader to [SchSpi94] for more extensive accounts on the theory of the Ramanujan expansions.

3. Lemmata

Here we state and prove two lemmas that are interesting in their own right. The first lemma is required to prove Theorem 1.1, while the second one is invoked within the proofs of Corollary 1.7 and Theorem 1.9. To this end, analogously to the exponential sums for the weights already introduced in Sect. 1., we set

$$\widehat{f}(\alpha) \stackrel{def}{=} \sum_{n\sim N} f(n)e(n\alpha) \quad (\alpha\in\mathbb{R}).$$

Notice that now we can write

$$T_{w,f}(q,N,H) = \sum_a w_H(a)\sum_{\substack{n\sim N\\ n\equiv a\,(q)}} f(n) - \widehat{w_H}(0)\frac{\widehat{f}(0)}{q},$$

$$T_{W,f}(q,N,H) = \sum_a W_H(a)\sum_{\substack{n\sim N\\ n\equiv a\,(q)}} f(n) - \widehat{W_H}(0)\frac{\widehat{f}(0)}{q},$$

while the formula (1..3) becomes

$$\widehat{f}(0) = R_1(f)N + O_\varepsilon(Q^{1+\varepsilon}).$$

The first lemma gives a similar relation between the $\ell-$th Ramanujan coefficient of f and $\widehat{f}(\alpha)$, when $\alpha = j/\ell$ is any non-integer rational with $(j,\ell)=1$. Note that such a formula is not a straightforward consequence of Wintner's criterion (see VIII.2 of [SchSpi94]).

Lemma 3.1. *Let f be a sieve function of range $Q \ll N$, with $Q, N \in \mathbb{N}$. Then*

$$\widehat{f}(j/\ell) = R_\ell(f)N + O_\varepsilon((\ell Q)^\varepsilon(Q+\ell)), \qquad \forall \ell > 1, \forall j \in \mathbb{Z}_\ell^*.$$

Proof. By hypothesis $f = g_Q * \mathbf{1}$ with an essentially bounded g. Hence

$$\begin{aligned}\widehat{f}(j/\ell) &= \sum_d g_Q(d) \sum_{v \sim \frac{N}{d}} e_\ell(jdv) \\ &= \sum_{d \equiv 0\,(\ell)} g_Q(d)\Big(\frac{N}{d} + O(1)\Big) + O\Big(\sum_{d \not\equiv 0\,(\ell)} \frac{|g_Q(d)|}{\|jd/\ell\|}\Big).\end{aligned}$$

Since

$$\sum_{d \equiv 0\,(\ell)} g_Q(d)\Big(\frac{N}{d} + O(1)\Big) = R_\ell(f)N + O_\varepsilon\Big(Q^\varepsilon\Big(\frac{Q}{\ell} + 1\Big)\Big)\ ,$$

then the lemma is proved whenever we show that

$$\sum_{\substack{d \le Q \\ d \not\equiv 0\,(\ell)}} \frac{1}{\|jd/\ell\|} \ll_\varepsilon \ell^\varepsilon(Q+\ell).$$

To this end, it suffices to observe that

$$\sum_{\substack{d \le Q \\ d \not\equiv 0\,(\ell)}} \frac{1}{\|jd/\ell\|} \le \sum_{0<|r|\le \ell/2} \sum_{\substack{d \le Q \\ jd \equiv r\,(\ell)}} \frac{1}{|r/\ell|} \ll \ell \sum_{r \le \ell/2} \frac{1}{r}\Big(\frac{Q}{\ell} + 1\Big).$$

The proof is completed. □

Remark 3.2. Note that the formula of the above lemma is non-trivial when $\ell, Q \ll N^{1-\delta}$, for some $\delta > 0$. Moreover, it is easy to see that it holds uniformly with respect to $j \in \mathbb{Z}_\ell^*$. ◇

Let us turn our attention to the next lemma. As already mentioned in Sect. 1., by means of an elementary dispersion method, in [CopLap14], Lemma 7, we established a link between weighted Selberg integrals and autocorrelations of an arithmetic function f gauged by a weight w such that w_H is bounded, as $H \to \infty$. Under the further hypothesis that the sieve function f and the weight w are real, the formula of the aforementioned lemma becomes

$$\begin{aligned}J_{w,f}(N,H) &= \sum_{0 \le |a| \ll H} W_H(a)\mathcal{C}_f(a) + \sum_{x \sim N} |\widehat{w_H}(0)R_1(f)|^2 \\ &\quad -2\widehat{w_H}(0)R_1(f) \sum_{n \le 3N} f(n) \sum_{x \sim N} w_H(n-x) + O_\varepsilon\left(H^3 N^\varepsilon\right).\end{aligned}$$

Similarly, for the mixed weighted Selberg integral of sieve functions f_1, f_2 we have

$$\begin{aligned}J_{w,(f_1,f_2)}(N,H) = &\sum_a W_H(a)\mathcal{C}_{f_1,f_2}(a) - \widehat{W_H}(0)R_1(f_1)R_1(f_2)N \\ &- \widehat{w_H}(0)\Big(R_1(f_1)\sum_{x \sim N}\Delta_2(x) + R_1(f_2)\sum_{x \sim N}\Delta_1(x)\Big) \\ &+ O_\varepsilon\left(H^3 N^\varepsilon\right), \qquad (3..1)\end{aligned}$$

where we set $\Delta_j(x) \stackrel{def}{=} \sum_n w_H(n-x)f_j(n) - \widehat{w_H}(0)R_1(f_j)$.
By using such a formula we prove the next lemma, where $J_{w,(f_1,f_2)}(N,H)$ is expressed in terms of arithmetic bands of f_1 or f_2.

Lemma 3.3. *Let g_1 and g_2 be real and essentially bounded arithmetic functions supported in $[1, Q_1]$ and $[1, Q_2]$, respectively, with $Q_1, Q_2 \in \mathbb{N}$ such that $Q_1 \leq Q_2 \ll N$, as $N \to \infty$. If $w : \mathbb{R} \to \mathbb{R}$ is such that w_H is uniformly bounded, as $H \to \infty$, then one has*

$$\begin{aligned} J_{w,(f_1,f_2)}(N,H) &= \sum_{q \leq Q_1} g_1(q) T_{W,f_2}(q,N,H) + O_\varepsilon\big(N^\varepsilon H^2 (Q_2 + H)\big) \\ &= \sum_{q \leq Q_2} g_2(q) T_{W,f_1}(q,N,H) + O_\varepsilon\big(N^\varepsilon H^2 (Q_2 + H)\big), \end{aligned}$$

where we set $f_j = g_j * \mathbf{1}$, and W_H is the correlation of w_H.

Proof. First, let us write

$$\begin{aligned} \sum_{x \sim N} \sum_n w_H(n-x) f_j(n) &= \sum_{n \sim N} f_j(n) \sum_{n-H \leq x \leq n+H} w(n-x) + O_\varepsilon\left(N^\varepsilon H^2\right) \\ &= \widehat{w_H}(0) \sum_{n \sim N} f_j(n) + O_\varepsilon\left(N^\varepsilon H^2\right). \end{aligned}$$

Then, by arguing as in (1..3) and recalling that $R_1(f_j) \ll_\varepsilon Q_j^\varepsilon$, we get

$$\begin{aligned} \sum_{x \sim N} \Delta_j(x) &= \widehat{w_H}(0) \Big(\sum_{n \sim N} f_j(n) - R_1(f_j) N \Big) + O_\varepsilon\left(N^\varepsilon H^2\right) \\ &\ll_\varepsilon N^\varepsilon H (Q_j + H). \end{aligned}$$

Since W_H is even and $Q_1 \leq Q_2 \ll N$, the above formula (3..1) yields

$$\begin{aligned} J_{w,(f_1,f_2)}(N,H) &= \sum_a W_H(a) \mathcal{C}_{f_1,f_2}(a) - \widehat{W_H}(0) R_1(f_1) R_1(f_2) N \\ &\quad + O_\varepsilon\left(N^\varepsilon H^2 (Q_2 + H)\right) \\ &= \sum_a W_H(a) \mathcal{C}_{f_2,f_1}(a) - \widehat{W_H}(0) R_1(f_1) R_1(f_2) N \\ &\quad + O_\varepsilon\left(N^\varepsilon H^2 (Q_2 + H)\right). \end{aligned}$$

Thus, we can stick to the first equality, apply (1..3) to f_1 and write

$$\begin{aligned} \sum_a W_H(a) \mathcal{C}_{f_1,f_2}(a) &- \widehat{W_H}(0) R_1(f_1) R_1(f_2) N \\ &= \sum_a W_H(a) \sum_{n \sim N} f_1(n) \sum_{\substack{q \mid n-a \\ q \leq Q_2}} g_2(q) \\ &\quad - \widehat{W_H}(0) \sum_{n \sim N} f_1(n) \sum_{q \leq Q_2} \frac{g_2(q)}{q} + O_\varepsilon\left(Q_2^{1+\varepsilon} H^2\right) \\ &= \sum_{q \leq Q_2} g_2(q) \Big(\sum_a W_H(a) \sum_{\substack{n \sim N \\ n \equiv a\ (q)}} f_1(n) - \widehat{W_H}(0) \frac{\widehat{f_1}(0)}{q} \Big) \\ &\quad + O_\varepsilon\left(Q_2^{1+\varepsilon} H^2\right). \end{aligned}$$

The lemma is completely proved. □

4. Proofs of Theorems 1.1, 1.9 and Corollary 1.7

Proof of Theorem 1.1. By the orthogonality of additive characters we get

$$\begin{aligned} T_{w,f}(q,N,H) &= \frac{1}{q}\sum_{a} w_H(a)\sum_{n\sim N} f(n)\sum_{j'\le q} e_q(j'(a-n)) - \frac{\widehat{w_H}(0)}{q}\widehat{f}(0) \\ &= \frac{1}{q}\sum_{j'<q}\sum_{a} w_H(a)e_q(j'a)\widehat{f}\,(-j'/q) \\ &= \frac{1}{q}\sum_{\substack{\ell>1\\ \ell|q}}\ \sideset{}{^*}\sum_{j\,(\ell)} \widehat{f}\,(-j/\ell)\,\widehat{w_H}(j/\ell), \end{aligned}$$

where we have set $\ell = q/(j',q)$. By applying Lemma 3.1 and (2..1) we see that

$$\begin{aligned} T_{w,f}(q,N,H) &\ll_\varepsilon \frac{1}{q}\sum_{\substack{\ell>1\\ \ell|q}}\Big(|R_\ell(f)|N + (\ell Q)^\varepsilon(Q+\ell)\Big)\sideset{}{^*}\sum_{j\,(\ell)}\Big|\widehat{w_H}\Big(\frac{j}{\ell}\Big)\Big| \\ &\ll_\varepsilon \frac{Q^\varepsilon}{q}\sum_{\substack{\ell>1\\ \ell|q}}\Big(\frac{N}{\ell} + Q\ell^\varepsilon + \ell^{1+\varepsilon}\Big)\ell\mathcal{L}^1_\ell(\widehat{w_H}) \\ &\ll_\varepsilon N^\varepsilon\Big(\frac{N}{q}+Q+q\Big)\max_{\substack{\ell>1\\ \ell|q}}\mathcal{L}^1_\ell(\widehat{w_H}). \end{aligned}$$

The theorem is completely proved. □

Proof of Corollary 1.7. As already noticed in the proof of Lemma 3.3, we can write

$$\begin{aligned} \mathcal{C}_{f_1,f_2}(a) &= \sum_{n\sim N} f_1(n)f_2(n-a) = \sum_{n\sim N} f_1(n)\sum_{\substack{q|n-a\\ q\le Q_2}} g_2(q) \\ &= \sum_{q\le Q_2} g_2(q)\sum_{\substack{n\sim N\\ n\equiv a\,(q)}} f_1(n). \end{aligned}$$

Thus, the formula (1..3) and Theorem 1.1 (more precisely, we apply (1..1) here) yield

$$\begin{aligned} \sum_{a\le H}\mathcal{C}_{f_1,f_2}(a) &= \sum_{q\le Q_2} g_2(q)\Big(\frac{H}{q}\widehat{f_1}(0) + T_{f_1}(q,N,H)\Big) \\ &= H\Big(\sum_{q\le Q_2}\frac{g_2(q)}{q}\Big)\left(R_1(f_1)N + O_\varepsilon\left(Q_1^{1+\varepsilon}\right)\right) \\ &\quad + O_\varepsilon\Big(N^\varepsilon\sum_{q\le Q_2}\Big(\frac{N}{q}+q+Q_1\Big)\Big) \\ &= R_1(f_1)R_1(f_2)NH + O_\varepsilon\big(N^\varepsilon(N+Q_2^2+Q_1H)\big), \end{aligned}$$

that is the first formula of Corollary 1.7. In order to prove the stated inequality for the mixed weighted Selberg integral, it is enough to observe that Lemma 3.3 and the hypothesis $Q_1 \le Q_2 \ll N$ imply

$$\begin{aligned} J_{w,(f_1,f_2)}(N,H) &= \sum_{q\le Q_2} g_2(q)T_{W,f_1}(q,N,H) + O_\varepsilon\big(N^\varepsilon H^2(Q_2+H)\big) \\ &\ll_\varepsilon N^\varepsilon\sum_{q\le Q_2}|T_{W,f_1}(q,N,H)| + N^\varepsilon H^2(Q_2+H). \end{aligned}$$

Hence the conclusion follows immediately from Corollary 1.3. □

Before going to the proof of Theorem 1.9, let us remark explicitly that (1..4) is plainly a particular case of the latter inequality. Moreover, it transpires from the previous proof that, given any real and essentially bounded arithmetic function g supported in $[1, Q]$, with $Q \ll N$, for $f = g * \mathbf{1}$ one has

$$\sum_{a \le H} \mathcal{C}_f(N) = R_1(f)^2 NH + \sum_{q \le Q} g(q) T_f(q, N, H) + O_\varepsilon\left(N^\varepsilon QH\right). \tag{4..1}$$

Proof of Theorem 1.9. For simplicity and without loss of generality, let us assume that, whatever the choice of an assertion among *i)-v)* as hypothesis, the gain of the non-trivial bound is always N^δ.
Part I. $i) \Longrightarrow ii)$: as we said, let us suppose that

$$\sum_{q \le Q} |T_{W,f}(q, N, H)| \ll N^{1-\delta} H^2.$$

Thus, $ii)$ follows immediately from (1..4), where $H^2(Q + H) \ll N^{1-\delta} H^2$ because of the hypotheses $H, Q \ll N^{1-\delta}$.
$ii) \Longrightarrow iii)$: since we assume that $J_{w,f}(N, H) \ll N^{1-\delta} H^2$, then by the Cauchy inequality and the trivial bound for $J_{w,f_1}(N, H)$ we get

$$\begin{aligned} J_{w,(f,f_1)}(N, H) \quad &\le \sqrt{J_{w,f}(N, H)} \sqrt{J_{w,f_1}(N, H)} \\ &\ll_\varepsilon N^\varepsilon \sqrt{N^{1-\delta} H^2} \sqrt{NH^2} \ll N^{1-\delta/3} H^2. \end{aligned}$$

$iii) \Longrightarrow i)$: after setting

$$s_{W,f}(q) \stackrel{def}{=} \begin{cases} \operatorname{sgn}(T_{W,f}(q, N, H)) & \text{if } 1 \le q \le Q \\ 0 & \text{otherwise,} \end{cases}$$

it is readily seen that $f_1 = s_{W,f} * \mathbf{1}$ is a sieve function of range Q. Thus, we can write

$$\sum_{q \le Q} |T_{W,f}(q, N, H)| = \sum_{q} s_{W,f}(q) T_{W,f}(q, N, H).$$

Now, by taking $g_1 = s_{W,f}$ and $f_2 = f$ in Lemma 3.3 we see that

$$\sum_{q \le Q} |T_{W,f}(q, N, H)| = J_{w,(f,f_1)}(N, H) + O_\varepsilon\left(N^\varepsilon H^2 (Q + H)\right),$$

where again $H^2(Q + H)$ is non-trivial. The first part of the theorem is completely proved.
Part II. $iv) \Longrightarrow v)$: since $Q \ll N^{1-\delta}$ and we assume that

$$\sum_{q \le Q} |T_f(q, N, H)| \ll N^{1-\delta} H,$$

then it is easily seen that the formula (4..1) yields

$$\sum_{a \le t} \mathcal{C}_f(a) = R_1(f)^2 N[t] + O_\varepsilon\left(N^{1-\delta+\varepsilon} t\right) \quad \text{for all } 1 \le t \le H,$$

where $[t]$ is the *integer part* of t. Thus, by partial summation we can write

$$\begin{aligned}\sum_{1\le a\le H}(H-a)\mathcal{C}_f(a) &= \int_1^H \sum_{a\le t}\mathcal{C}_f(a)dt\\ &= \int_1^H \left(R_1(f)^2N[t]+O_\varepsilon\left(N^{1-\delta+\varepsilon}t\right)\right)dt\\ &= \frac{R_1(f)^2}{2}NH^2+O_\varepsilon\left(N^{1+\varepsilon}H\right)+O_\varepsilon\left(N^{1-\delta+\varepsilon}H^2\right).\end{aligned}$$

Now, since $\mathcal{C}_f(0)\ll_\varepsilon N^{1+\varepsilon}$, and for $1\le a\le H$ one has

$$\begin{aligned}\mathcal{C}_f(-a) &= \sum_{n\sim N}f(n)f(n+a)=\sum_{N+a<m\le 2N+a}f(m-a)f(m)\\ &= \mathcal{C}_f(a)+O_\varepsilon\left(N^\varepsilon H\right),\end{aligned}$$

then

$$\sum_{0\le |a|\le H}(H-|a|)\mathcal{C}_f(a)=R_1(f)^2NH^2+O\Big(N^{1-\delta/3}H^2\Big).$$

By using this formula in (3..1), where we take

$$W_H(a)=HC_H(a)=\max(H-|a|,0)$$

(see Remark 1.5), we immediately obtain $J_f(N,H)\ll N^{1-\delta/3}H^2$.

$v)\Longrightarrow iv)$: we suppose that $J_f(N,H)\ll N^{1-\delta}H^2$ and set

$$s_f(q)\overset{def}{=}\begin{cases}\operatorname{sgn}(T_f(q,N,H)) & \text{if } 1\le q\le Q,\\ 0 & \text{otherwise,}\end{cases}\qquad f_1\overset{def}{=}s_f*\mathbf{1}.$$

Thus, we can write

$$\begin{aligned}\sum_{q\le Q}|T_f(q,N,H)| &= \sum_q s_f(q)\Big(\sum_{a\le H}\sum_{\substack{n\sim N\\ n\equiv a\,(q)}}f(n)-\frac{H}{q}\sum_{n\sim N}f(n)\Big)\\ &= \sum_{a\le H}\Big(\sum_{n\sim N}f(n)f_1(n-a)-R_1(f_1)\sum_{n\sim N}f(n)\Big)\\ &= \sum_{a\le H}\sum_{N-a<x\le 2N-a}f(x+a)f_1(x)-R_1(f_1)R_1(f)NH\\ &\quad +O_\varepsilon(N^\varepsilon QH)\\ &= \sum_{x\sim N}f_1(x)\sum_{x<m\le x+H}f(m)-R_1(f_1)R_1(f)NH\\ &\quad +O_\varepsilon(N^\varepsilon(Q+H)H)\\ &= \sum_{x\sim N}f_1(x)\Big(\sum_{x<m\le x+H}f(m)-R_1(f)H\Big)\\ &\quad +O_\varepsilon(N^\varepsilon(Q+H)H),\end{aligned}$$

where we have applied (1..3) to both f and f_1. Note that the O-term contribution is non-trivial because of hypotheses on Q and H. In order to deal with the main term of the latter formula, after recalling that f_1 is essentially bounded, we apply the Cauchy inequality and the above assumption on $J_f(N,H)$ to get

$$\begin{aligned}\sum_{x\sim N}f_1(x)\Big(\sum_{x<m\le x+H}f(m)-R_1(f)H\Big) &\ll_\varepsilon N^{1/2+\varepsilon}\sqrt{J_f(N,H)}\\ &\ll_\varepsilon N^{1+\varepsilon-\delta/2}H,\end{aligned}$$

which in turn yields

$$\sum_{q\le Q} |T_f(q,N,H)| \ll N^{1-\delta/3}H.$$

Theorem 1.9 is completely proved. □

5. A remarkable truncated divisor sum

Let us recall that the truncated divisor sum defined in [Gol92] is

$$\Lambda_R(n) \overset{def}{=} \sum_{\substack{d|n\\ d\le R}} \mu(d)\log(R/d) = (\log R)\sum_{\substack{d|n\\ d\le R}} \mu(d) - \sum_{\substack{d|n\\ d\le R}} \mu(d)\log d,$$

so that Λ_R is plainly a linear combination (with relatively *small* coefficients) of two sieve functions, whose Eratosthenes transforms are respectively the *restricted* Möbius function, $\mu_R \overset{def}{=} \mu \cdot \mathbf{1}_{[1,R]}$, and $\mu_R \cdot \log$.

After recalling also the well-known relations (see [Da00])

$$\sum_{d=1}^{\infty} \frac{\mu(d)\log d}{d} = -1 \quad \text{and} \quad \sum_{d\le R} \frac{\mu(d)}{d}, \sum_{d>R} \frac{\mu(d)\log d}{d} \ll \exp\big(-c\sqrt{\log R}\big),$$

(hereafter, $c > 0$ is an unspecified constant), we see that

$$\begin{aligned} R_1(\Lambda_R) &= \sum_{d\le R} \frac{\mu(d)\log(R/d)}{d} \\ &= (\log R)\sum_{d\le R}\frac{\mu(d)}{d} - \sum_{d\le R}\frac{\mu(d)\log d}{d} = 1 + O\big(\exp\big(-c\sqrt{\log R}\big)\big). \end{aligned}$$

Thus, the mean value formula (1..3) gives

$$\sum_{n\sim N} \Lambda_R(n) = N + O\big(N\exp\big(-c\sqrt{\log R}\big)\big) + O_\varepsilon(N^\varepsilon R),$$

while, if $R \ll N$, a straightforward application of (1..1) yields

$$\sum_{a\le H} \sum_{\substack{n\sim N\\ n\equiv a\,(q)}} \Lambda_R(n) = \frac{NH}{q} + O_\varepsilon\Big(N^\varepsilon\Big(\frac{N}{q} + q + R\Big)\Big) + O\big(N\exp\big(-c\sqrt{\log R}\big)\big).$$

In case the *level* $\lambda \overset{def}{=} (\log R)/(\log N)$ is positive, i.e. $0 < \lambda_0 \le \lambda < 1$ (for a fixed λ_0), we may replace $\log R$ by $\log N$ in the above formulæ, where now $c = c(\lambda)$. Assuming that this is the case, Corollary 1.7 provides the following *first generation* formula for the correlation of Λ_R:

$$\begin{aligned} \sum_{a\le H}\sum_{n\sim N} \Lambda_R(n)\Lambda_R(n-a) &= NH + O\big(NH\exp\big(-c\sqrt{\log N}\big)\big) \\ &\quad + O_\varepsilon\big(N^\varepsilon(N + R^2 + RH)\big). \end{aligned}$$

It is worthwhile to remark that by following the classical approach in the literature the remainder term for the single correlation is $\ll_\varepsilon N^\varepsilon R^2$, that trivially yields a remainder $\ll_\varepsilon N^\varepsilon R^2 H$ in the first generation formula above, whereas by our method we save H.

6. Further comments

The key of the present approach is that the correlation of a real sieve function $f = g_Q * \mathbf{1}$ can be written as

$$\mathcal{C}_f(a) = \sum_{q \le Q} g(q) \sum_{\substack{n \sim N \\ n \equiv a\,(q)}} f(n).$$

In the literature (see [IwaKow04], Ch.17), we find several studies of the distribution of an arithmetic function f (not necessarily a sieve function) over primitive residue classes. Most results are focused on non-trivial bounds for the *error term*

$$E_f(N;q,a) \stackrel{def}{=} \sum_{\substack{n \sim N \\ n \equiv a\,(q)}} f(n) - M_f(N;q,a)$$

for all $(a,q)=1$, provided q is not too large. Here, $M_f(N;q,a)$ is the expected *mean value* term. Let us recall two major variants of the problem. The first one concerns the *Bombieri-Vinogradov* type mean

$$\sum_{q \le Q} \max_{(a,q)=1} |E_f(N;q,a)|\,,$$

for which we refer the reader to [Mot76]. The second classical variant is the *Barban-Davenport-Halberstam* type quadratic mean

$$\sum_{q \le Q} \sum_{\substack{a \le q \\ (a,q)=1}} E_f(N;q,a)^2.$$

The latter has also a short interval version introduced by Hooley [Hoo99], that is

$$\sum_{q \le Q} \sum_{\substack{a \le \rho q \\ (a,q)=1}} E_f(N;q,a)^2, \quad \text{where } \rho \to 0.$$

In all such problems, the challenging issue is the *level* $\lambda \stackrel{def}{=} (\log Q)/(\log N)$ of distribution of f in arithmetic progressions (see [FriIwa10], §9.8 and §22.1). For example, the celebrated Bombieri-Vinogradov Theorem gives a non-trivial bound for

$$\sum_{q \le Q} \max_{(a,q)=1} \Big| \sum_{\substack{n \sim N \\ n \equiv a\,(q)}} \Lambda(n) - \frac{N}{\varphi(q)} \Big|, \quad \text{where } \varphi(q) \stackrel{def}{=} |\{a \le q, (a,q)=1\}|,$$

essentially with a level $\lambda = 1/2$ (which seems to be a structural barrier at least for the distribution of primes). However, for many applications one can just deal with individual reduced class a and take the sum over $q \le Q$, $(q,a)=1$. Indeed, by assuming that $a \ne 0$, one can see that it is possible to break the level $1/2$ for the *Bombieri-Friedlander-Iwaniec* type mean (see [FriIwa10], Theorem 22.1)

$$\sum_{\substack{q \le Q \\ (a,q)=1}} \Big| \sum_{\substack{n \sim N \\ n \equiv a\,(q)}} \Lambda(n) - \frac{N}{\varphi(q)} \Big|.$$

Consistently with the present notation, the above formula for the correlation of a sieve function becomes

$$\mathcal{C}_f(a) = \sum_{q \le Q} g(q) M_f(N;q,a) + \sum_{q \le Q} g(q) E_f(N;q,a),$$

where, by recalling that $g(q) \ll_\varepsilon q^\varepsilon$, one has

$$\sum_{q \le Q} g(q) E_f(N;q,a) \ll_\varepsilon Q^\varepsilon \sum_{q \le Q} |E_f(N;q,a)|\,.$$

Thus, here for each individual residue a we deal with a sum over $q \le Q$ without any further restriction. Then, it is not surprising that a straight asymptotic

$$\mathcal{C}_f(a) \sim \sum_{q \le Q} g(q) M_f(N; q, a)$$

has been proved for very few interesting instances of f, including the noteworthy case of the divisor function (see the third version of [CopLap14] on `arXiv` for a brief account on this matter). Better expectations for the *first generation* of correlation averages,

$$\sum_{a \le H} \mathcal{C}_f(a),$$

are given substance by Corollary 1.7 (and by the alternative approach of Lemma 12 in [CopLap14]). Furthermore, note that Theorem 1.9 concerns the average

$$\sum_{q \le Q} \Big| \sum_{a \le H} E_f(N; q, a) \Big|,$$

where, unlike the aforementioned means, the sums are taken over all the moduli $q \le Q$ and over a short interval of residue classes a, when f is a sieve function of range $Q \ll N^{1-\delta}$ and $H \ll N^{1-\delta}$. The bound for the weighted Selberg integral given in Corollary 1.7 and its application through Theorem 1.9 allow $Q \ll \sqrt{NH} N^{-\varepsilon}$, that is to say, the level might go beyond $1/2$ when we deal with *not too short* intervals, e.g., $H \gg N^{3\varepsilon}$.

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