Hardy-Ramanujan Journal 39 (2016), 63-64 submitted 28/11/2016, accepted 01/12/2016, revised 11/12/2016

Book Review

Lectures on the Riemann Zeta Function, by Henryk Iwaniec, University Lecture Series, Volume: 62, American Mathematical Society, Providence, RI, 2014, viii+119 pp., Softcover, ISBN 978-1-4704-1851.

Keywords. Riemann zeta-function, Levinson-Conrey method.
2010 Mathematics Subject Classification. 11M06 (primary); 11M41 (secondary).

These lecture notes arise from a course given by the author to graduate students at Rutgers University in the fall of 2012. The book is divided in two parts, having different aims and levels.

The first part introduces the basic theory of the Riemann zeta function, defined for $\Re s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

as well as some of its applications to the distribution of prime numbers. Actually, $\zeta(s)$ was first introduced for real values of s by Euler, who used it to give a new proof of the existence of infinitely many primes. In contrast with Euclid's purely arithmetical reasoning, Euler's proof combines analytic and arithmetical arguments, and may be regarded as the starting point of analytic number theory.

Part 1 begins with four chapters reviewing the approach to the study of the distribution of primes by elementary methods, where complex function theory is not used. In particular, Chapter 4 presents an interesting short elementary proof of the Prime Number Theorem, and discusses related questions. Chapters 5-11 contain the basic theory of the Riemann zeta function, starting with a brief description of Riemann's 1859 original paper and ending with the proof of the classical zero-free region and the Prime Number Theorem. Chapters 12-15, which conclude Part 1, enter the finer theory of $\zeta(s)$. The first two chapters present several forms of the approximate functional equation and some basic facts about Dirichlet polynomials. Such results are then applied, in the next two chapters, to the study of the horizontal distribution of the zeta zeros. Indeed, the famous Riemann Hypothesis (1859) asserts that all complex zeros of $\zeta(s)$ lie on the critical line $\Re s = 1/2$. This would completely solve the problem of the horizontal distribution, and many approximations to the Riemann Hypothesis have been obtained by several authors, starting with the famous theorems of Hardy and of Bohr & Landau published in 1914. Indeed, Hardy's theorem shows that there are infinitely many zeta zeros on the critical line, while the Bohr-Landau theorem asserts that almost all zeros lie in an arbitrarily small strip around such a line. In these directions, Chapter 14 contains an account of Carlson's density theorem, the first result giving sharp quantitative upper bounds for the number of zeros of $\zeta(s)$ off the critical line, and Chapter 15 presents a well known achievement of Hardy & Littlewood on the zeros on the critical line. Here, the approximate functional equation is used to show that there are $\gg T$ zeros on the critical line up to height T. We refer to Sankaranarayanan's survey, in this volume, of Ramachandra's contributions to the Riemann zeta function for several related problems and results.

Part 2 is devoted to the more recent work on the zeros of $\zeta(s)$ on the critical line, leading to sharp improvements of Hardy-Littlewood's theorem mentioned above. The first result showing that a positive proportion of the zeros lie on the critical line is a famous theorem due to Selberg in 1942, obtained by an ingenious refinement of the Hardy-Littlewood approach involving several important

We thank episciences.org for providing open access hosting of the electronic journal *Hardy-Ramanujan Journal*

innovations. Denoting by $N_0(T)$ the number of zeros $\rho = 1/2 + i\gamma$ with $0 < \gamma \leq T$, Selberg showed that

$$N_0(T) \geq cT \log T$$

for some $c > 0$ and T sufficiently large. Though the constant c is computable by Selberg's method, it will be very small. A lot of work has been done subsequently in order to get significantly better constants c, starting with a well known result by Levinson in 1974, proving that at least 1/3 of the zeros lie on the critical line. Levinson introduced a different approach to the problem, and further improvements and innovations have been obtained later, notably by Conrey who managed to show that at least 2/5 of the zeros are on the 1/2-line. Part 2 of Iwaniec's book presents the Levinson-Conrey method in a detailed and motivated form. Moreover, the results are often given in greater generality than actually needed, and several detours are also included, in order to offer a broader view on important ideas and techniques which may be useful for further research in this and other directions. The second part of the book is therefore more advanced, and contains original insights as well.

I wish to stress once more the richness of Iwaniec's book; the standard material is offered in a concise but instructive way, and from the more advanced part one can feel the deep insight of the author. This is therefore a highly welcome and a very useful addition to the literature on the Riemann zeta function and related topics.

Alberto Perelli
Dipartimento di Matematica
Via Dodecaneso 35
16146 Genova, ITALY.
e-mail: perelli@dima.unige.it