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On ℓ-Regular Bipartitions Modulo m

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In memory of Professor S. Srinivasan

Abstract. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n and $B_\ell(n)$ denote the number of ℓ-regular bipartitions of n. In this paper, we establish several infinite families of congruences satisfied by $B_\ell(n)$ for $\ell \in \{2,4,7\}$. We also establish a relation between $b_9(2n)$ and $B_3(n)$.

Keywords. Congruence, partition, regular partition, regular bipartition, theta functions

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1. Introduction and Notations

A partition of a positive integer n is any non-increasing sequence of positive integers whose sum is n. The number of such partitions is denoted by $p(n)$, and the number of partitions where the summands are distinct is denoted by $q(n)$. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n, where an $\ell-$regular partition of n is a partition of n such that none of its parts is divisible by ℓ. It is known that $b_2(n) = q(n)$. The generating function for the number of $\ell-$regular partitions of n is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1},$$

where f_k is defined by $f_k := \prod_{m=1}^{\infty}(1-q^{km})$, k, a positive integer.

In 1997 Gordon and Ono [GoOn97] obtained some divisibility properties of $b_\ell(n)$ by powers of certain special primes. In fact they have proved the following results:

1. Let $\ell = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ be the prime factorization of a positive integer ℓ and let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. If $p_i^{\alpha_i} \geq \sqrt{\ell}$, then for every positive integer j

$$\lim_{N\to\infty} \frac{S_\ell(N; p_i^j)}{N} = 1,$$

   where $S_\ell(N;M)$ is the number of positive integers $n \leq N$ for which $b_\ell(n) \equiv 0 \pmod{M}$. In other words the set of those positive integers n for which $b_\ell(n) \equiv 0 (\text{mod } p_i^j)$ has arithmetic density one. In fact there exists a positive constant α depending on p_i, j and ℓ such that there are at most $O(\frac{N}{\log^a N})$ many integers $n \leq N$ for which $b_\ell(n)$ is not divisible by p_i^j.

2. Let $\ell = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ be the prime factorization of a positive integer ℓ and let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. If $p_i^{\alpha_i} \geq \sqrt{\ell}$, then there are infinitely many integers n for which

$$b_\ell(n) \not\equiv 0 \ (mod\ p_i).$$

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These results have strongly influenced many authors to study the arithmetic properties of $b_\ell(n)$, its divisibility and the distribution. In fact Andrews, Hirschhorn and Sellers [AnHiSe10] derived some infinite families of congruences for $b_4(n)$ modulo 2. Hirschhorn and Sellers [HiSe10] obtained many Ramanujan-type congruences for $b_5(n)$ modulo 2. Webb [We11] established an infinite family of congruences for $b_{13}(n)$ modulo 3. Xia and Yao [XiYa14a] established several infinite families of congruences for $b_9(n)$ modulo 2. Cui and Gu [CuGu13] derived congruences for $b_\ell(n)$ modulo 2 for certain values of ℓ by employing the p-dissection formulas of Ramanujan's theta functions. In [Xi14], Xia found congruences for $b_4(n)$ modulo 8. Keith [Ke14] obtained the following conjecture which was proved by Xia and Yao in [XiYa14b]:

Theorem 1.1. *For $k = 0, 2, 3, 4, \alpha \geq 1$ and $n \geq 0$,*

$$b_9\left(5^{2\alpha}n + \frac{(3k+2)5^{2\alpha-1}-1}{3}\right) \equiv 0 \pmod{3}.$$

Dandurand and Penniston [DaPe09] gave the exact criteria for the divisibility of $b_\ell(n)$ for $\ell \in \{5, 7, 11\}$. Xia [Xi15] showed that $b_\ell(A(k)n + B(k)) \equiv C(k)b_\ell(n) \pmod{\ell}$, where $A(k)$, $B(k)$ and $C(k)$ are functions in k and $\ell \in \{13, 17, 19\}$ and derived several strange congruences for $b_\ell(n)$ modulo ℓ. Wang [Wa17a][Wa17b] established several infinite families of congruences modulo powers of 5 for $b_5(n)$. Recently, in [AdRa18], Adiga and Ranganatha proved Ramanujan-type congruences modulo powers of 7 for $b_7(n)$ and $b_{49}(n)$.

An ℓ-regular bipartition of n is an ordered pair of ℓ-regular partitions (λ_1, λ_2) such that the sum of all the parts equals n. Denote the number of ℓ-regular bipartitions of n by $B_\ell(n)$. Then, the generating function of $B_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} B_\ell(n)q^n = \frac{f_\ell^2}{f_1^2}. \tag{1.1}$$

Using Ramanujan's two modular equations of degree 7, Lin [Li15] established an infinite family of congruences for $B_7(n)$ modulo 3 and in [Li16], he established infinite families of congruences for $B_{13}(n)$ modulo 3. In [KaFa17], Kathiravan and Fathima proved several infinite families of congruences satisfied by $B_\ell(n)$ for $\ell \in \{5, 7, 13\}$. They showed that for all $\alpha > 0$,

$$\begin{aligned} B_5\left(4^\alpha n + \frac{5\times 4^\alpha - 2}{6}\right) &\equiv 0 \pmod{5}, \\ B_7\left(5^{8\alpha}n + \frac{5^{8\alpha}-1}{2}\right) &\equiv 3^\alpha B_7(n) \pmod{7} \text{ and} \\ B_{13}\left(5^{12\alpha}n + 5^{12\alpha} - 1\right) &\equiv B_{13}(n) \pmod{13}. \end{aligned}$$

A (k, ℓ)-regular bipartition of n is a bipartition (λ, μ) of n such that λ is a k-regular partition and μ is an ℓ-regular partition. Let $B_{k,\ell}(n)$ denote the number of (k, ℓ)-regular bipartitions of n. Then the generating function of $B_{k,\ell}(n)$ is given by

$$\sum_{n=0}^{\infty} B_{k,\ell}(n)q^n = \frac{f_k f_\ell}{f_1^2}.$$

Dou [Do16] proved an infinite family of congruences modulo 11: for $\alpha \geq 2$ and $n \geq 0$

$$B_{3,11}\left(3^\alpha n + \frac{5\cdot 3^{\alpha-1}-1}{2}\right) \equiv 0 \pmod{11}.$$

She stated two conjectures:

Conjecture 1: For any $n \geq 0$, $B_{5,7}(7n+6) \equiv 0 \pmod 7$.

Conjecture 2: For any $n \geq 0$,

$$\begin{aligned} B_{3,7}(An+B) &\equiv 0 \pmod 2, \\ B_{3,7}(Cn+D) &\equiv 0 \pmod 3, \\ B_{3,7}(En+F) &\equiv 0 \pmod 9, \end{aligned}$$

where $(A,B) \in \{(14,4),(14,10),(16,1),(28,6),(32,21)\}$, $(C,D)=(4,3)$, and $(E,F) \in \{(7,3),(7,4),(14,13),(21,6),(21,20),(25,3),(25,13),(25,18),(25,23)\}$. In [Wa16], Wang studied the arithmetic properties of $B_{3,\ell}(n)$ and $B_{5,\ell}(n)$ and confirmed the conjectures proposed by Dou. Xia and Yao [XiYa18] also confirmed the conjectures of Dou and proved several infinite families of congruences for $B_{s,t}(n)$ modulo 3, 5 and 7. Adiga and Ranganatha [AdRa17] provided a simple proof for Ramanujan type congruence for the (3, 7)-regular bipartitions modulo 3 which was conjectured by Dou and also found some new infinite families of congruences for (3, 7)-regular bipartitions modulo 3.

In this sequel, we establish an infinite family of congruences modulo m for ℓ-regular bipartitions, where $\ell \in \{2,4,7\}$. Also, we establish a relation between $b_9(2n)$ and $B_3(n)$.

2. Preliminary Lemmas

Ramanujan's [Ber91][Ra57] general theta function $f(a,b)$ is given by

$$\begin{aligned} f(a,b) &:= 1+\sum_{n=1}^{\infty}(ab)^{n(n-1)/2}(a^n+b^n) \\ &= \sum_{-\infty}^{\infty}(ab)^{n(n+1)/2}(b)^{n(n-1)/2}, \quad |ab|<1. \end{aligned}$$

The special cases of $f(a,b)$ are given by

$$\varphi(q) := f(q,q) = 1+2\sum_{k=1}^{\infty}q^{k^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty}q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2} = (q;q)_\infty, \tag{2.4}$$

$$\chi(q) := (-q;q^2)_\infty. \tag{2.5}$$

Cui and Gu[CuGu13] found the following p-dissections of $\psi(q)$ and $f(-q)$.

Lemma 2.1. *(Cui and Gu[CuGu13, Theorem(2.1)])For any odd prime p,*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}), \tag{2.6}$$

where $\frac{k^2+k}{2}$ *and* $\frac{p^2-1}{8}$ *are not in the same residue class modulo* p *for* $0 \le k \le (p-3)/2$.

Lemma 2.2. *(Cui and Gu[CuGu13, Theorem(2.2)])For any prime* $p \ge 5$,

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2-(6k+1)p}{2}}, -q^{\frac{3p^2+(6k+1)p}{2}}) + (-1)^{\pm\frac{p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}), \tag{2.7}$$

where $\pm$ *depends on the conditions that* $(\pm p-1)/6$ *should be an integer. Moreover, note that* $(3k^2+k)/2 \not\equiv (p^2-1)/24 \pmod p$ *as* k *runs through the range of the summation.*

3. Main Results

Theorem 3.1. *We have*

$$b_9(2n) \equiv B_3(n) (mod\ 3). \tag{3.8}$$

Proof. By the binomial theorem, it is easy to see that for any prime ℓ,

$$f_\ell \equiv f_1^\ell\ (mod\ \ell). \tag{3.9}$$

Putting $l = 3$ in (3.9) and then changing q to q^3 in the resulting identity, we obtain

$$f_9 \equiv f_3^3\ (mod\ 3). \tag{3.10}$$

Changing q to q^2 in (3.10) we obtain

$$f_{18} \equiv f_6^3\ (mod\ 3). \tag{3.11}$$

Xia and Yao[XiYa12] proved that

$$\sum_{n=0}^{\infty} b_9(n) q^n = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}.$$

Extracting those terms in which the power of q is congruent to 0 modulo 2 in the above equation and then changing q^2 to q, we obtain

$$\sum_{n=0}^{\infty} b_9(2n) q^n = \frac{f_6^3 f_9}{f_1^2 f_3 f_{18}} \equiv \frac{f_3^2}{f_1^2}\quad (mod\ 3)$$

$$\equiv \sum_{n=0}^{\infty} B_3(n) q^n\quad (mod\ 3),$$

on using (3.10) and (3.11). This gives (3.8) on comparing coefficients of q^n.

Theorem 3.2. *For any prime* $p \ge 5$ *and non negative integers* α *and* n, *we have*

$$\sum_{n=0}^{\infty} B_2\left(p^{2\alpha} n + \frac{p^{2\alpha}-1}{12}\right) q^n \equiv f^2(-q)\quad (mod\ 2). \tag{3.12}$$

Proof. We prove the Theorem by induction on α.
When $\alpha = 0$, we have by definition,

$$\sum_{n=0}^{\infty} B_2(n)q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2} = (-q;q)_\infty^2 \equiv f^2(-q) \pmod 2. \tag{3.13}$$

Suppose that the result is true for $\alpha > 0$. Then we have

$$\sum_{n=0}^{\infty} B_2\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{12}\right) q^n \equiv f^2(-q) \pmod 2. \tag{3.14}$$

Now we prove the case for $\alpha + 1$. Squaring both the sides of (2.7), substituting the resulting identity to the right of (3.14), extracting those terms in which the power of q is congruent to $\frac{p^2-1}{12}$ modulo p in the resulting identity and then changing q^p to q, we obtain

$$\sum_{n=0}^{\infty} B_2\left(p^{2\alpha}\left(pn + \frac{p^2-1}{12}\right) + \frac{p^{2\alpha}-1}{12}\right) q^n \equiv f^2(-q^p) \pmod 2. \tag{3.15}$$

Changing n to pn and then changing q^p to q in the above identity, we obtain

$$\sum_{n=0}^{\infty} B_2\left(p^{2\alpha}\left(p^2n + \frac{p^2-1}{12}\right) + \frac{p^{2\alpha}-1}{12}\right) q^n = \sum_{n=0}^{\infty} B_2\left(p^{2\alpha+2}n + \frac{p^{2\alpha+2}-1}{12}\right) q^n \equiv f^2(-q) \pmod 2.$$

Therefore, the result is true for $\alpha + 1$ and hence for all $\alpha \geq 0$.

Corollary 3.3. *For any prime $p \geq 5$, non negative integers α and n and for $i = 1, 2, \cdots, p-1$, we have*

$$B_2\left(p^{2\alpha+2}n + \frac{(12i+p)p^{2\alpha+1}-1}{12}\right) \equiv 0 \pmod 2. \tag{3.16}$$

Proof. From (3.15), we have

$$\sum_{n=0}^{\infty} B_2\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{12}\right) q^n \equiv f^2(-q^p) \pmod 2.$$

Since there are no terms on the right of the above equation in which the powers of q are congruent to $1, 2, \cdots, p-1$ modulo p, (3.16) follows.

Theorem 3.4. *For any odd prime p and for non negative integers α and n, we have*

$$\sum_{n=0}^{\infty} B_4\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{4}\right) q^n \equiv \psi^2(q) \pmod 2. \tag{3.17}$$

Proof. When $\alpha = 0$, we have by definition,

$$\sum_{n=0}^{\infty} B_4(n)q^n = \frac{(q^4;q^4)_\infty^2}{(q;q)_\infty^2} \equiv [(q;q)_\infty^3]^2 \pmod 2. \tag{3.18}$$

From [Ja1881] we recall Jacobi's identity

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{3.19}$$

Hence we have

$$(q;q)_\infty^3 \equiv \sum_{n=0}^{\infty} q^{n(n+1)/2} \equiv \psi(q) \quad (mod\ 2).$$

Therefore from (3.18) we have

$$\sum_{n=0}^{\infty} B_4(n)q^n = \psi^2(q) \quad (mod\ 2), \tag{3.20}$$

which is the case $\alpha = 0$ of (3.17). Suppose the result holds for $\alpha > 0$. Then we have

$$\sum_{n=0}^{\infty} B_4\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{4}\right) q^n \equiv \psi^2(q) \ \ (mod\ 2). \tag{3.21}$$

Now we prove the case for $\alpha+1$. Squaring both the sides of (2.6), substituting the resulting identity to the right of (3.21), extracting those terms in which the power of q is congruent to $\frac{p^2-1}{4}$ modulo p in the resulting identity and then changing q^p to q, we obtain

$$\sum_{n=0}^{\infty} B_4\left(p^{2\alpha}\left(pn + \frac{p^2-1}{4}\right) + \frac{p^{2\alpha}-1}{4}\right) q^n \equiv \psi^2(q^p) \quad (mod\ 2). \tag{3.22}$$

On changing n to pn and then changing q^p to q, we obtain

$$\sum_{n=0}^{\infty} B_4\left(p^{2\alpha}\left(p^2n + \frac{p^2-1}{4}\right) + \frac{p^{2\alpha}-1}{4}\right) q^n \equiv \psi^2(q) \quad (mod\ 2),$$

which is same as

$$\sum_{n=0}^{\infty} B_4\left(p^{2(\alpha+1)}n + \frac{p^{2(\alpha+1)}-1}{4}\right) q^n \equiv \psi^2(q) \quad (mod\ 2).$$

Therefore, the result is true for $\alpha+1$ and hence for all $\alpha \geq 0$.

Corollary 3.5. *For any odd prime p, non negative integers α and n, and for $i = 1, 2, \cdots, p-1$, we have*

$$B_4\left(p^{(2\alpha+1)}n + \frac{(4i+p)p^{(2\alpha+1)}-1}{4}\right) \equiv 0 \quad (mod\ 2). \tag{3.23}$$

Proof. From (3.22) we have

$$\sum_{n=0}^{\infty} B_4\left(p^{(2\alpha+1)}n + \frac{p^{2\alpha+2}-1}{4}\right) q^n \equiv \psi^2(q^p) \quad (mod\ 2).$$

Since there are no terms on the right of the above equation in which the powers of q are congruent to $1, 2, \cdots, p-1$ modulo p, (3.23) follows.

Theorem 3.6. *For $\alpha \geq 1$, we have*

$$\sum_{n=0}^{\infty} B_7\left(3^{(2\alpha-1)}n + \frac{3^{2\alpha}-1}{2}\right) q^n \equiv 3^{4\alpha} f_3^{12} \qquad (mod\ 7). \tag{3.24}$$

Proof. We have by (3.19)

$$\begin{aligned}
(q;q)_\infty^3 &= \sum_{k=0}^{2}\sum_{n=0}^{\infty}(-1)^{3n+k}(2(3n+k)+1)q^{\frac{(3n+k)((3n+k)+1)}{2}} \\
&= \sum_{k=0}^{2}(-1)^k q^{k(k+1)/2}\sum_{n=0}^{\infty}(-1)^n(6n+2k+1)q^{3n.\frac{3n+2k+1}{2}} \\
&= \sum_{\substack{k=0\\k\neq 1}}^{2}(-1)^k q^{k(k+1)/2}\sum_{n=0}^{\infty}(-1)^n(6n+2k+1)q^{3n.\frac{3n+2k+1}{2}} - 3q(q^9;q^9)^3.
\end{aligned}$$

Therefore, we have

$$(q;q)_\infty^6 = \left[\sum_{\substack{k=0\\k\neq 1}}^{2}(-1)^k q^{k(k+1)/2}\sum_{n=0}^{\infty}(-1)^n(6n+2k+1)q^{3n.\frac{3n+2k+1}{2}} - 3q(q^9;q^9)^3\right]^2. \tag{3.25}$$

Now we have by definition,

$$\sum_{n=0}^{\infty} B_7(n)q^n = \frac{(q^7;q^7)_\infty^2}{(q;q)_\infty^2} \equiv (q;q)_\infty^{12} \quad (mod\ 7). \tag{3.26}$$

From (3.25), we have

$$\sum_{n=0}^{\infty} B_7\left(3n + \frac{3^2-1}{2}\right) q^n \equiv 3^4(q^3;q^3)_\infty^{12} \quad (mod\ 7), \tag{3.27}$$

which is the case $\alpha = 1$ of (3.24). Changing n to $3n$ in (3.27) and then changing q^3 to q in the resulting identity, we obtain

$$\sum_{n=0}^{\infty} B_7\left(3^2n + \frac{3^2-1}{2}\right) q^n \equiv 3^4(q;q)_\infty^{12} \quad (mod\ 7). \tag{3.28}$$

From (3.26) and (3.28), we deduce that

$$B_7\left(3^2n + \frac{3^2-1}{2}\right) q^n \equiv 3^4 B_7(n) \quad (mod\ 7). \tag{3.29}$$

The theorem then follows from (3.27), (3.29) and induction on α.

Corollary 3.7. *For $\alpha \geq 1$ and $i = 1,2$, we have*

$$B_7\left(3^{2\alpha}n + \frac{(2i+3)3^{2\alpha-1}-1}{2}\right) \equiv 0 \quad (mod\ 7). \tag{3.30}$$

Proof. The result follows directly from the fact that the right-hand side of (3.24) is a power series in q^3.

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