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Non-vanishing of Dirichlet series without Euler products

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Abstract. We give a new proof that the Riemann zeta function is nonzero in the half-plane $\{s \in \mathbb{C} : \sigma > 1\}$. A novel feature of this proof is that it makes no use of the Euler product for $\zeta(s)$.

Keywords. Riemann zeta function, Euler product, zeros.

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1. Introduction

Let $s = \sigma + it$ be a complex variable. In the half-plane

$$\mathscr{H} := \{s \in \mathbb{C} : \sigma > 1\}$$

the Riemann zeta function can be defined either as a Dirichlet series

$$\zeta(s) := \sum_{n \in \mathbb{N}} n^{-s}$$

or (equivalently) as an Euler product

$$\zeta(s) := \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

Since a convergent infinite product of nonzero factors is not zero, the zeta function does not vanish in $\mathscr{H}$. This can also be seen by applying the logarithm to the Euler product:

$$\log \zeta(s) = \sum_{p} \sum_{m \in \mathbb{N}} (mp^s)^{-1}.$$

Indeed, since the double sum on the right converges absolutely in $\mathscr{H}$, it follows that $\zeta(s) \neq 0$ for all $s \in \mathscr{H}$. Alternatively, since the Möbius function μ is bounded, it follows that the series

$$\zeta(s)^{-1} = \sum_{n=1}^{\infty} \mu(n) n^{-s}$$

converges absolutely when $\sigma > 1$, so $\zeta(s)$ cannot vanish. Of course, to prove that the Möbius function is bounded, one exploits the multiplicativity of μ, so this argument also relies (albeit implicitly) on the Euler product for $\zeta(s)$.

It is crucial to our understanding of the primes to extend the zero-free region for $\zeta(s)$ as far to the left of $\sigma = 1$ as possible.[1] According to Titchmarsh [Ti86, §3.1] this means extending the "sphere of influence" of the Euler product:

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[1] At present, the strongest result in this direction is due to Mossinghoff and Trudjian [MosTru15]; see also the earlier papers [Fo00, JaKw14, Kad05] and references therein.

> *The problem of the zero-free region appears to be a question of extending the sphere of influence of the Euler product beyond its actual region of convergence; for examples are known of functions which are extremely like the zeta-function in their representation by Dirichlet series, functional equation, and so on, but which have no Euler product, and for which the analogue of the Riemann hypothesis is false. In fact the deepest theorems on the distribution of the zeros of $\zeta(s)$ are obtained in the way suggested. But the problem of extending the sphere of influence of [the Euler product] to the left of $\sigma = 1$ in any effective way appears to be of extreme difficulty.*

But let's play the devil's advocate for a moment! Is it really the case that the non-vanishing of the Riemann zeta function in $\mathscr{H}$ (and in wider regions) *fundamentally relies on* the existence of an Euler product? Our aim in this paper is to provide some evidence to the contrary.

2. Statement of results

For a given arithmetical function F with $F(1) \neq 0$, let $\widetilde{F}$ denote the *Dirichlet inverse* of F; this can be defined via the Möbius relation

$$\sum_{ab=n} F(a)\widetilde{F}(b) = I(n) := \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{otherwise.} \end{cases}$$

To prove that a Dirichlet series $D(s) := \sum F(n)n^{-s}$ is nonzero in $\mathscr{H}$, it is enough to show that $D(s)^{-1} = \sum \widetilde{F}(n)n^{-s}$ converges in $\mathscr{H}$. Using partial summation, this is a consequence of any bound of the form

$$\sum_{n \le x} \widetilde{F}(n) \ll x^{1+o(1)} \qquad (x \to \infty). \tag{2.1}$$

Our proof of the next theorem establishes (2.1) whenever $\widetilde{F}$ is supported on a set of κ-free numbers.[2]

Theorem 2.1. *Let $D(s) := \sum_{n\in\mathbb{N}} F(n)n^{-s}$ be a Dirichlet series such that F is bounded, $F(1) \neq 0$, and the Dirichlet inverse $\widetilde{F}$ is supported on the set of κ-free numbers for some $\kappa \geq 2$. Then $D(s) \neq 0$ in $\mathscr{H}$.*

This theorem is proved in §5. It establishes the property of non-vanishing in $\mathscr{H}$ for a large class of Dirichlet series, almost all of which do not have an Euler product (but some do).

For a Dirichlet series $D(s)$ attached to a bounded completely multiplicative function F (for example, the Riemann zeta function), Theorem 2.1 provides a novel route to showing that $D(s)$ is nonzero in $\mathscr{H}$. For such F, one can easily show that $\widetilde{F}$ is supported on the set of *squarefree* numbers provided that one has the luxury of using the Euler product for $D(s)$. For this reason, it is important to note that our proof of the next theorem *makes no use of the Euler product for* $D(s)$. Instead, a combinatorial identity is employed to show that $\widetilde{F}$ has the required support.

Theorem 2.2. *Let F be an arithmetical function that is bounded and completely multiplicative. Then the Dirichlet inverse $\widetilde{F}$ is supported on the set of squarefree numbers.*

This theorem is proved in §6.

In particular, Theorems 2.1 and 2.2 together yield the following result without any use of the Euler product for $\zeta(s)$.

Corollary 2.3. *The Riemann zeta function does not vanish in $\mathscr{H}$.*

[2] For a given integer $\kappa \geq 2$, a natural number n is said to be *κ-free* if $p^k \nmid n$ for every prime p.

To further illustrate how our results can be applied, we introduce and study a special family of Dirichlet series $\mathscr{D} := \{D_z(s) : z \in \mathbb{C}\}$ with the following properties:

(i) The Riemann zeta function belongs to $\mathscr{D}$;

(ii) Every series $D_z(s)$ is meromorphic and *nonzero* in the region $\mathscr{H}$;

(iii) Only two series in $\mathscr{D}$ have an Euler product, namely the Riemann zeta function and the constant function $\mathbf{1}_{\mathbb{C}}(s) = 1$ for all $s \in \mathbb{C}$.

Viewing $\zeta(s)$ in relation to the other members of $\mathscr{D}$, the existence of an Euler product seems quite unusual, whereas non-vanishing in the half-plane $\mathscr{H}$ is a property enjoyed by every member of $\mathscr{D}$.

3. Preliminaries

Throughout the paper, we fix an integer parameter $\kappa \ge 2$ and denote by $\mathbb{N}_\kappa$ the set of κ-free numbers. We denote by $\mathbf{1}_{\mathbb{N}_\kappa}$ the indicator function of $\mathbb{N}_\kappa$:

$$\mathbf{1}_{\mathbb{N}_\kappa}(n) := \begin{cases} 1 & \text{if } n \in \mathbb{N}_\kappa; \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\omega(n)$ the number of distinct prime factors of n and by $\Omega(n)$ the number of prime factors of n, counted with multiplicity.

For any integer $k \ge 2$ we denote by $\log_k x$ the k-th iterate of the function $x \mapsto \max\{\log x, 1\}$. In particular, $\log_2 x = \log\log x$ and $\log_3 x = \log\log\log x$ when x is sufficiently large.

We use the equivalent notations $f(x) = O(g(x))$ and $f(x) \ll g(x)$ to mean that the inequality $|f(x)| \le c\,g(x)$ holds with some constant c. Throughout the paper, any implied constants in the symbols O and $\ll$ may depend (where obvious) on the parameters κ, ε but are absolute otherwise.

Two classical results of Hardy and Ramanujan [HR17, Lemmas B and C] assert the existence of constants $c_1, c_2 > 0$ such that the inequalities

$$\big|\{n \le x : \omega(n) = \ell\}\big| \le \frac{c_1 x}{\log x} \frac{(\log_2 x + c_2)^{\ell-1}}{(\ell-1)!} \tag{3.2}$$

and

$$\big|\{n \le x : \Omega(n) = \ell\}\big| \le \frac{c_1 x}{\log x} \sum_{j=0}^{\ell-1} \big(\tfrac{9}{10}\big)^{\ell-1-j} \frac{(\log_2 x + c_2)^j}{j!} \tag{3.3}$$

hold for all real $x \ge 2$. In the next lemma, we study the counting function

$$N_{\kappa,\ell}(x) := \big|\{n \le x : n \in \mathbb{N}_\kappa \text{ and } \Omega(n) = \ell\}\big|. \tag{3.4}$$

Although this function might seem closely related to that on the left side of (3.3), we prove that it satisfies a bound nearly as strong as (3.2).

Lemma 3.1. *There are absolute constants $C_1, C_2 > 0$ with the following property. For any integers $\kappa \ge 2$ and $\ell \ge 1$, the counting function defined by (3.4) satisfies the upper bound*

$$N_{\kappa,\ell}(x) \le \frac{C_1 x}{\log x} \frac{((\kappa-1)\log_2 x + (\kappa-1)C_2)^{\ell-1}}{(\ell-1)!} \qquad (x \ge 2). \tag{3.5}$$

Proof. Our proof is an adaptation of arguments from [HR17].

When $\ell \le \kappa$, the condition $\Omega(n) = \ell$ implies that $n \in \mathbb{N}_\kappa$. Using (3.3) it follows that

$$N_{\kappa,\ell}(x) = \big|\{n \le x : \Omega(n) = \ell\}\big| \le \frac{ec_1 x(\log_2 x + c_2)^{\ell-1}}{\log x} \qquad (x \ge 2),$$

hence (3.5) holds for any choice of $C_1 \ge ec_1$ and $C_2 \ge c_2$.

From now on, we assume that $\ell > \kappa$. To simplify the notation slightly, we put $\kappa_1 := \kappa - 1$.

Let $p(1) := 2 < p(2) := 3 < p(3) := 5 < \cdots$ be the sequence of all primes, and put

$$\tilde{p}(j) := p(\lceil j/\kappa_1 \rceil) \qquad (j \in \mathbb{N}),$$

where for any $t > 0$, $\lceil t \rceil$ is the least integer that is $\ge t$. In other words, $\big(\tilde{p}(j)\big)_{j\in\mathbb{N}}$ is the sequence

$$\underbrace{2,\ldots,2}_{\kappa_1 \text{ copies}},\underbrace{3,\ldots,3}_{\kappa_1 \text{ copies}},\underbrace{5,\ldots,5}_{\kappa_1 \text{ copies}},\ldots$$

in which the primes appear in increasing order, each being repeated κ_1 times.

Let $n \in \mathbb{N}_\kappa$, $n \ge 2$, and suppose that $\Omega(n) = \ell$. Among all of the ordered ℓ-tuples $(j_1, \ldots, j_\ell)$ having $j_1 < \cdots < j_\ell$ and for which

$$n = \tilde{p}(j_1) \cdots \tilde{p}(j_\ell), \tag{3.6}$$

let $\Psi(n)$ be the unique ℓ-tuple $(j_1, \ldots, j_\ell)$ that minimizes the sum $j_1 + \cdots + j_\ell$. For any such n we also put

$$J(n) := j_\ell,$$

and we set $J(1) := 0$. For example, if $\kappa = 5$, then $4400 \in \mathbb{N}_\kappa$ and we have

$$\Psi(4400) = (1,2,3,4,9,10,17) \qquad \text{and} \qquad J(4400) = 17.$$

Let $\mathcal{S}$ be the set of ordered pairs (j, m) such that $j \le J(m)$, $m \in \mathbb{N}_\kappa$, and $\tilde{p}(j)m \le x$. The condition $j \le J(m)$ implies that the prime $\tilde{p}(j)$ does not exceed the largest prime factor of m, and thus $\tilde{p}(j) \le m \le x/\tilde{p}(j)$; consequently,

$$|\mathcal{S}| \le \sum_{j\,:\,\tilde{p}(j)^2 \le x} N_{\kappa,\ell-1}(x/\tilde{p}(j)). \tag{3.7}$$

On the other hand, suppose that $n \le x$, $n \in \mathbb{N}_\kappa$ and $\Omega(n) = \ell$. Factoring n as in (3.6) with $(j_1, \ldots, j_\ell) = \Psi(n)$, one verifies that the pair $(j_i, n/\tilde{p}(j_i))$ lies in $\mathcal{S}$ for each $i = 1, \ldots, \ell - 1$ Hence, n can be expressed in $\ell - 1$ different ways as the product of the entries of an ordered pair in $\mathcal{S}$, which implies that

$$(\ell - 1)N_{\kappa,\ell}(x) \le |\mathcal{S}|. \tag{3.8}$$

Combining (3.7) and (3.8), and using induction, we have

$$\begin{aligned}
N_{\kappa,\ell}(x) &\le \frac{1}{\ell-1} \sum_{j\,:\,\tilde{p}(j)^2 \le x} N_{\kappa,\ell-1}(x/\tilde{p}(j)) \\
&\le \frac{1}{\ell-1} \sum_{j\,:\,\tilde{p}(j)^2 \le x} \frac{C_1(x/\tilde{p}(j))}{\log(x/\tilde{p}(j))} \frac{(\kappa_1 \log_2(x/\tilde{p}(j)) + \kappa_1 C_2)^{\ell-2}}{(\ell-2)!} \\
&\le \frac{C_1 x(\kappa_1 \log_2 x + \kappa_1 C_2)^{\ell-2}}{(\ell-1)!} \sum_{j\,:\,\tilde{p}(j)^2 \le x} \frac{1}{\tilde{p}(j)\log(x/\tilde{p}(j))}.
\end{aligned}$$

As each prime is repeated precisely κ_1 times in $\big(\tilde{p}(j)\big)_{j\in\mathbb{N}}$ we have

$$\sum_{j\,:\,\tilde{p}(j)^2\le x}\frac{1}{\tilde{p}(j)\log(x/\tilde{p}(j))}=\kappa_1\sum_{p\,:\,p^2\le x}\frac{1}{p\log(x/p)}.$$

The proof is completed using the fact that

$$\sum_{p\,:\,p^2<x}\frac{1}{p\log(x/p)}<\frac{\log_2 x+C_2}{\log x}\qquad(x\ge 2)$$

holds for a sufficiently large choice of C_2; see the proof of [HR17, Lemma C].

4. Factorisatio numerorum

For any $n\ge 2$, let $f(n)$ denote the number of representations of n as a product of integers exceeding one, where two representations are considered equal only if they contain the same factors in the same order. For technical reasons, we also set $f(1):=1$.

One can define $f(n)$ as follows. For every positive integer k, let $f_k(n)$ denote the number of ordered k-tuples $(n_1,\dots,n_k)$ such that each $n_j\ge 2$ and the product $n_1\cdots n_k$ equals n. Then

$$f(n):=I(n)+\sum_{k\ge1}f_k(n)\qquad(n\in\mathbb{N}),\tag{4.9}$$

where

$$I(n):=\begin{cases}1&\text{if } n=1;\\0&\text{otherwise.}\end{cases}$$

Note that the sum in (4.9) is finite since $f_k(n)=0$ when $k>\Omega(n)$.

In one of the earliest papers about the function $f(n)$, Kalmár [Kal30] establishes the asymptotic formula

$$\sum_{n\le x}f(n)\sim-\frac{x^\beta}{\beta\zeta'(\beta)}\qquad(x\to\infty),\tag{4.10}$$

where $\beta=1.728647\cdots$ is the unique positive root of $\zeta(\beta)=2$. In particular, this implies that

$$f(n)\ll n^\beta\qquad(n\in\mathbb{N}).\tag{4.11}$$

The bound (4.11) is essentially optimal since Hille [Hi36] has shown that for every $\varepsilon>0$ the lower bound $f(n)\gg n^{\beta-\varepsilon}$ holds for infinitely many n; see also Erdős [Er41].

The next proposition is fundamental in the sequel as it leads to a significant strengthening of (4.11) for κ-free numbers n.

Proposition 4.1. *For any integer $n\ge 2$ we have*

$$f(n)\le\exp(\ell\log\ell+O(\ell\log_2\ell\log_3\ell))\qquad\textit{with}\quad \ell:=\Omega(n).$$

Proof. Let $\mathcal{T}(n)$ be the set of ordered tuples $(n_1,\dots,n_r)$ of any length r such that every $n_j\ge 2$ and $n_1\cdots n_r=n$. Thus, $|\mathcal{T}(n)|=f(n)$.

Let $\mathcal{P}(\ell)$ be the set of ordered partitions λ of ℓ; these are ordered tuples $\lambda=(\lambda_1,\dots,\lambda_r)$ of any length r such that $1\le\lambda_1\le\cdots\le\lambda_r$ and $\lambda_1+\cdots+\lambda_r=\ell$.

We begin by constructing a map $\Phi:\mathcal{T}(n)\to\mathcal{P}(\ell)$ as follows. For any given $\eta=(n_1,\dots,n_r)$ in $\mathcal{T}(n)$, let $\Phi_\Omega(\eta)$ denote the tuple $(\Omega(n_1),\dots,\Omega(n_r))$, and set

$$\mathcal{U}(n):=\{\Phi_\Omega(\eta):\eta\in\mathcal{T}(n)\};$$

thus, $\Phi_\Omega : \mathcal{T}(n) \to \mathcal{U}(n)$. Next, for any $w = (w_1, \dots, w_r)$ in $\mathcal{U}(n)$, let $\Phi_\sigma(w)$ be the tuple $(\lambda_1, \dots, \lambda_r)$ that is obtained by rearranging the entries of w into nondecreasing order; then $\Phi_\sigma : \mathcal{U}(n) \to \mathcal{P}(\ell)$. The map $\Phi : \mathcal{T}(n) \to \mathcal{P}(\ell)$ is defined to be the composition $\Phi_\sigma \circ \Phi_\Omega$.

Next, for any $\lambda \in \mathcal{P}(\ell)$ let $d_\lambda(n)$ be the cardinality of the set $\Phi^{-1}(\{\lambda\})$ of preimages of λ in $\mathcal{T}(n)$.[3] Since any product counted by $f(n)$ gives rise to a unique partition λ via the map Φ, we have

$$f(n) := I(n) + \sum_{\lambda \in \mathcal{P}(\ell)} d_\lambda(n).$$

In view of the celebrated estimate of Hardy and Ramanujan [HR18]

$$|\mathcal{P}(\ell)| \sim (4\ell\sqrt{3})^{-1} \exp\left(\pi\sqrt{2\ell/3}\,\right) \qquad (\ell \to \infty),$$

to prove the proposition it suffices to show that the individual bound

$$d_\lambda(n) \le \exp(\ell \log \ell + O(\ell \log_2 \ell \log_3 \ell)) \tag{4.12}$$

holds for every $\lambda \in \mathcal{P}(\ell)$.

To this end, let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a fixed element of $\mathcal{P}(\ell)$. For any natural number k, let m_k be the multiplicity with which k occurs in the partition λ, i.e.,

$$m_k := \big|\{j : \lambda_j = k\}\big| \qquad (k \in \mathbb{N}).$$

Note that $\ell = \sum_k k m_k$. Setting $m := \sum_k m_k$, a simple combinatorial argument shows that

$$d_\lambda(n) \le \frac{\ell!}{\prod_k (k!)^{m_k}} \cdot \frac{m!}{\prod_k m_k!} \tag{4.13}$$

(roughly speaking, the second factor is the cardinality of the set $\Phi_\sigma^{-1}(\{\lambda\})$ of preimages of λ in $\mathcal{U}(n)$, whereas for any such preimage w the first factor bounds the cardinality of the set $\Phi_\Omega^{-1}(\{w\})$ of preimages of w in $\mathcal{T}(n)$). We remark that (4.13) holds with equality whenever n is squarefree. Since $\ell! \le \ell^\ell$, to establish (4.12) it is enough to show that

$$\log\left(\frac{m!}{\prod_k (k!)^{m_k} \prod_k m_k!}\right) \ll \ell \log_2 \ell \log_3 \ell. \tag{4.14}$$

Since $m \le \ell$ and $\log j! = j \log j + O(j)$ for all positive integers j, the left side of (4.14) is

$$\begin{aligned} &\le m \log \ell - \sum_k m_k(k \log k + O(k)) - \sum_k (m_k \log m_k + O(m_k)) \\ &= \sum_{k\,:\,m_k \ne 0} m_k \log\left(\frac{\ell}{k^k m_k}\right) + O(\ell) = S_1 + S_2 + O(\ell), \quad \text{(say)} \end{aligned}$$

where

$$S_1 := \sum_{\substack{k\,:\,m_k \ne 0 \\ m_k > \ell/g(\ell)}} m_k \log\left(\frac{\ell}{k^k m_k}\right) \qquad \text{and} \qquad S_2 := \sum_{\substack{k\,:\,m_k \ne 0 \\ m_k \le \ell/g(\ell)}} m_k \log\left(\frac{\ell}{k^k m_k}\right)$$

and

$$g(\ell) := \frac{(\log \ell)^2}{(\log_2 \ell)^2 \log_3 \ell}.$$

[3] A more descriptive but less precise definition is the following. If $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}(\ell)$, then $d_\lambda(n)$ is the number of r-tuples $(n_1, \dots, n_r)$ for which the product $n_1 \cdots n_r$ equals n and such that, after a suitable permutation of the indices, one has $\Omega(n_j) = \lambda_j$ for each j (that is, the multisets $\{\Omega(n_j)\}$ and $\{\lambda_j\}$ are the same).

For each k in the sum S_1, we have $m_k > \ell/g(\ell)$ and $km_k \le \sum_{j\le\ell} jm_j = \ell$; therefore,

$$S_1 \le \sum_k \frac{\ell}{k}\log\Big(\frac{g(\ell)}{k^k}\Big) \le \ell\log g(\ell) \sum_{k\,:\,k^k\le g(\ell)} \frac{1}{k}$$
$$\ll \ell\log g(\ell)\log_2 g(\ell) \ll \ell\log_2\ell\log_3\ell.$$

For each k in the sum S_2, we have $1 \le m_k \le \ell/g(\ell)$; thus,

$$S_2 \le \frac{\ell}{g(\ell)}\sum_k \log\Big(\frac{\ell}{k^k}\Big) \le \frac{\ell\log\ell}{g(\ell)} \sum_{k\,:\,k^k\le\ell} 1 \ll \frac{\ell(\log\ell)^2}{g(\ell)\log_2\ell} \ll \ell\log_2\ell\log_3\ell.$$

Combining the above bounds on S_1 and S_2, we derive (4.14), and in turn (4.12), finishing the proof.

The following corollary is crucial in the next section.

Corollary 4.2. *For any constant $C > 0$ we have*

$$\Big|\sum_{n\le x} C^{\Omega(n)} f(n)\mathbf{1}_{\mathbb{N}_\kappa}(n)\Big| \le x^{1+o(1)} \qquad (x\to\infty), \tag{4.15}$$

where the function implied by $o(1)$ depends only on C and κ.

Proof. Let Q denote the quantity on the left side of (4.15).

For any $n \in \mathbb{N}_\kappa$ we have $\Omega(n) \le \kappa\,\omega(n)$. Also, $\omega(n) \le 2(\log x)/\log_2 x$ for all $n \le x$ once x is sufficiently large. Hence, defining $B_\kappa(x) := 2\kappa(\log x)/\log_2 x$ it follows from Proposition 4.1 that

$$Q \le \sum_{\ell\le B_\kappa(x)} C^\ell \exp(\ell\log\ell + O(\ell\log_2\ell\log_3\ell))\cdot N_{\kappa,\ell}(x),$$

where $N_{\kappa,\ell}(x)$ is the counting function given by (3.4). By Lemma 3.1 we have

$$Q \le \sum_{\ell\le B_\kappa(x)} C^\ell \exp(\ell\log\ell + O(\ell\log_2\ell\log_3\ell))\cdot \frac{C_1 x}{\log x}\,\frac{((\kappa-1)\log_2 x + (\kappa-1)C_2)^{\ell-1}}{(\ell-1)!}$$
$$\le x^{1+o(1)} \sum_{\ell\le B_\kappa(x)} \exp(\ell\log\ell)\cdot \frac{((\kappa-1)\log_2 x + (\kappa-1)C_2)^{\ell-1}}{(\ell-1)!} \qquad (x\to\infty).$$

Using the estimates

$$(\ell-1)! = \exp(\ell\log\ell + O(\ell)) = x^{o(1)}\exp(\ell\log\ell)$$

and

$$((\kappa-1)\log_2 x + (\kappa-1)C_2)^{\ell-1} = \exp(O(\ell\log_3 x)) = x^{o(1)},$$

which hold uniformly for all $\ell \le B_\kappa(x)$, the result follows.

5. Reciprocal of a Dirichlet series

Theorem 5.1. *Suppose that F is bounded on $\mathbb{N}$, and $F(1) \ne 0$. Then $\sum_{n\in\mathbb{N}} \widetilde{F}(n)\mathbf{1}_{\mathbb{N}_\kappa}(n)n^{-s}$ converges absolutely in $\mathscr{H}$.*

Proof. Without loss of generality, we can assume that $F(1) = \widetilde{F}(1) = 1$. Let $C \ge 1$ be a number such that

$$|F(n)| \le C \qquad (n \in \mathbb{N}). \tag{5.16}$$

For every positive integer k, let $\mathcal{T}_k(n)$ be the set of ordered k-tuples $(n_1, \dots, n_k)$ such that every $n_j \ge 2$ and $n_1 \cdots n_k = n$. Then $|\mathcal{T}_k(n)| = f_k(n)$ in the notation of §4. We denote

$$f_k(F;n) := \sum_{(n_1,\dots,n_k)\in\mathcal{T}_k(n)} F(n_1)\cdots F(n_k). \tag{5.17}$$

Using (5.16) we derive that

$$|f_k(F;n)| \le C^k f_k(n). \tag{5.18}$$

Since $f_k(F;n) = 0$ for all $k > \Omega(n)$, and the inequality $\Omega(n) \le (\log n)/\log 2$ holds for all n, it follows that

$$|f_k(F;n)| \le n^B f_k(n) \qquad \text{with} \quad B := \max\{0, (\log C)/\log 2\}.$$

Summing over k and using (4.11), we see that

$$\sum_{k\ge1} |f_k(F;n)| \ll n^{B+\beta} \qquad (n \in \mathbb{N}). \tag{5.19}$$

Next, put

$$D(s) := \sum_{n=1}^{\infty} F(n)n^{-s} = 1 + Z(F;s) \qquad \text{with} \quad Z(F;s) := \sum_{n\ge2} F(n)n^{-s}.$$

For every positive integer k we have

$$Z(F;s)^k = \sum_{n\ge2} f_k(F;n)n^{-s}.$$

In view of (5.19) the identity

$$\frac{1}{1+Z(F;s)} = 1 + \sum_{k\ge1}(-1)^k Z(F;s)^k = 1 + \sum_{n\ge2}\sum_{k\ge1}(-1)^k f_k(F;n)n^{-s} \tag{5.20}$$

holds throughout the half-plane $\{\sigma > B+\beta\}$ since all sums converge absolutely in that region. Noting that the left side of (5.20) is $D(s)^{-1} = \sum_{n\in\mathbb{N}} \widetilde{F}(n)n^{-s}$, we conclude that

$$\widetilde{F}(n) = I(n) + \sum_{k\ge1}(-1)^k f_k(F;n) \qquad (n \in \mathbb{N}). \tag{5.21}$$

To prove the theorem, we need to show that the Dirichlet series

$$\sum_{n\in\mathbb{N}} \widetilde{F}(n)\mathbf{1}_{\mathbb{N}_\kappa}(n)n^{-s} = 1 + \sum_{n\ge2}\sum_{k\ge1}(-1)^k f_k(F;n)\mathbf{1}_{\mathbb{N}_\kappa}(n)n^{-s}$$

converges absolutely in $\mathscr{H}$. For any natural number n it is clear that $f_k(F;n) = 0$ whenever $k > \Omega(n)$, hence using (5.18) we see that

$$\left|\sum_{k\ge1}(-1)^k f_k(F;n)\mathbf{1}_{\mathbb{N}_\kappa}(n)\right| \le I(n) + \sum_{k\le\Omega(n)} C^k f_k(n)\mathbf{1}_{\mathbb{N}_\kappa}(n) \le C^{\Omega(n)} f(n)\mathbf{1}_{\mathbb{N}_\kappa}(n).$$

Consequently, it suffices to show that the sum

$$\sum_{n\in\mathbb{N}} C^{\Omega(n)} f(n)\mathbf{1}_{\mathbb{N}_\kappa}(n) n^{-\sigma} \tag{5.22}$$

converges for $\sigma > 1$. However, since the summatory function

$$S(x) := \sum_{n\le x} C^{\Omega(n)} f(n)\mathbf{1}_{\mathbb{N}_\kappa}(n)$$

satisfies the bound $S(x) \ll x^{1+\varepsilon}$ for every $\varepsilon > 0$ by Corollary 4.2, the convergence of (5.22) for $\sigma > 1$ follows by partial summation.

Proof of Theorem 2.1. Since the Dirichlet inverse $\widetilde{F}$ has its support in $\mathbb{N}_\kappa$ for some $\kappa \ge 2$, we have $\widetilde{F}(n) = \widetilde{F}(n)\mathbf{1}_{\mathbb{N}_\kappa}(n)$ for all n. By Theorem 5.1,

$$\sum_{n\in\mathbb{N}} \widetilde{F}(n)\mathbf{1}_{\mathbb{N}_\kappa}(n) n^{-s} = \sum_{n\in\mathbb{N}} \widetilde{F}(n) n^{-s} = D(s)^{-1}$$

converges absolutely in $\mathscr{H}$, and the result follows.

6. The family $\mathscr{D}$

For any fixed $z \in \mathbb{C}$, let F_z be the arithmetical function defined by

$$F_z(n) := \begin{cases} 1 & \text{if } n = 1; \\ -z & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n\in\mathbb{N}} F_z(n) n^{-s} = 1 - z(\zeta(s) - 1) \qquad (s \in \mathscr{H}).$$

Taking $F := F_z$ in (5.19) and (5.21) we have $\widetilde{F}_z(n) \ll n^{B+\beta}$, where $B := \max\{0, (\log|z|)/\log 2\}$ and $\beta = 1.728647\cdots$ as before. This implies that the formal identity

$$\sum_{n\in\mathbb{N}} \widetilde{F}_z(n) n^{-s} = \frac{1}{1 - z(\zeta(s) - 1)} \tag{6.23}$$

holds rigorously when $\sigma > B + \beta + 1$. Moreover, it is clear that the Dirichlet series can be analytically continued to the region $\{\sigma > \beta_z\}$, where β_z is the unique positive root of $\zeta(\beta_z) = 1 + |z|^{-1}$ if $z \ne 0$, and $\beta_0 := -\infty$. It is worth mentioning that for any fixed $z \ne 0$ or -1, the function on the right side of (6.23) has infinitely many poles in $\mathscr{H}$ since the equation $\zeta(s) = 1 + z^{-1}$ has infinitely many solutions in any strip $\{1 < \sigma < 1 + \varepsilon\}$; see, e.g., Titchmarsh [Ti86, Theorem 11.6 (C)].

Next, we introduce two Dirichlet series given by

$$D_z^\dagger(s) := \sum_{n\in\mathbb{N}} \widetilde{F}_z(n)\mathbf{1}_{\mathbb{N}_2}(n) n^{-s}$$

and

$$D_z(s) := D_z^\dagger(s)^{-1} = \sum_{n\in\mathbb{N}} G_z(n) n^{-s},$$

where G_z is the Dirichlet inverse of $\widetilde{F}_z \cdot \mathbf{1}_{\mathbb{N}_2}$. According to Theorem 5.1, $D_z^\dagger(s)$ converges absolutely in $\mathscr{H}$, hence it is analytic in that region. This implies that $D_z(s)$ has a meromorphic extension to $\mathscr{H}$, and $D_z(s) \ne 0$ in $\mathscr{H}$; thus, we have verified property (ii) of §1.

From the above definitions, one sees that $F_{-1}(n) = \mathbf{1}_{\mathbb{N}}(n)$, and thus $D_{-1}(s) = \zeta(s)$. This establishes property (i) of §1. We also have $F_0(n) = I(n)$, so that $D_0(s) = \mathbf{1}_{\mathbb{C}}(s)$.

Finally, we establish property (iii) of §1. Observe that the Möbius relations

$$\sum_{ab=n} \widetilde{F}_z(a)F_z(b) = I(n) \qquad \text{and} \qquad \sum_{ab=n} \widetilde{F}_z(a)\mathbf{1}_{\mathbb{N}_2}(a)^2 G_z(b) = I(n)$$

immediately imply that $G_z(p) = G_z(q) = G_z(pq) = -z$ for any two different primes p and q.[4] If $D_z(s)$ has an Euler product, then G_z is multiplicative, and therefore

$$(-z)^2 = G_z(p)G_z(q) = G_z(pq) = -z,$$

which is only possible for $z = 0$ or -1.

Lemma 6.1. *Let z be a complex number, n a natural number, and p a prime number not dividing n. For any integer $\alpha \geq 1$ we have*

$$\widetilde{F}_z(p^\alpha n) = (z+1)^{\alpha-1} \sum_{\ell \geq 1} z^\ell \big(z + \ell\alpha^{-1}(z+1)\big) \binom{\alpha+\ell-1}{\ell} f_\ell(n).$$

Proof. Using (5.17), (5.21) and the definition of F_z, it follows that

$$\widetilde{F}_z(p^\alpha n) = \sum_{k \geq 1} z^k f_k(p^\alpha n). \tag{6.24}$$

The quantity $f_k(p^\alpha n)$ is the number of ordered k-tuples $(m_1, \dots, m_k)$ such that $m_j = p^{\alpha_j} n_j$ for each j, with $\alpha_j \geq 1$ or $n_j \geq 2$, $\alpha_1 + \cdots + \alpha_k = \alpha$, and $n_1 \cdots n_k = n$. To construct such a k-tuple, first choose an integer ℓ in the range $1 \leq \ell \leq k$ and an ordered ℓ-tuple $(\hat{n}_1, \dots, \hat{n}_\ell)$ with each $\hat{n}_j \geq 2$ and $\hat{n}_1 \cdots \hat{n}_\ell = n$; for any choice of ℓ there are precisely $f_\ell(n)$ such ℓ-tuples. Next, maintaining the ordering of the integers $\hat{n}_j$ in $(\hat{n}_1, \dots, \hat{n}_\ell)$, we construct $(n_1, \dots, n_k)$ by inserting $k - \ell$ extra entries, each equal to one (thus, every n_i in the resulting k-tuple is one of the numbers $\hat{n}_j$, or else $n_i = 1$); there are $\binom{k}{k-\ell} = \binom{k}{\ell}$ ways to insert these extra ones to form $(n_1, \dots, n_k)$. To guarantee that every $m_j = p^{\alpha_j} n_j \geq 2$ in the final k-tuple $(m_1, \dots, m_k)$, so it is counted in the computation of $f_k(p^\alpha n)$, we have to replace each entry $n_i = 1$ in $(n_1, \dots, n_k)$ with a copy of the prime p. As there are only α copies of p available, it must be the case that $\alpha \geq k - \ell$ else this choice of ℓ is unacceptable. The remaining $\alpha - k + \ell$ copies of the prime p can be distributed arbitrarily. As the number of ways that one can distribute $\alpha - k + \ell$ objects into k boxes is $\binom{\alpha+\ell-1}{k-1}$, putting everything together we have

$$f_k(p^\alpha n) = \sum_{\ell=\max\{1,k-\alpha\}}^{k} \binom{k}{\ell} \binom{\alpha+\ell-1}{k-1} f_\ell(n).$$

Combining this result with (6.24), we derive that

$$\widetilde{F}_z(p^\alpha n) = \sum_{\ell \geq 1} f_\ell(n) \sum_{k=\ell}^{\alpha+\ell} z^k \binom{k}{\ell} \binom{\alpha+\ell-1}{k-1}$$

Making the change of variables $k \mapsto k + \ell$ in the inner summation, it follows that

$$\widetilde{F}_z(p^\alpha n) = \sum_{\ell \geq 1} z^\ell f_\ell(n) B(z, \alpha, \ell),$$

[4] A more elaborate argument shows that $F_z(n) = G_z(n) = -z$ for all squarefree numbers n.

where

$$B(z,\alpha,\ell) := \sum_{k=0}^{\alpha} z^k \binom{k+\ell}{\ell}\binom{\alpha+\ell-1}{k+\ell-1}.$$

In view of the combinatorial identity

$$\binom{k+\ell}{\ell}\binom{\alpha+\ell-1}{k+\ell-1} = \binom{\alpha+\ell-1}{\ell}\left(\binom{\alpha-1}{k-1} + \frac{\ell}{\alpha}\binom{\alpha}{k}\right)$$

(where $\binom{\alpha-1}{k-1} = 0$ when $k=0$), it follows that

$$\begin{aligned} B(z,\alpha,\ell) &= \binom{\alpha+\ell-1}{\ell}\sum_{k=0}^{\alpha} z^k\left(\binom{\alpha-1}{k-1} + \frac{\ell}{\alpha}\binom{\alpha}{k}\right) \\ &= \binom{\alpha+\ell-1}{\ell}\big(z(z+1)^{\alpha-1} + \ell\alpha^{-1}(z+1)^{\alpha}\big), \end{aligned}$$

and we obtain the stated result.

Proof of Theorem 2.2. Since F is completely multiplicative, from (5.17) it follows that

$$f_k(F;n) = F(n) f_k(n) \qquad (k,n\in\mathbb{N}).$$

With two applications of (5.21) we deduce that

$$\widetilde{F}(n) = I(n) + F(n)\sum_{k\ge 1}(-1)^k f_k(n) = F(n)\widetilde{F}_{-1}(n) \qquad (n\in\mathbb{N}). \tag{6.25}$$

Applying Lemma 6.1 with $z=-1$, we see that $\widetilde{F}_{-1}(p^{\alpha}n) = 0$ for any $n\in\mathbb{N}$, any prime p not dividing n, and all $\alpha\ge 2$. This implies that $\widetilde{F}_{-1}$ is supported on the set of squarefree numbers,[5] and (6.25) shows that the same is true of $\widetilde{F}$.

7. Remarks

Theorems 2.1 and 5.1 can be extended to cover all functions satisfying the polynomial growth condition $F(n) \ll n^A$ provided that one is willing to replace $\mathscr{H}$ with the half-plane $\{s\in\mathbb{C} : \sigma > A+1\}$ in those theorems. It would be interesting to see whether the ideas of this paper can be developed to produce zero-free regions for $\zeta(s)$ and other Dirichlet series inside the critical strip.

Sarnak [Sa11] has recently considered a general pseudo-randomness principle related to a famous conjecture of Chowla [Ch65]. Roughly speaking, the principle asserts that the Möbius function $\mu(n)$ does not correlate with any function $\xi(n)$ of low complexity. In other words,

$$\sum_{n\le x}\mu(n)\xi(n) = o\bigg(\sum_{n\le x}|\xi(n)|\bigg) \qquad (x\to\infty). \tag{7.26}$$

Combining Kalmár's result (4.10) with Corollary 4.2, we see that (7.26) is verified for the function $\xi(n) := f(n)$. However, this is not due to the randomness of $\mu(n)$ but instead to the fact $f(n)$ takes smaller values on squarefree numbers than it does on natural numbers in general. It would be interesting see whether (7.26) holds for $\xi(n) := f(n)\mathbf{1}_{\mathbb{N}_2}(n)^2$.

[5] As we have already seen, $\widetilde{F}_{-1}$ is the Möbius function μ.

Let $f_{even}(n)$ [resp. $f_{odd}(n)$] denote the number of representations of n as a product of an *even* [resp. *odd*] number of integers exceeding one, where two representations are considered equal only if they contain the same factors in the same order. In other words,

$$f_{even}(n) := I(n) + \sum_{\substack{k\geq 1\\ k \text{ even}}} f_k(n) \qquad \text{and} \qquad f_{odd}(n) := \sum_{\substack{k\geq 1\\ k \text{ odd}}} f_k(n).$$

Clearly, $f(n) = f_{even}(n) + f_{odd}(n)$, but it is somewhat less obvious that

$$\mu(n) = f_{even}(n) - f_{odd}(n) \qquad (n \in \mathbb{N}). \tag{7.27}$$

Indeed, taking $F := F_{-1} = \mathbf{1}_{\mathbb{N}}$ we have $f_k(F; n) = f_k(n)$ for all n by (5.17), and then (7.27) follows immediately from (5.21). The identity (7.27) is originally due to Linnik [Li63]; see also Friedlander and Iwaniec [FrIw10, Chapter 17].

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