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# On an identity of Ramanujan

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*To the memory of S. Srinivasan*

**Abstract.** Proofs published so far in articles and books, of the Ramanujan identity presented in this note, which depend on Euler products, are essentially the same as Ramanujan's original proof. In contrast, the proof given here is short and independent of the use of Euler products.

**Keywords.** Ramanujan's identity, Möbius inversion, Selberg's $\Lambda^2$-sieve.

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This article is essentially a reproduction of the page 4 of [Mo09], for the sake of a wider audience. The identity of Ramanujan that we discuss here is

$$\sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)\sigma_\beta(n)}{n^s} = \frac{\zeta(s)\zeta(s-\alpha)\zeta(s-\beta)\zeta(s-\alpha-\beta)}{\zeta(2s-\alpha-\beta)}, \tag{0.1}$$

where $\Re(s) > 1 + \max\{0, \Re\alpha, \Re\beta, \Re(\alpha+\beta)\}$; notations are as usual.

This famous formula is stated in [Ra1916] by Ramanujan; and his proof is in [Ra62, p. 135]. My aim is to give a new proof. It is short and, unlike Ramanujan's proof, avoids having to consider Euler products.

*Proof.* Note first that

$$\sigma_\alpha(n)\sigma_\beta(n) = \sum_{u|n, v|n} u^\alpha v^\beta = \sum_{[u,v]|n} u^\alpha v^\beta, \tag{0.2}$$

where $[u, v]$ is the least common multiple of $u, v$. Thus the left side of (0.1) equals

$$\zeta(s) \sum_{u,v} \frac{u^\alpha v^\beta}{[u,v]^s} = \zeta(s) \sum_{u,v} \frac{\langle u,v\rangle^s}{u^{s-\alpha} v^{s-\beta}}, \tag{0.3}$$

where $\langle u, v\rangle$ is the greatest common divisor of $u, v$. The Möbius inversion gives $\sum_{f|g} \eta_s(f) = g^s$ with $\eta_s(m) = \sum_{d|m} \mu(d)(m/d)^s$. Hence (0.3) equals

$$\zeta(s) \sum_{u,v} \frac{1}{u^{s-\alpha} v^{s-\beta}} \sum_{d|u, d|v} \eta_s(d) = \zeta(s)\zeta(s-\alpha)\zeta(s-\beta) \sum_d \frac{\eta_s(d)}{d^{2s-\alpha-\beta}}. \tag{0.4}$$

The last sum is obviously equal to $\zeta(s-\alpha-\beta)/\zeta(2s-\alpha-\beta)$. This completes the proof. □

**Remarks**: Following kind recommendations of the reviewer I shall add some points closely related to the present subject.

(I) *Methods*: Treatments of (0.1) published so far in articles and books, save for the present one, are essentially the same as Ramanujan's original proof; thus they depend on Euler products. In contrast, the above proof is independent of Euler products; note that the application of the Möbius inversion immediately after (0.3) plays partly an alternative rôle. Any extension of the present method to general situations such as those dealt with in the recent article [SaRam16] which depends on Euler products appears to be an interesting arithmetical issue, especially in the automorphic context.

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(II) *Applications*: Ramanujan's identity extends readily to Dirichlet $L$-functions, which gives rise to a strikingly simple proof of Dirichlet's prime number theorem, a well known historical fact. This approach is pursued in [BaRa76] so that zero-free regions for Dirichlet $L$-functions are obtained without employing the Hadamard theory of entire functions. In [Mo83, Part II] their argument is combined with Selberg's $\Lambda^2$-sieve, which has yielded, among other things, an *elementary* proof of Vinogradov's zero-free region for the zeta-function and a drastically simplified (in fact, a new) proof of Linnik's famous zero-density theorem and Deuring–Heilbronn phenomenon for Dirichlet $L$-functions. Also, extensions of (0.1) that involve $(\sigma_\alpha(n)\sigma_\beta(n))^2$ and alike have important applications; for instance, in [4] a hypothetical improvement of Vinogradov's zero-free region is obtained; the hypothesis is of intriguing nature. As indicated above, in [SaRam16] an extension of Ramanujan's idea to automorphic $L$-functions is discussed in a general setting. Relevant to this, it should be remarked that the case of symmetric power $L$-functions is treated in [Mo15], and combined again with the $\Lambda^2$-sieve new assertions are established on the distribution of zeros of those $L$-functions and consequently of Hecke eigenvalues over primes in short intervals.

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