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An elementary property of correlations

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To the memory of S. Srinivasan

Abstract. We study the "shift-Ramanujan expansion" to obtain a formulae for the shifted convolution sum $C_{f,g}(N,a)$ of general functions f,g satisfying Ramanujan Conjecture; here, the shift-Ramanujan expansion is with respect to a shift factor $a>0$. Assuming Delange Hypothesis for the correlation, we get the "Ramanujan exact explicit formula", a kind of *finite* shift-Ramanujan expansion. A noteworthy case is when $f=g=\Lambda$, the von Mangoldt function; so $C_{\Lambda,\Lambda}(N,2k)$, for natural k, corresponds to $2k$-twin primes; under the assumption of Delange Hypothesis, we easily obtain the proof of Hardy-Littlewood Conjecture for this case.

Keywords. correlation, shift Ramanujan expansion, $2k$-twin primes

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1. Introduction and statement of the results

We define for any arithmetic functions $f,g:\mathbb{N}\to\mathbb{C}$ their *correlation* (or shifted convolution sum) of *shift* a:

$$C_{f,g}(N,a)\overset{def}{=}\sum_{n\le N}f(n)g(n+a),\quad \text{for all } a\in\mathbb{N}.$$

Notice in passing that it is an arithmetic function itself, of argument $a\in\mathbb{N}$, the shift. In fact, in §5 of [CoMu18] we introduced the *shift-Ramanujan expansion*, i.e. (see (1) in [CoMu18] for $c_\ell(a)$, the *Ramanujan sum*):

$$C_{f,g}(N,a)=\sum_{\ell=1}^{\infty}\widehat{C_{f,g}}(N,\ell)c_\ell(a),\quad \text{for all } a\in\mathbb{N}.$$

Any arithmetic function $F:\mathbb{N}\to\mathbb{C}$ may be written as $F(n)=\sum_{d|n}F'(d)$, by Möbius inversion [Ten95], with a uniquely determined $F'\overset{def}{=}F*\mu$ (see [Ten95] for $*$, μ), its *Eratosthenes transform* (Wintner's [Win43] terminology).

We shall, hereafter, truncate $g(m)=\sum_{q|m}g'(q)$ as $g_N(m)\overset{def}{=}\sum_{q|m,q\le N}g'(q)$; in fact, our calculations will be shorter, with an a-independent truncation at a small cost, i.e. the error is small:

$$C_{f,g}(N,a)-C_{f,g_N}(N,a)=\sum_{N<q\le N+a}g'(q)\sum_{\substack{n\le N\\ n\equiv -a \bmod q}}f(n)\ll\max_{n\le N}|f(n)|\cdot\max_{N<q\le N+a}|g'(q)|\cdot a,\quad \text{for all } a\in\mathbb{N},\quad (1)$$

which, in the case f and g satisfy the Ramanujan Conjecture,[1] is $O_\varepsilon\left(N^\varepsilon(N+a)^\varepsilon a\right)$, uniformly for all $a\in\mathbb{N}$.

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[1] Ramanujan Conjecture for f says: $f(n)\ll_\varepsilon n^\varepsilon$, as $n\to\infty$. Hereafter Vinogradov's $\ll$ is equivalent to Landau's O-notation, [Ten95], also, $\ll_\varepsilon$ says, like O_ε, that the constant may depend on arbitrarily small $\varepsilon>0$.

We say, by definition, that a correlation $C_{f,g}(N,a)$ is *fair* when the dependence on the shift a is only inside the argument of g, $n+a$, but not in f, g, neither in their supports. Assuming "*g has range Q*", i.e.,

$$g(m) = g_Q(m) \stackrel{def}{=} \sum_{q|m, q\le Q} g'(q) = \sum_{\ell \le Q} \hat{g}(\ell) c_\ell(m), \text{ where } \hat{g}(\ell) \stackrel{def}{=} \sum_{q\equiv 0 \bmod \ell} \frac{g'(q)}{q}$$

(that is, compare [CoMuSa17], g_Q finite Ramanujan expansion), with Q independent of a, then $C_{f,g}(N,a)$ is

$$C_{f,g}(N,a) = C_{f,g_Q}(N,a) = \sum_{q\le Q} \hat{g}(q) \sum_{n\le N} f(n) c_q(n+a), \text{ for all } a \in \mathbb{N}, \tag{2}$$

where the $\hat{g}(q)$ are above Ramanujan coefficients of g. This correlation is fair if and only if all the $f(n)$, the $\hat{g}(q)$ and their supports don't depend on a, i.e.: a-dependence is only in $c_q(n+a)$! We define:

$$C'_{f,g_N}(N,\ell) \stackrel{def}{=} \sum_{t|\ell} C_{f,g_N}(N,t) \mu\left(\frac{\ell}{t}\right),$$

which has part in the following Delange Hypothesis, for the truncated correlation $C_{f,g_N}(N,a)$:

$$\sum_{d=1}^{\infty} \frac{2^{\omega(d)}}{d} \left| C'_{f,g_N}(N,d) \right| < \infty, \tag{DH}$$

where the arithmetic function $\omega(d)$ counts the prime factors of d, whence $2^{\omega(d)}$ is the number of square-free divisors of d, that has bound

$$2^{\omega(d)} \ll_\varepsilon d^\varepsilon, \text{ as } d \to \infty,$$

since it is bounded by the number of divisors of d (and divisor function also satisfies Ramanujan Conjecture).

It is very well known that for general arithmetic functions (see following result) Delange Hypothesis implies Carmichael's Formula: here we apply this to our truncated correlation, getting for it, from the above hypothesis (DH), the formula

$$\widehat{C_{f,g_N}}(N,\ell) = \frac{1}{\varphi(\ell)} \lim_{x\to\infty} \frac{1}{x} \sum_{a\le x} C_{f,g_N}(N,a) c_\ell(a), \tag{CF}$$

where $\varphi(\ell) \stackrel{def}{=} |\{n \le \ell : (n,\ell)=1\}|$ is the Euler function. Actually, the implication (DH)⇒(CF) follows from a result of Wintner (of 1943 [Win43]) and a result of Delange (published in 1976, [De76]) that we quote here from [ScSp94] Theorem 2.1 in Chapter VIII on Ramanujan expansions (restating and selecting properties), for all arithmetic functions F:

Wintner-Delange Formula. *Let* $F : \mathbb{N} \to \mathbb{C}$ *satisfy Delange Hypothesis, namely*

$$\sum_{d=1}^{\infty} \frac{2^{\omega(d)}}{d} \left| F'(d) \right| < \infty.$$

Then the Ramanujan expansion

$$\sum_{q=1}^{\infty} \widehat{F}(q) c_q(n)$$

converges pointwise to $F(n)$, *for all* $n \in \mathbb{N}$, *with coefficients given by the formula*

$$\widehat{F}(q) = \sum_{d \equiv 0 \bmod q} \frac{F'(d)}{d}, \quad \textit{for all } q \in \mathbb{N}$$

(where the series on RHS, right hand side, converges pointwise, for all $q \in \mathbb{N}$*) and also by Carmichael*[2] *formula*

$$\widehat{F}(q) = \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} F(n) c_q(n), \quad \textit{for all } q \in \mathbb{N}$$

(*where the limit on RHS exists, for all* $q \in \mathbb{N}$)

We don't need, actually, to prove this result, as it follows from (quoted) Th.2.1 of [ScSp94]. In the case $F(a) = C_{f,g_N}(N,a)$, assuming the above (DH) (i.e., Delange Hypothesis for present F), then Wintner-Delange formula implies the above (CF) (i.e., Carmichael Formula for F); this, in turn, is condition (ii) of Theorem 1 in [CoMu18] which is equivalent, choosing $Q = N$, to the following **R***amanujan* **e***xact* **e***xplicit* **f***ormula* (as we named condition (iii) in Theorem 1 [CoMu18]) for C_{f,g_N}, that is also uniform in $a \in \mathbb{N}$:

$$C_{f,g_N}(N,a) = \sum_{\ell \le N} \left(\frac{\hat{g}(\ell)}{\varphi(\ell)} \sum_{n \le N} f(n) c_\ell(n) \right) c_\ell(a). \qquad \text{R.e.e.f.}$$

This part of our original correlation $C_{f,g}$, for general $f, g : \mathbb{N} \to \mathbb{C}$ satisfying Ramanujan Conjecture, has a lot of structure (it's a truncated divisor sum!); adding the other part, we estimated above in (1), we get, for fair correlations with (DH), the following "structure + small error"−elementary property (that gives name to the paper).

Theorem. *Let* $f, g : \mathbb{N} \to \mathbb{C}$ *satisfy the Ramanujan Conjecture and be such that, for the* N*-truncated divisor sum* $g_N(m)$ *defined above, the correlation* C_{f,g_N} *is fair and satisfies (DH). Then*

$$C_{f,g}(N,a) = \sum_{\ell \le N} \left(\frac{\hat{g}(\ell)}{\varphi(\ell)} \sum_{n \le N} f(n) c_\ell(n) \right) c_\ell(a) + O_\varepsilon\left(N^\varepsilon (N+a)^\varepsilon a\right),$$

uniformly in $a \in \mathbb{N}$.

What we said up to now suffices to prove the Theorem (notice: (1) & (2), Wintner-Delange result above and Theorem 1 in [CoMu18] are the whole **proof**). QED[3]

However, thanks to the importance and generality (in §3 we have, say, a huge application too) we will provide a step-by-step proof in next section, §2.

In a perfectly similar fashion to the proof of Corollary 1 [CoMu18], from Theorem 1 [CoMu18], we can prove (but we will not do) the following consequence.

Corollary 1. *Assume* $f, g : \mathbb{N} \to \mathbb{C}$ *satisfy Ramanujan Conjecture, where furthermore* f *is a* D*-truncated divisor sum, say* $f(n) = f_D(n) \stackrel{def}{=} \sum_{d|n, d \le D} f'(d)$, *with* $\frac{\log D}{\log N} < 1 - \delta$. *Also, let the correlation* C_{f,g_N} *be fair, with* (DH). *Then*

$$C_{f,g}(N,a) = \mathfrak{S}_{f,g}(a) N + O\left(N^{1-\delta}\right) + O_\varepsilon\left(N^\varepsilon (N+a)^\varepsilon a\right),$$

[2] The name given here is in honour of Carmichael [Ca32]: compare [Mu13, pp.26-27] for details

[3] In this paper, QED(=Quod Erat Demonstrandum=What was to be shown) is not the end of the story, in a proof (we use □ for it); also, in the following, it will indicate an involved, smaller, part of proof ending

uniformly in $a \in \mathbb{N}$, *where the "singular sum", here, is defined with* f, g *Ramanujan coefficients as*

$$\mathfrak{S}_{f,g}(a) \stackrel{def}{=} \sum_{q \le N} \hat{f}(q)\hat{g}(q)c_q(a), \quad \text{for all } a \in \mathbb{N}.$$

Before an "unnecessary", but beautiful, proof of our Theorem (that, actually, will prove even the above Wintner-Delange formula), we apply our Theorem, in section §3, to the noteworthy case of $2k$-twin primes, assuming (DH) for them. Also, we realised later that this noteworthy case also comes from our Theorem 1 [CoMu18]. In fact, truncating g at $Q = N$ (in Theorem 1) and considering a kind of approximation, to original correlation, as given above in equation (1), everything works fine!

2. The detailed proof of our Theorem

Proof. Starting from (1), we are left with the task of proving the R.e.e.f. above, i.e.,

$$\sum_{n \le N} f(n) \sum_{q|n+a, q \le N} g'(q) = \sum_{\ell \le N} \left(\frac{\hat{g}(\ell)}{\varphi(\ell)} \sum_{n \le N} f(n) c_\ell(n) \right) c_\ell(a).$$

The hypotheses of our Theorem ensure that the LHS, namely $C_{f,g_N}(N,a)$, satisfies (DH) above. Now, we need to infer (DH) $\Rightarrow$ (CF) (see the above), namely, get the Carmichael formula for our $C_{f,g_N}(N,a)$, so to have in the following, say, a way to infer the R.e.e.f.! However, we'll supply even more, by providing a proof, for the above "Wintner-Delange formula". (Hence, in the immediate following we'll import arguments from [De76] & [ScSp94].)

In order to prove it, we wish to prove that the following double series, over ℓ, d summations, is absolutely convergent; so, we may write the equation expressing it in two ways (first summing over ℓ, then d and the vice versa):

$$\sum_{d=1}^{\infty} \sum_{\ell|d} \frac{F'(d)}{d} c_\ell(n) = \sum_{\ell=1}^{\infty} \sum_{d \equiv 0 \bmod \ell} \frac{F'(d)}{d} c_\ell(n), \quad \text{for all } n \in \mathbb{N}, \qquad (*)$$

namely, exchange sums. In fact, $\frac{1}{d}\sum_{\ell|d} c_\ell(n) = \mathbf{1}_{d|n}$, for $\mathbf{1}_{\wp} = 1$ if and only if $\wp$ is true (0 otherwise), [CoMuSa17, Lemma 1] gives LHS

$$\sum_{d=1}^{\infty} \frac{F'(d)}{d} \sum_{\ell|d} c_\ell(n) = \sum_{d|n} F'(d) = F(n),$$

with on RHS the Wintner-Delange coefficients

$$\sum_{d \equiv 0 \bmod \ell} \frac{F'(d)}{d}, \quad \text{for all } \ell \in \mathbb{N}$$

thus supplying a proof of the first (Wintner-Delange's!) formula and also ensuring pointwise convergence of Ramanujan expansion, with these coefficients:

$$(*) \quad \Rightarrow \quad F(n) = \sum_{\ell=1}^{\infty} \left(\sum_{d \equiv 0 \bmod \ell} \frac{F'(d)}{d} \right) c_\ell(n), \text{ for all } n \in \mathbb{N}.$$

Absolute convergence of double series comes from the fact that LHS with moduli, for all $d,\ell\in\mathbb{N}$, are bounded by

$$\sum_{d=1}^{\infty}\frac{|F'(d)|}{d}\sum_{\ell|d}|c_\ell(n)|\le n\sum_{d=1}^{\infty}\frac{|F'(d)|}{d}2^{\omega(d)}<\infty,\quad\text{for all } n\in\mathbb{N},$$

coming as we know from Delange Hypothesis, starting from the optimal bound, proved by Hubert Delange:

$$\sum_{\ell|d}|c_\ell(n)|\le n\cdot 2^{\omega(d)},$$

for which we refer to Delange's original paper [De76] (also, for comments about optimality).

Left to prove, for Wintner-Delange formula above, is the fact that above coefficients (Wintner-Delange's, which we know, now, to be our Ramanujan coefficients!) are given also by the Carmichael formula:

$$\frac{1}{\varphi(q)}\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}F(n)c_q(n)=\sum_{d\equiv 0 \bmod q}\frac{F'(d)}{d},$$

our task, now; for which we plug (in LHS), for a large $K\in\mathbb{N}$, the decomposition:

$$F(n)=\sum_{d|n,d\le K}F'(d)+\sum_{d|n,d>K}F'(d)$$

rendering in the LHS the following (again, sums exchange is possible because F' can't depend on n):

$$\frac{1}{x}\sum_{n\le x}F(n)c_q(n)=\sum_{d\le K}F'(d)\frac{1}{x}\sum_{m\le x/d}c_q(dm)+\sum_{d>K}F'(d)\frac{1}{x}\sum_{m\le x/d}c_q(dm),$$

in which, now, we apply two different treatments, depending on $d\le K$ or $d>K$. For low divisors d,

$$\sum_{d\le K}F'(d)\frac{1}{x}\sum_{m\le x/d}c_q(dm)=\sum_{d\le K}F'(d)\sum_{j\le q,(j,q)=1}\frac{1}{x}\sum_{m\le x/d}e_q(jdm)$$

$$=\sum_{d\le K}F'(d)\sum_{j\le q,(j,q)=1}\left(\frac{1}{d}\cdot\mathbf{1}_{d\equiv 0 \bmod q}+O\left(\frac{1}{x}\left(1+\frac{\mathbf{1}_{d\not\equiv 0 \bmod q}}{\left\|\frac{jd}{q}\right\|}\right)\right)\right)=\varphi(q)\sum_{\substack{d\le K\\ d\equiv 0 \bmod q}}\frac{F'(d)}{d}+O(1/x),$$

from familiar exponential sums cancellations, with a final O-constant not affecting the x-decay, while for high divisors d:

$$\sum_{d>K}F'(d)\frac{1}{x}\sum_{m\le x/d}c_q(dm)\ll\varphi(q)\sum_{d>K}\frac{|F'(d)|}{d},$$

uniformly in $x>0$, using the trivial bound $|c_q(n)|\le\varphi(q)$, for all $n\in\mathbb{Z}$. In all,

$$\frac{1}{x}\sum_{n\le x}F(n)c_q(n)=\varphi(q)\sum_{\substack{d\le K\\ d\equiv 0 \bmod q}}\frac{F'(d)}{d}+O(1/x)+O\left(\varphi(q)\sum_{d>K}\frac{|F'(d)|}{d}\right),$$

entailing

$$\frac{1}{\varphi(q)}\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}F(n)c_q(n)=\sum_{\substack{d\le K\\ d\equiv 0 \bmod q}}\frac{F'(d)}{d}+O\left(\sum_{d>K}\frac{|F'(d)|}{d}\right),$$

actually, giving the required equation, since from Delange Hypothesis the series $\sum_{d=1}^{\infty}\frac{|F'(d)|}{d}$ converges, so errors in O are infinitesimal with K, an arbitrarily large natural number (also, present LHS doesn't depend on it!). At last, this also proves the convergence in RHS of these, say, $d \le K$-coefficients (as $K \to \infty$).

QED (Wintner-Delange Formula)

Let's turn to the application of this formula to our case $F(a) = C_{f,g_N}(N,a)$, getting that (since we are assuming (DH) in hypotheses) we have the Carmichael formula, (CF) above. Now (mimicking the proof of [CoMu18] Theorem 1, $(ii) \Rightarrow (iii)$, exactly) we'll get the Reef above; in fact, let's calculate, since we know that the shift Ramanujan expansion converges (again, from (DH) implying this by just proved Wintner-Delange), its shift-Ramanujan coefficients, for correlation $C_{f,g_N}(N,a)$, namely

$$\widehat{C_{f,g_N}}(N,\ell) = \frac{1}{\varphi(\ell)} \lim_{x\to\infty} \frac{1}{x} \sum_{a\le x} C_{f,g_N}(N,a) c_\ell(a).$$

Plugging (2) with $Q = N$ inside this RHS, we get for it:

$$\frac{1}{x} \sum_{a\le x} C_{f,g_N}(N,a) c_\ell(a) = \sum_{q\le N} \hat{g}(q) \sum_{n\le N} f(n) \frac{1}{x} \sum_{a\le x} c_q(n+a) c_\ell(a),$$

present exchange of sums being possible thanks to the hypothesis: $C_{f,g_N}(N,a)$ is fair. Then,

$$\frac{1}{\varphi(\ell)} \lim_{x\to\infty} \frac{1}{x} \sum_{a\le x} C_{f,g_N}(N,a) c_\ell(a) = \frac{1}{\varphi(\ell)} \sum_{q\le N} \hat{g}(q) \sum_{n\le N} f(n) \lim_{x\to\infty} \frac{1}{x} \sum_{a\le x} c_q(n+a) c_\ell(a), \qquad (**)$$

since all we are exchanging with $\lim\limits_{x\to\infty}$ are finite sums (again, we're implicitly using fairness); then, the orthogonality of Ramanujan sums (first proved by Carmichael in [Ca32], that's why (CF) bears his name), namely Theorem 1 in [Mu13]:

$$\lim_{x\to\infty} \frac{1}{x} \sum_{a\le x} c_q(n+a) c_\ell(a) = \mathbf{1}_{q=\ell} \cdot c_q(n), \quad \text{for all } \ell, n, q \in \mathbb{N},$$

gives inside $(**)$ whence for quoted (CF) the shift-Ramanujan coefficients

$$\widehat{C_{f,g_N}}(N,\ell) = \frac{1}{\varphi(\ell)} \hat{g}(\ell) \sum_{n\le N} f(n) c_\ell(n)$$

and this, thanks to the finite support of $\hat{g}$, up to $Q = N$, here, gives the R.e.e.f.! QED

One last detail: equation (2), actually, we didn't prove; but it follows from $m = n + a$ in (another unproven)

$$\sum_{q|m, q\le Q} g'(q) = \sum_{\ell\le Q} \hat{g}(\ell) c_\ell(m),$$

that is: the g_Q (see the beginning of the paper) **f**inite **R**amanujan **e**xpansion (f.R.e.) (for which we referred to [CoMuSa17], of course), with Ramanujan coefficients

$$\hat{g}(\ell) \stackrel{def}{=} \sum_{q\equiv 0 \bmod \ell} \frac{g'(q)}{q}.$$

This can be proved at once, from quoted Lemma 1 of [CoMuSa17], that we also prove (briefly) here:

$$\mathbf{1}_{q|m} = \frac{1}{q} \sum_{\ell|q} c_\ell(m),$$

because: the orthogonality of additive characters [Da00] (rearranging by g.c.d.) gives

$$\mathbf{1}_{q|m} = \frac{1}{q}\sum_{r\le q} e_q(rm) = \frac{1}{q}\sum_{\ell|q}\sum_{r\le q,(r,q)=q/\ell} e_q(rm) = \frac{1}{q}\sum_{\ell|q}\sum_{j\le\ell,(j,\ell)=1} e_\ell(jm), \quad \text{with } c_\ell(n) \overset{def}{=} \sum_{\substack{j\le\ell\\(j,\ell)=1}} e_\ell(jn).$$

Then from this divisibility condition we prove g_Q f.R.e.:

$$\sum_{q|m,q\le Q} g'(q) = \sum_{q\le Q}\frac{g'(q)}{q}\sum_{\ell|q} c_\ell(m) = \sum_{\ell\le Q}\hat{g}(\ell)c_\ell(m),$$

simply exchanging sums and using above definition of f.R.e. coefficients, $\hat{g}(q)$. QED (for equation (2), too.) □

3. The well-known case $f = g = \Lambda$, $a = 2k > 0$ of our Theorem: $2k$-prime-twins

Assuming (DH) for $f = g = \Lambda$, Hardy-Littlewood heuristic (Conjecture B and (5.26) [HaLi23]) is a Theorem.

Corollary 2. *Assuming Delange Hypothesis for $C_{\Lambda,\Lambda_N}(N,a)$, i.e.*

$$\sum_{d=1}^{\infty}\frac{2^{\omega(d)}}{d}\left|C'_{\Lambda,\Lambda_N}(N,d)\right| < \infty,$$

we get a kind of Hardy-Littlewood asymptotic formula, with an absolute constant $c > 0$, for all fixed $k \in \mathbb{N}$

$$C_{\Lambda,\Lambda}(N,2k) = \mathfrak{S}_{\Lambda,\Lambda}(2k)N + O\left(Ne^{-c\sqrt{\log N}}\right).$$

Proof. We apply the calculations for Ramanujan coefficients of the N-truncated von Mangoldt function Λ_N, from the classical [Da00] von Mangoldt $\Lambda = (-\mu\log) * \mathbf{1}$, [Ten95], defined as usual in terms of primes $p \in \mathbb{P}$:

$$\Lambda(n) \overset{def}{=} \sum_{k\in\mathbb{N}}\sum_{p\in\mathbb{P}}\mathbf{1}_{n=p^k}\log p \;\Rightarrow\; \Lambda(n) = \sum_{d|n}(-\mu(d)\log d), \;\; \Lambda_N(n) = \sum_{d|n,d\le N}(-\mu(d)\log d),$$

entailing

$$\Lambda_N(n) = \sum_{q\le N}\widehat{\Lambda_N}(q)c_q(n), \quad \widehat{\Lambda_N}(q) \overset{def}{=} -\sum_{\substack{d\le N\\ d\equiv 0 \bmod q}}\frac{\mu(d)\log d}{d} \ll \frac{\log^2 N}{q},$$

where now these are, thanks to §4 of [CoMu18], with an absolute $c > 0$,

$$\widehat{\Lambda_N}(q) = \frac{\mu(q)}{\varphi(q)} + O\left(\frac{1}{q}\exp\left(-c\sqrt{\log N}\right)\right), \quad \text{for all } q \le \sqrt{N},$$

thanks to the zero-free region of Riemann zeta-function (actually, we are not using most recent one). Now,

$$C_{\Lambda,\Lambda}(N,a) = \sum_{\ell\le N}\frac{\widehat{\Lambda_N}(\ell)}{\varphi(\ell)}\left(\sum_{n\le N}\Lambda(n)c_\ell(n)\right)c_\ell(a) + O_\varepsilon\left(N^\varepsilon (N+a)^\varepsilon a\right),$$

from our Theorem: C_{Λ,Λ_N} is fair & assume (DH), $f = g = \Lambda$. Set $a = 2k > 0$, $\frac{\log k}{\log N} < 1 - \delta$, $\delta \in (0, 1/2)$ fixed:

$$C_{\Lambda,\Lambda}(N,a) = \sum_{\ell \le \sqrt{N}} \frac{\mu(\ell)}{\varphi^2(\ell)} \left(\sum_{n \le N} \Lambda(n) c_\ell(n) \right) c_\ell(a) + O\left(\exp\left(-c\sqrt{L}\right) \sum_{\ell \le \sqrt{N}} \frac{(a,\ell)}{\ell\varphi(\ell)} \sum_{n \le N} \Lambda(n)(n,\ell) \right)$$

$$+ O\left(L^2 \sum_{\sqrt{N} < \ell \le N} \frac{(a,\ell)}{\ell\varphi(\ell)} \sum_{n \le N} \Lambda(n)(n,\ell) \right) + O\left(N^{1-\delta}\right),$$

where we have applied well-known $|c_q(n)| \le (q,n)$, see Lemma A.1 in [CoMu18], and above bounds for Λ_N, abbreviating hereafter $L \stackrel{def}{=} \log N$. In the main term, applying PNT(Prime Number Theorem) [Da00], [Ten95]:

$$\sum_{n \le N} \Lambda(n) c_\ell(n) = \mu(\ell) \sum_{\substack{n \le N \\ (n,\ell)=1}} \Lambda(n) + O\Big(L\varphi(\ell) \sum_{p|\ell} \log p \Big) \stackrel{\text{PNT}}{=} \mu(\ell)N + O\left(Ne^{-c\sqrt{L}}\right) + O\left(L\varphi(\ell)\log\ell\right),$$

from well known [Da00]: $\sum_{p|\ell} \log p \le \sum_{n|\ell} \Lambda(n) = \log \ell$; here, we need to bound the n-sum in remainders as

$$\sum_{n \le N} \Lambda(n)(n,\ell) = \sum_{d|\ell} d \sum_{\substack{n \le N \\ (n,\ell)=d}} \Lambda(n) \ll \sum_{d|\ell} d \sum_{\substack{n \le N \\ n \equiv 0 \bmod d}} \Lambda(n) \ll N + \ell L \sum_{k \in \mathbb{N}} \sum_{p^k|\ell} \log p \ll NL^2, \quad \text{for all } \ell \le N,$$

by Čebičev bound [Ten95]: $\sum_{n \le N} \Lambda(n) \ll N$. Then, using [Ten95]: $\varphi(\ell) \gg \ell/\log \ell$, changing time to time $c > 0$,

$$C_{\Lambda,\Lambda}(N,a) = N \sum_{\ell \le \sqrt{N}} \frac{\mu^2(\ell)}{\varphi^2(\ell)} c_\ell(a) + O\left(Ne^{-c\sqrt{L}} \sum_{\ell \le \sqrt{N}} \frac{(a,\ell)}{\ell^2} + NL^5 \sum_{\sqrt{N} < \ell \le N} \frac{(a,\ell)}{\ell^2} + N^{1-\delta} \right)$$

$$= N \sum_{\ell=1}^{\infty} \frac{\mu^2(\ell)}{\varphi^2(\ell)} c_\ell(a) + O\left(N \sum_{\ell > \sqrt{N}} \frac{\log^2 \ell}{\ell^2} (a,\ell) \right) + O\left(Ne^{-c\sqrt{L}} \sum_{\ell \le \sqrt{N}} \frac{(a,\ell)}{\ell^2} + NL^5 \sum_{\sqrt{N} < \ell \le N} \frac{(a,\ell)}{\ell^2} + N^{1-\delta} \right)$$

being, by the definition of classic *singular series* for $a = 2k$-twin primes,

$$\mathfrak{S}_{\Lambda,\Lambda}(a) \stackrel{def}{=} \sum_{\ell=1}^{\infty} \frac{\mu^2(\ell)}{\varphi^2(\ell)} c_\ell(a)$$

and, also, by following bounds: (use $(A+B)^2 \ll A^2 + B^2$, then, [Ten95]: $\sum_{d|a} 1 \ll_\varepsilon a^\varepsilon$ and $\sum_{d \le x} 1/d \ll \log x$)

$$\sum_{\ell > \sqrt{N}} \frac{\log^2 \ell}{\ell^2} (a,\ell) \ll \sum_{\substack{d|a \\ d \le \sqrt{N}}} \frac{1}{d} \sum_{m > \sqrt{N}/d} \frac{\log^2 d + \log^2 m}{m^2} + \sum_{\substack{d|a \\ d > \sqrt{N}}} \frac{1}{d} \sum_{m=1}^{\infty} \frac{\log^2 d + \log^2 m}{m^2} \ll_\varepsilon a^\varepsilon \frac{L^2}{\sqrt{N}},$$

$$\sum_{\ell \le \sqrt{N}} \frac{(a,\ell)}{\ell^2} \ll \sum_{\substack{d|a \\ d \le \sqrt{N}}} \frac{1}{d} \sum_{m \le \sqrt{N}/d} \frac{1}{m^2} \ll L,$$

$$\sum_{\sqrt{N} < \ell \le N} \frac{(a,\ell)}{\ell^2} \ll \sum_{\substack{d|a \\ d \le \sqrt{N}}} \frac{1}{d} \sum_{\sqrt{N}/d < m \le N/d} \frac{1}{m^2} + \sum_{\substack{d|a \\ d > \sqrt{N}}} \frac{1}{d} \sum_{m \le N/d} \frac{1}{m^2} \ll_\varepsilon \frac{a^\varepsilon}{\sqrt{N}},$$

uniformly in $a = 2k$, $k \in \mathbb{N}$, with $\frac{\log k}{\log N} < 1 - \delta$, for a fixed $\delta \in (0, 1/2)$, proves Hardy-Littlewood Conjecture[4]

$$C_{\Lambda,\Lambda}(N, 2k) = \mathfrak{S}_{\Lambda,\Lambda}(2k)N + O\left(Ne^{-c\sqrt{\log N}}\right).$$

□

We don't have time to go deeper (but we've plenty of margins[5]).

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[4] In this version, it's actually a strong form of famous Conjecture!
[5] In 1637 Fermat wrote "... Hanc marginis exiguitas non caperet."