Hardy-Ramanujan Journal 41 (2018), 104-117 submitted 06/03/2018, accepted 26/10/2018, revised 06/11/2018

Hybrid level aspect subconvexity for $GL(2) \times GL(1)$ Rankin-Selberg L-Functions

Keshav Aggarwal, Yeongseong Jo, and Kevin Nowland

To the memory of S. Srinivasan

Abstract. Let M be a squarefree positive integer and P a prime number coprime to M such that $P \sim M^\eta$ with $0 < \eta < 2/5$. We simplify the proof of subconvexity bounds for $L\left(\frac{1}{2}, f \otimes \chi\right)$ when f is a primitive holomorphic cusp form of level P and χ is a primitive Dirichlet character modulo M. These bounds are attained through an unamplified second moment method using a modified version of the delta method due to R. Munshi. The technique is similar to that used by Duke-Friedlander-Iwaniec save for the modification of the delta method.

Keywords. Special values of L-functions, Rankin-Selberg convolution, subconvexity, δ-method

2010 Mathematics Subject Classification. 11F11, 11F67, 11L05

1. Introduction

Let f be a primitive holomorphic cusp form of level P and χ a primitive Dirichlet character modulo M with $(P, M) = 1$, M a square-free positive integer, and P a prime number. The Rankin-Selberg convolution $L(s, f \otimes \chi)$ is given by

$$L(s, f \otimes \chi) = \sum_{n \geq 1} \frac{\lambda_f(n)\chi(n)}{n^s},$$

at least for $\mathrm{Re}(s)$ sufficiently large. Let $\mathcal{Q} = \mathcal{Q}(f \otimes \chi) = PM^2$ denote the size of the conductor of L-function $L(s, f \otimes \chi)$. For $\mathrm{Re}(s) = 1/2$ and any $\varepsilon > 0$, the functional equation, Stirling's estimate for the gamma function along with the Phragmen-Lindelöf convexity principle yields the following bound

$$L(s, f \otimes \chi) \ll_\varepsilon \mathcal{Q}^{1/4+\varepsilon}$$

which is known as the convexity bound in [IwKo04, (5.21)], whereas the generalized Riemann hypothesis implies the Lindelöf hypothesis [IwKo04, corollary 5.20]:

$$L(s, f \otimes \chi) \ll_\varepsilon \mathcal{Q}^{\varepsilon}.$$

One classical problem, subconvexity problem, is to establish the bound of the form

$$L(s, f \otimes \chi) \ll \mathcal{Q}^{1/4-\delta}$$

for some $\delta > 0$ when $\mathrm{Re}(s) = 1/2$. Although the convexity bound is far from the expected Lindelöf hypothesis, any power saving for this conductor has applications in proving certain equidistribution problems. We refer to [Mi04, Sa92, Sa01] for the discussion of the link between the quantum unique ergodicity problem and subconvexity bounds for certain higher degree L-functions.

We thank episciences.org for providing open access hosting of the electronic journal *Hardy-Ramanujan Journal*

In studying the subconvexity problem for character twists of holomorphic modular forms of full level, Duke-Friedlander-Iwaniec [DFIw93] introduced a simple yet powerful decomposition of the delta symbol which detects when an integer is zero. The starting point for their method was an amplified second moment average over primitive Dirichlet characters of a given level. The subconvexity problem for twisted L-functions $L(s, f \otimes \chi)$ in the conductor-aspect was also solved in their paper for the first time for f a holomorphic cusp form of level one, with subconvexity exponent $1/2 - 1/22$ (following the computation of section 4.3 in [Mi07] for example).

Recent works achieving hybrid subconvexity bounds for Rankin-Selberg convolution L-functions of large level include Kowalski-Michel-Vanderkam [KoMiV02], Michel [Mi04] and Harcos-Michel [HaMi06]. Michel-Venkatesh [MiVe10] solve the subconvexity problem for the L-functions of $GL(1)$ and $GL(2)$ automorphic representations over a fixed number field, uniformly in all aspects. Holowinsky-Munshi [HoMu13] prove a hybrid level aspect bound for the L-function coming from the convolution of two holomorphic modular forms of nontrivial levels, one being squarefree and the other being prime. Ye [Zh14] relaxed the level conditions in [HoMu13] to both levels square-free. Moreover, the author in [Zh14] used a large sieve inequality to achieve a subconvexity bound for the full range of levels when both forms are holomorphic. The works of Holowinsky-Munshi and Ye relied on an application of Heath-Brown's refinement [He96] of the classical delta method due to Duke-Friedlander-Iwaniec [DFIw93]. Browning-Munshi in [BrMu13] introduce a modification of the delta method with factorization moduli to obtain a structural advantage.

In this paper we follow the work of Duke-Friedlander-Iwaniec [DFIw93] who studied the Rankin-Selberg convolution of a primitive Dirichlet character with a holomorphic modular form for the full modular group. Their paper, as ours, uses a second moment average over primitive Dirichlet characters of a given level. In this paper, we allow for holomorphic forms with a range of permissible but nontrivial levels relative to the conductor of the Dirichlet character. It is here that we make use of the modified delta method with a conductor lowering trick, based on the work of Munshi in [Mu15].

The subconvexity of L-functions for the twisting modular form has already been established in [Co03, DFIw93, Ha03, Mi07]. The main point of this paper, however, is to demonstrate how using a modified application of the delta method simplifies the arithmetic structure and lengthens the admissible hybrid range of the level parameters M and P. To be precise, we present a simple method which ends with a trivial application of the Weil bound for Kloosterman sums and establishes subconvexity for the hybrid range $P = M^\eta$ for $0 < \eta < 2/5$. Using the classical delta method without the conductor lowering trick and following the same process, one would obtain a hybrid range of $0 < \eta < 2/7$. In the meanwhile, Blomer and Harcos in theorem 2 of [BlHa14] establish the following hybrid estimate

$$L\left(\frac{1}{2}, f \otimes \chi\right) \ll_{k,\varepsilon} P^{\frac{1}{4}+\varepsilon} M^{\frac{3}{8}+\varepsilon} + P^{\frac{1}{2}+\varepsilon} M^{\frac{1}{4}+\varepsilon}.$$

For $P = M^\eta$, we obtain that

$$L\left(\frac{1}{2}, f \otimes \chi\right) \ll_{k,\varepsilon} \mathcal{Q}^{\frac{1}{4}+\varepsilon}\left(\mathcal{Q}^{-\frac{1}{8(2+\eta)}-\frac{1-\eta}{4(2+\eta)}}\right),$$

The hybrid range $0 < \eta < 1$ of Blomer and Harcos is stronger than our range $0 < \eta < 2/5$. However we emphasize that our technique does not require amplification or the large sieve inequality and recently this method is adopted in [HoZh17] to extend a hybrid subconvexity range bound for $L(1/2, g \otimes h)$ where g is a primitive holomorphic cusp form of level M and h is a primitive either holomorphic or Mass cusp form of level P with $(M, P) = 1$, M a squarefree integer, and P a prime.

Of course, one has the ability to push the analysis further, in either method, by analyzing the resulting sum of Kloosterman sums through the large sieve inequality similar to the work of Ye. Again, however, there is an advantage in the modified delta method in that we obtain sums of standard Kloosterman sums for the group $\Gamma_0(P)$ associated to the cusp at ∞. Without the modified delta method, one would instead get Kloosterman sums associated to the cusps 0 and ∞ and then more work is required (using the work of Deshoullier-Iwaniec [DesIw82] for example).

We provide a sketch of these arguments below and note that our methods may also be applied to analogous Rankin-Selberg convolutions.

1.A. Holomorphic cusp forms

In this section, we review the theory of holomorphic modular forms. The definitive reference for the theory is in [IwKo04, chapter 14]. Let $P > 0$ be a prime number and $k > 0$ an even integer. Let $S_k(P)$ be the linear space of holomorphic cusp forms of weight k, level P, and trivial nebentypus. We let $\Gamma_0(P)$ be the Hecke congruence subgroups defined by

$$\Gamma_0(P) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, c \equiv 0 \mod P \right\}$$

If $f \in S_k(P)$, then $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and satisfies

$$f(gz) = (cz+d)^k f(z)$$

for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(P)$ acting as a linear fractional transformation on z in the upper half-plane $\mathbb{H}$. Additionally, f vanishes at every cusp. Such an f has Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \psi_f(n) n^{\frac{k-1}{2}} e(nz),$$

where $e(x) := \exp(2\pi i x)$, and the Fourier coefficients $\psi_f(n)$ satisfy

$$\psi_f(n) \ll n^{\varepsilon}$$

for $\varepsilon > 0$ arbitrary with the implied constant depending on ε. This was proved by Deligne [Del74].

$S_k(P)$ is a finite dimensional Hilbert space with respect to the Petersson inner product. $S_k(P)$ has an orthogonal basis $B_k(P)$ which consists of eigenfunctions of all Hecke operators T_n such that $(n,P) = 1$, where T_n acts on f by the formula

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ (a,P)=1}} \left(\frac{a}{d}\right)^{k/2} \sum_{b \mod d} f\left(\frac{az+b}{d}\right) =: \lambda_f(n) f(z).$$

Such an f is called a *Hecke eigen cusp form.* The Hecke operators are multiplicative and satisfy

$$\psi_f(m)\lambda_f(n) = \sum_{d|(m,n)} \psi_f\left(\frac{mn}{d^2}\right)$$

for n coprime with P. In particular, $\psi_f(1)\lambda_f(n) = \psi_f(n)$ for $(n,P) = 1$. We normalize such that $\psi_f(1) = 1$ and we have $\lambda_f(n) = \psi_f(n)$ for $(n,P) = 1$. There exists a subset $B_k^*(P)$ of $B_k(P)$ of *newforms* or *primitive holomorphic cusp forms* which are eigenfunctions of all Hecke operators T_n for $n \geq 1$ with $\lambda_f(n) = \psi_f(n)$. Let $f \in B_k^*(P)$ be a newform and χ a primitive Dirichlet character modulo M with $(P,M) = 1$. Let $f \otimes \chi$ be a twisted modular form on $\mathbb{H}$ given by the Fourier expansion

$$(f \otimes \chi)(z) = \sum_{n=1}^{\infty} \chi(n)\lambda_f(n) n^{\frac{k-1}{2}} e(nz).$$

Then $f \otimes \chi$ is a newform of level PM^2.

1.B. Rankin-Selberg $L-$functions

Let $f \in B_k^*(P)$ be a newform and χ a primitive Dirichlet character modulo M where M and P are coprime, M is squarefree, and P is a prime. Then the Rankin-Selberg convolution L-function associated to $f \otimes \chi$ is

$$L(s, f \otimes \chi) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s}.$$

The associated completed L-function is

$$\Lambda(s, f \otimes \chi) = \mathcal{Q}^{s/2} L_\infty(s, f \otimes \chi) L(s, f \otimes \chi),$$

where $\mathcal{Q} = \mathcal{Q}(f \otimes \chi) = PM^2$ and the local factor at infinity L_∞ is a product of gamma functions. The approximate functional equation shows that the special value $L(1/2, f \otimes \chi)$ is given by

$$L\left(\frac{1}{2}, f \otimes \chi\right) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{\sqrt{n}} V\left(\frac{n}{\sqrt{\mathcal{Q}}}\right) + \epsilon(f \otimes \chi) \sum_{m=1}^{\infty} \frac{\lambda_f(m)\overline{\chi(m)}}{\sqrt{m}} V\left(\frac{m}{\sqrt{\mathcal{Q}}}\right)$$

(see chapter 5 of [IwKo04]) where V is a smooth function with rapid decay at infinity, and for any positive integer A, the derivatives of $V(y)$ satisfy

$$y^j V^{(j)}(y) \ll_k \mathcal{Q}^\varepsilon (1+y)^{-A} \log(2 + y^{-1})$$

for any $\varepsilon > 0$. We also have the asymptotic

$$V(y) = 1 + O\left(\left(\frac{y}{\sqrt{\mathcal{Q}}}\right)^\alpha\right)$$

for $\alpha > 0$. Let h be a smooth function which is compactly supported on $[1/2, 5/2]$ with bounded derivatives and suppose that X runs over 2^ν with $\nu = -1, 0, 1, 2, 3, \cdots$. Applying a smooth partition of unity and the asymptotic for V, we are left with

$$\left|L\left(\frac{1}{2}, f \otimes \chi\right)\right| \ll_{k,\varepsilon} \mathcal{Q}^\varepsilon \left\{ \max_{\mathcal{Q}^{\frac{1}{2}-\delta} \le X \le \mathcal{Q}^{\frac{1}{2}+\varepsilon}} |L_{f\otimes\chi}(X)| + \mathcal{Q}^{\frac{1}{4}-\frac{\delta}{2}} \right\},$$

where

$$L_{f\otimes\chi}(X) := \sum_n \frac{\lambda_f(n)\chi(n)}{\sqrt{n}} h\left(\frac{n}{X}\right). \tag{1.1}$$

2. Statement of main results

We prove the hybrid range and subconvexity bounds for $L(1/2, f \otimes \chi)$. We average over primitive characters modulo M through a second moment method to achieve subconvexity bounds.

Theorem 2.1. (Second Moment) *Let P and M be coprime positive integers with P prime and M squarefree. Let k be a fixed positive even integer. Set $\mathcal{Q} = PM^2$. Let h be a smooth function with support in $[1/2, 5/2]$ and bounded derivatives. Let $\varepsilon, \delta > 0$ and choose any X such that $\mathcal{Q}^{\frac{1}{2}-\delta} \le X \le \mathcal{Q}^{\frac{1}{2}+\varepsilon}$. If $f \in B_k^*(P)$ and χ is a primitive Dirichlet character modulo M, then*

$$\frac{1}{\varphi^\star(M)} \sideset{}{^\star}\sum_{\chi \bmod M} |L_{f\otimes\chi}(X)|^2 \ll_{k,\varepsilon} \mathcal{Q}^\varepsilon \left(1 + \frac{\mathcal{Q}^{\frac{1}{2}}}{M} \cdot \frac{P^{\frac{5}{8}+\frac{\delta}{4}}}{M^{\frac{1}{4}-\frac{\delta}{2}}}\right). \tag{2.2}$$

$\sum^\star$ means summation over primitive characters or over integers coprime with the specified modulus, and $\varphi^\star$ is the number of primitive multiplicative characters modulo M. We apply the theorem below to $F(x,y) = h(x)h(y)$, $f_1 = f_2$, and $X = Y$ for any $\mathcal{Q}^{\frac{1}{2}-\delta} \le X \le \mathcal{Q}^{\frac{1}{2}+\varepsilon}$ to obtain the above second moment bound. The first and second terms on the right hand side in (2.2) are the contributions from a zero shift and theorem 2.2, respectively.

In analytic number theory, we are often concerned with the following shifted convolution sums:

$$\sum_{\ell_1 m - \ell_2 n = h} \lambda_f(m)\lambda_g(n) H(\ell_1 m, \ell_2 n)$$

for f, g two primitive cusp forms, $h \neq 0$, ℓ_1, ℓ_2 two comprime integers. Here $H(x,y)$ is a bounded, smooth, compactly supported function in $[X, 2X] \times [Y, 2Y]$ for some $X, Y \ge 1/2$. The works involving estimates of these sums with the divisor functions, known as the additive divisor problems, have a long history (See [DFIw94, Mo94] and the reference given there). Shifted convolution sums with two Fourier coefficients have been extensively investigated in [BlHa14, DFIw93, Ha03, HaMi06, Mi04] for the case when $\lambda_g(n) = \overline{\lambda_f(n)}$ or $\lambda_g(n) = \lambda_f(n)$ and [HoMu13, HoZh17] for the general case. Thus it is natural to compare the strategies of the existing approaches and ponder about the tools necessary to study the hybrid level aspect problem. Is an application of spectral methods and the large sieve required or are there other more direct or softer methods available? Here we study the following shifted convolution sum by using a delta method with a conductor lowering trick, Voronoi summation formula and only the Weil bound for individual Kloosterman sums.

Theorem 2.2. (Shifted Convolution Sums) *Let f_1, f_2 be newforms in $B_k^*(P)$. Let M be a positive squarefree integer coprime with P. Let $r \neq 0$ be an integer coprime with P. Let $X, Y \ge 1$ and F a smooth function supported on $[1/2, 5/2] \times [1/2, 5/2]$ satisfying*

$$x^i y^j \partial^i x^i \partial^j y^j F\left(\frac{x}{X}, \frac{y}{Y}\right) \ll Z Z_x^i Z_y^j,$$

for $Z > 0$ and $Z_x, Z_y \ge 1$. Then

$$\sum_n \sum_{m=n+rM} \frac{\lambda_{f_1}(n)\lambda_{f_2}(m)}{\sqrt{nm}} F\left(\frac{n}{X}, \frac{m}{Y}\right) \ll_{k,\varepsilon} (PMr)^\varepsilon Z (Z_x Z_y)^{\frac{1}{2}} \max\{Z_x, Z_y\}^2 P^{\frac{3}{4}} \frac{\max\{X, Y\}^{\frac{3}{4}}}{(XY)^{\frac{1}{2}}}. \tag{2.3}$$

The second moment bound which is better by a power of $\mathcal{Q}$ than the convexity bound means that we must have a bound which is better than

$$\frac{1}{\varphi^\star(M)} \sum_{\chi \mod M}^{\star} |L_{f\otimes\chi}(X)|^2 \ll \frac{\mathcal{Q}^{\frac{1}{2}+\varepsilon}}{M}.$$

Therefore the bounds in (2.2) induce the hybrid subconvexity range $P \sim M^\eta$ for $0 < \eta < 2/5$.

Corollary 2.3. (Subconvexity) *Let f and χ be as above with $\eta = \frac{\log P}{\log M}$. Then*

$$L\left(\frac{1}{2}, f \otimes \chi\right) \ll_{k,\varepsilon} \frac{\mathcal{Q}^{\frac{1}{4}+\varepsilon}}{\mathcal{Q}^{\frac{2-5\eta}{20(2+\eta)}}}.$$

This produces a subconvexity bound for $0 < \eta < 2/5$.

Proof. From the reduction in section 1,

$$\left| L\left(\frac{1}{2}, f \otimes \chi\right) \right| \ll_{k,\varepsilon} \mathcal{Q}^\varepsilon \left\{ \mathcal{Q}^{\frac{1}{4}-\frac{\delta}{2}} + \max_{\mathcal{Q}^{\frac{1}{2}-\delta} \le X \le \mathcal{Q}^{\frac{1}{2}+\varepsilon}} |L_{f\otimes\chi}(X)| \right\}. \tag{2.4}$$

To bound this, weaken the bound in the above theorem by just taking the second term in (2.2) as the first provides saving for any P and M having size. This gives

$$|L_{f\otimes\chi}(X)| \ll \mathcal{Q}^{\frac{1}{4}+\varepsilon}\frac{P^{\frac{5+2\delta}{16}}}{M^{\frac{2-4\delta}{16}}}.$$

Equate the first term and second terms on the right hand side of (2.4) while replacing P and M with powers of $\mathcal{Q}$ by using $P = M^\eta$. We have $P = \mathcal{Q}^{\frac{\eta}{2+\eta}}$ and $M = \mathcal{Q}^{\frac{1}{2+\eta}}$. Therefore the optimal choice of δ satisfies

$$-\frac{\delta}{2} = \frac{5+2\delta}{16}\cdot\frac{\eta}{2+\eta} - \frac{2-4\delta}{16}\cdot\frac{1}{2+\eta}.$$

Then the saving δ is then explicitly calculated to be $\delta = \frac{2-5\eta}{10(2+\eta)}$.

3. Proof Sketch

We give a quick sketch of the proof which results in the $0 < \eta < 2/7$ bound when we do not use the conductor lowering trick followed by the sketch to get the improved range $0 < \eta < 2/5$.

The standard approximate functional equation argument in section 1 reduces our L-function to the analysis of the following sum

$$\sum_{n\sim\sqrt{P}M}\frac{\lambda_f(n)\chi(n)}{\sqrt{n}}.$$

A second moment average over primitive characters leads us to needing to understand the sum

$$\sum_{n\sim\sqrt{P}M}\frac{\lambda_f(n)}{\sqrt{n}}\sum_{\substack{m\sim\sqrt{P}M\\ m\equiv n \mod M}}\frac{\lambda_f(m)}{\sqrt{m}}.$$

Using the delta symbol, we rewrite the above as

$$\sum_{0\neq|r|\leq\sqrt{P}}\sum_{n\sim\sqrt{P}M}\frac{\lambda_f(n)}{\sqrt{n}}\sum_{m\sim\sqrt{P}M}\frac{\lambda_f(m)}{\sqrt{m}}\delta(m-n+rM,0),$$

where the $r=0$ term can be bounded trivially. Applying the usual delta method and using Voronoi summation on both the n- and m-sums, one can obtain Kloosterman sums of the form

$$\sum_{0\neq|r|\leq\sqrt{P}}\frac{1}{Q}\sum_{q\leq Q}\frac{1}{q}\sum_{n\sim P}\frac{\lambda_f(n)}{\sqrt{n}}\sum_{m\sim P}\frac{\lambda_f(m)}{\sqrt{m}}S(rM,(n-m)\bar{P};q). \tag{3.5}$$

with $Q = M^{1/2}P^{1/4}$ the square root of the size of the equation. Applying the Weil bound for each Kloosterman sum leads to the of second moment average being bounded by $\sqrt{P}\left(\frac{P^7}{M^2}\right)^{1/8}$, such that $0 < P < M^{2/7}$ is a range for subconvexity, as a bound of $\sqrt{P}$ would produce the convexity bound.

However, using the conductor lowering trick by instead using $\delta((n-m+rM)/P,0)$ with the condition $n-m+rM \equiv 0 \mod P$ followed by Voronoi summation as in the previous case, one instead obtains the sum of Kloosterman sums

$$\sum_{0\neq|r|\leq\sqrt{P}}\frac{1}{Q}\sum_{q\leq Q}\frac{1}{qP}\sum_{n\sim P}\frac{\lambda_f(n)}{\sqrt{n}}\sum_{m\sim P}\frac{\lambda_f(m)}{\sqrt{m}}S(rM,n-m;qP). \tag{3.6}$$

The length of the n- and m-sums is the same as before, whereas we have additional saving P in the denominator on the q-sum, which comes from detecting the congruence condition $n-m+rM \equiv 0$

mod P. The Weil bound for Kloosterman sums allows us to have a better bound by $P^{1/2}$. However, Q has also changed and is now $M^{1/2}/P^{1/4}$ as the equation in the delta symbol has changed and the division by Q – which ends up a division by $Q^{1/2}$ because of the Weil bound – means we only save $P^{1/4}$. Dividing the previous bound by this gives $\sqrt{P}\left(\frac{P^5}{M^2}\right)^{1/8}$. The subconvexity range has been improved to $0 < P < M^{2/5}$.

We conclude this section with two remarks. Let σ be the element in $\Gamma_0(P)$. If $\mathbf{a}$ is a cusp of $\Gamma_0(P)$, its stabilizer is defined by $\Gamma_{\mathbf{a}} = \{\sigma \in \Gamma_0(P) \mid \sigma \cdot \mathbf{a} = \mathbf{a}\}$. In particular $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{Z} \right\}$. Let $\mathbf{a}$ and $\mathbf{b}$ be two cusps of $\Gamma_0(P)$. Let $\sigma_{\mathbf{a}} \in SL(2,\mathbb{R})$ be a scaling matrix such that $\sigma_{\mathbf{a}} \cdot \infty = \mathbf{a}$ and $\sigma_{\mathbf{a}}^{-1}\Gamma_{\mathbf{a}}\sigma_{\mathbf{a}} = \Gamma_\infty$. We define similarly for $\sigma_{\mathbf{b}}$. The Kloosterman sum attached to two cusps $\mathbf{a}$ and $\mathbf{b}$ is defined in [DesIw82] as

$$S_{\mathbf{ab}}(\alpha, \beta; c) = \sum_{\begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \sigma_{\mathbf{a}}^{-1}\Gamma_0(P)\sigma_{\mathbf{b}}/\Gamma_\infty} e\left(\frac{\alpha a + \beta d}{c}\right).$$

Resuming our analysis, we obtain the standard Kloosterman sum associated simply to the cusp at ∞ for the group $\Gamma_0(P)$ in (3.6). Previously, P was attached to $(n - m)$ and not the modulus q, which gave a Kloosterman sum associated with the cusps at 0 and ∞ for $\Gamma_0(P)$ in (3.5).

Finally, we should mention that we can improve the hybrid range further by using a large sieve inequality as in [DesIw82] to estimate the sums of Kloosterman sums instead of bounding them individually. Even using the Kuznetsov formula with a large sieve inequality would improve the subconvexity estimate if not the range of subconvexity. One could also introduce an amplifier to improve the the range $M^\eta \ll P \ll M^{2/5-\eta}$ to the left to cover the case $P \sim 1$. Since our purpose is simply to demonstrate the utility of the modified delta method in improving the range of subconvexity and simplifying the Kloosterman sum structure, we do not continue the argument in these ways.

4. Lemmas: summation formulas, the delta method

Before beginning the proof of the theorem, we collect several lemmas which will be employed in the proof. The crux of the work is an application of the delta method, which we state below. The delta method was used in [DFIw93]. This is a decomposition of the δ-symbol via a character sum. We utilize the version given by Heath-Brown [He96].

Lemma 4.1. (The delta method, [He96]) *For any $Q > 1$ there exist $c_Q > 0$ and a smooth function $g(x, y)$ defined on $(0, \infty) \times \mathbb{R}$, such that*

$$\delta(n, 0) = \frac{c_Q}{Q^2} \sum_{q=1}^{\infty} \sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{an}{q}\right) g\left(\frac{q}{Q}, \frac{n}{Q^2}\right).$$

The constant c_Q satisfies $c_Q = 1 + O(Q^{-A})$ for any $A > 0$. Moreover, $g(x, y) \ll x^{-1}$ for all y, and $g(x, y)$ is non-zero only for $x \le \max\{1, 2|y|\}$. If $x \le 1$ and $2|y| \le x$, then

$$x^i \partial^i x^i g(x, y) \ll x^{-1}, \qquad \partial y g(x, y) = 0.$$

If $2|y| > x$, then

$$x^i y^j \partial^i x^j \partial^j y^j g(x, y) \ll x^{-1}.$$

Let k be a positive integer. We recall elementary properties of the Bessel functions as can be seen in [WhWa96]. The Bessel function of the first kind with order k can be decomposed as

$$J_k(x) = e^{ix} W_k(x) + e^{-ix} \overline{W_k(x)}$$

where

$$W_k(x) = \frac{e^{i(\frac{\pi}{2}k - \frac{\pi}{4})}}{\Gamma(k+\frac{1}{2})}\sqrt{\frac{2}{\pi x}}\int_0^\infty e^{-y}\left(y\left(1+\frac{iy}{2x}\right)\right)^{k-\frac{1}{2}} dy.$$

Applying the asymptotic expansion for Whittaker functions W for $x \gg 1$, we have

$$x^j W_k^{(j)}(x) \ll_{k,j} \frac{1}{(1+x)^{\frac{1}{2}}}$$

with $j \geq 0$. Using the Taylor series expansion for $x \ll 1$, we obtain the bound

$$x^j J_k^{(j)}(x) \ll_{k,j} x^k$$

with $j \geq 0$. The Bessel function is used in the integral transform found in Voronoi summation.

Lemma 4.2. (Voronoi summation, [KoMiV02]) *Let $(a,q)=1$ and let h be a smooth function compactly supported in $(0,\infty)$. Let P denote the prime and let f denote a holomorphic newform of level P and weight k. Set $P_2 := P/(P,q)$. Then there exists a complex number η of modulus 1 (depending on a,q,f) and a newform f^* of the same level P and same weight k such that*

$$\sum_n \lambda_f(n) e\left(n\frac{a}{q}\right) h(n) = \frac{2\pi\eta}{q\sqrt{P_2}} \sum_n \lambda_{f^*}(n) e\left(-n\frac{\overline{aP_2}}{q}\right) \int_0^\infty h(y) J_{k-1}\left(\frac{4\pi\sqrt{ny}}{q\sqrt{P_2}}\right) dy.$$

In general, given $e(\bar{a}/q)$ the overline on the numerator a indicates the multiplicative inverse modulo the denominator q. In order to truncate and bound the sums which result after Voronoi summation, we use the following lemma.

Lemma 4.3. *Let k, M, P be positive integers with $k \geq 2$ and let r be a nonzero integer. Take $Q > 1$ and $X, Y \geq 1$. For any $a, b > 0$, define*

$$J(a,b\,;c) := \int_0^\infty \int_0^\infty \frac{1}{\sqrt{xy}} F\left(\frac{x}{X},\frac{y}{Y}\right) g\left(\frac{qc}{Q}, \frac{x-y+rM}{PQ^2}\right) J_{k-1}(4\pi a\sqrt{x}) J_{k-1}(4\pi b\sqrt{y})\, dx dy,$$

where $g\left(\frac{qc}{Q}, \frac{x-y+rM}{PQ^2}\right)$ is the function in lemma 4.1 and F is a smooth function supported in $[1/2, 5/2] \times [1/2, 5/2]$ with partial derivatives satisfying

$$x^i y^j \partial^i x^i \partial^j y^j F(x,y) \ll Z Z_x^i Z_y^j$$

for some $Z > 0$, $Z_x, Z_y \gg 1$. We have

$$J(a,b\,;c) \ll Z\sqrt{XY}\frac{Q}{qc}\frac{1}{(1+a\sqrt{X})^{1/2}}\frac{1}{(1+b\sqrt{Y})^{1/2}}\left[\frac{1}{a\sqrt{X}}\left(Z_x + \frac{X}{qcQP}\right)\right]^i \left[\frac{1}{b\sqrt{Y}}\left(Z_y + \frac{Y}{qcQP}\right)\right]^j. \tag{4.7}$$

Also,

$$J(a,b\,;c) \ll \frac{Z}{ab(1+a\sqrt{X})^{1/2}(1+b\sqrt{Y})^{1/2}}\frac{Q}{qc}\min\left\{Z_x b\sqrt{Y}, Z_y a\sqrt{X}\right\}(XY|r|MQqP)^{\varepsilon}. \tag{4.8}$$

Proof. Starting with the change of variables $x \mapsto xX$ and $y \mapsto yY$ and then integrating by parts and using the bound of the Bessel function gives (4.7). For example, integrating by parts once in the x integral leads to

$$J(a,b\,;c) \ll Z\sqrt{XY}\frac{Q}{qc}\frac{1}{(1+a\sqrt{X})^{1/2}}\frac{1}{(1+b\sqrt{Y})^{1/2}}\left[\frac{1}{a\sqrt{X}}(Z_x + XI)\right],$$

where

$$I := \int_{1/2}^{5/2}\int_{1/2}^{5/2}_{2|xX-yY+rM|>qQP} \frac{1}{|xX-yY+rM|}\,dxdy.$$

The condition on the integral comes from where the g function is nonzero. However, we have $|xX - yY + rm|^{-1} \ll (qcQP)^{-1}$, which gives (4.7) for $i = 1$ and $j = 0$. Repeated integration by parts gives the same result for higher values of i and j.

For (4.8), we treat the integral I differently. Let $u = xX - yYrM$ to get

$$I \ll \frac{1}{X}\int_{1/2}^{5/2}\int_{qQP/2}^{(X+Y+|r|M)} \frac{1}{u}dudy \ll \frac{(XY|r|MqQP)^{\varepsilon}}{X}.$$

Doing the same thing with $i = 0$ and $j = 1$ and taking the minimum of the two bounds finishes the proof.

5. Reduction of the second moment to shifted sums

Let $\varepsilon,\ \delta > 0$ and choose any X such that $\mathcal{Q}^{1/2-\delta} \le X \le \mathcal{Q}^{1/2+\varepsilon}$. Define

$$S_f(X) := \frac{1}{\varphi^{\star}(M)} \sideset{}{^\star}\sum_{\chi \mod M} |L_{f\otimes\chi}(X)|^2,$$

where $L_{f\otimes\chi}(X)$ is given in (1.1). The proof of the reduction to shifted sums is similar to [DFIw93]. We open the square and write the primitive characters in terms of Gauss sums to obtain

$$S_f(X) = \frac{1}{M\varphi^{\star}(M)} \sideset{}{^\star}\sum_{\chi \mod M} \left| \sum_{b \mod M} \overline{\chi(b)} \sum_n \frac{\lambda_f(n)}{\sqrt{n}} h\left(\frac{n}{X}\right) e\left(\frac{nb}{M}\right)\right|^2.$$

Adding the nonprimitive characters to this sum produces the summation over all characters modulo M. By the orthogonality of multiplicative characters,

$$S_f(X) \le \frac{\varphi(M)}{M\varphi^{\star}(M)} \sideset{}{^\star}\sum_{b \mod M} \left|\sum_n \frac{\lambda_f(n)}{\sqrt{n}} h\left(\frac{n}{X}\right) e\left(\frac{nb}{M}\right)\right|^2.$$

We extend the summation to all residue classes M by adding the residues which are not coprime with M, and open the square. Using the orthogonality of additive characters,

$$S_f(X) \ll M^{\varepsilon} \left|\sum_n \frac{\lambda_f(n)}{\sqrt{n}} h\left(\frac{n}{X}\right) \sum_{m\equiv n \mod M} \frac{\lambda_f(m)}{\sqrt{m}} h\left(\frac{m}{X}\right)\right|.$$

Write $m = n + rM$ and note that $r \ll \frac{X}{M} \le \frac{\mathcal{Q}^{\frac{1}{2}+\varepsilon}}{M}$. The diagonal term $m = n$ satisfies

$$\sum_n \frac{\lambda_f(n)^2}{n} h\left(\frac{n}{X}\right)^2 \ll \mathcal{Q}^{\varepsilon}.$$

We are left to consider the off-diagonal terms

$$R_f(X) := \sum_{0\neq|r|\ll\frac{\mathcal{Q}^{1/2+\varepsilon}}{M}} \sum_n \frac{\lambda_f(n)}{\sqrt{n}} h\left(\frac{n}{X}\right) \sum_{m=n+rM} \frac{\lambda_f(m)}{\sqrt{m}} h\left(\frac{m}{X}\right).$$

6. Treatment of shifted convolution sums

Let $X, Y \geq 1$. Motivated by the reduction of the second moment problem to bounding the shifted convolution sums, we now take f_1, f_2 to newforms in $B_k^*(P)$ and consider

$$S_{f_1,f_2}(X,Y) := \sum_n \sum_{m=n+rM} \frac{\lambda_{f_1}(n)\lambda_{f_2}(m)}{\sqrt{nm}} F\left(\frac{n}{X}, \frac{m}{Y}\right),$$

where r is a nonzero integer coprime to P (valid in our specific application, where $|r| \ll P^{1/2+\varepsilon}$). In this section we establish theorem 2.2.

6.A. Modified delta method

We start with detecting the equation $m = n + rM$ in $S_{f_1,f_2}(X,Y)$. In this regard, we note that $n - m + rM = 0$ is equivalent to $(n - m + rM)/P = 0$ and $n - m + rM \equiv 0$ modulo P. Therefore,

$$S_{f_1,f_2}(X,Y) = \sum\sum_{n-m+rM\equiv 0 \mod P} \frac{\lambda_{f_1}(n)\,\lambda_{f_2}(m)}{\sqrt{nm}} F\left(\frac{n}{X}, \frac{m}{Y}\right) \delta\left(\frac{n-m+rM}{P}, 0\right).$$

Using lemma 4.1 to detect $(n - m + rM)/P = 0$ and a sum of additive characters to detect the congruence gives

$$\begin{aligned} S_{f_1,f_2}(X,Y) =& \sum_n \frac{\lambda_{f_1}(n)}{\sqrt{n}} \sum_m \frac{\lambda_{f_2}(m)}{\sqrt{m}} F\left(\frac{n}{X}, \frac{m}{Y}\right) \frac{1}{P} \sum_{b \mod P} e\left(\frac{b(n-m+rM)}{P}\right) \\ &\times \frac{c_Q}{Q^2} \sum_{q=1}^{\infty} \sideset{}{^\star}\sum_{a \mod q} e\left(\frac{a(n-m+rM)}{Pq}\right) g\left(\frac{q}{Q}, \frac{n-m+rM}{PQ^2}\right) \\ =& \frac{c_Q}{PQ^2} \sum_{q=1}^{\infty} \sideset{}{^\star}\sum_{a \mod q} \sum_{b \mod P} e\left(\frac{rM(a+bq)}{qP}\right) \sum_n \frac{\lambda_{f_1}(n)}{\sqrt{n}} e\left(\frac{n(a+bq)}{qP}\right) \\ &\times \sum_m \frac{\lambda_{f_2}(m)}{\sqrt{m}} e\left(-\frac{m(a+bq)}{qP}\right) F\left(\frac{n}{X}, \frac{m}{Y}\right) g\left(\frac{q}{Q}, \frac{n-m+rM}{PQ^2}\right). \end{aligned}$$

From lemma 4.1, $g\left(\frac{q}{Q}, \frac{n-m+rM}{PQ^2}\right) \neq 0$ for $\frac{q}{Q} \leq \max\left\{1, \frac{2|n-m+rM|}{PQ^2}\right\}$. We want to choose Q such that the max is always 1. To do this, note that

$$2\frac{|n-m+rM|}{PQ^2} \leq 2\frac{X+Y+|r|M}{PQ^2}.$$

For there to be any solutions to $m - n + rM = 0$, we need $|r| \leq (X + Y)/M$. Therefore it is sufficient to choose Q such that

$$8\frac{\max\{X,Y\}}{PQ^2} \leq 1.$$

Set $Q^2 := 8\frac{\max\{X,Y\}}{P}$ so that the outer q-sum only extends to Q. We write $\gamma = a + bq$. Since the a- and b-sums yield a complete set of residue γ modulo qP with $(\gamma, q) = 1$, $S_{f_1,f_2}(X,Y)$ reduces to

$$\frac{c_Q}{PQ^2} \sum_{q=1}^{Q} \sum_{\substack{\gamma \mod qP \\ (\gamma,q)=1}} e\left(\frac{rM\gamma}{qP}\right) \sum_n \frac{\lambda_{f_1}(n)}{\sqrt{n}} e\left(\frac{n\gamma}{qP}\right) \sum_m \frac{\lambda_{f_2}(m)}{\sqrt{m}} e\left(-\frac{m\gamma}{qP}\right) F\left(\frac{n}{X}, \frac{m}{Y}\right) g\left(\frac{q}{Q}, \frac{n-m+rM}{PQ^2}\right).$$

6.B. Voronoi summation

We apply Voronoi summation to $S_{f_1,f_2}(X,Y)$ in the m-sum, then in the n-sum. Since we are dealing with forms of prime level P, we break the q-sum above as $S_{f_1,f_2}(X,Y)=\mathcal{S}+\mathcal{T}$, where $\mathcal{S}$ is the sum over $(q,P)=1$ and $\mathcal{T}$ is the sum over $P\mid q$.

6.B.a. $(q,P)=1$ case

We assume that $(q,P)=1$. We proceed with the argument by considering two cases according to the divisibility of γ by P. To this end, we write $\mathcal{S}$ as $\mathcal{S}=\mathcal{S}_1+\mathcal{S}_2$, where the γ-sum in $\mathcal{S}_1$ runs over $(P,\gamma)=1$ and the γ-sum in $\mathcal{S}_2$ is taken over $P\mid\gamma$. When $(P,\gamma)=1$, we get the sum

$$\begin{aligned}\mathcal{S}_1 :=& \frac{c_Q}{PQ^2}\sum_{\substack{q=1\\(q,P)=1}}^{Q}\ \sideset{}{^\star}\sum_{\gamma \mod qP} e\left(\frac{rM\gamma}{qP}\right)\sum_n \frac{\lambda_{f_1}(n)}{\sqrt{n}}e\left(\frac{n\gamma}{qP}\right)\\ &\times\sum_m \frac{\lambda_{f_2}(m)}{\sqrt{m}}e\left(-\frac{m\gamma}{qP}\right)F\left(\frac{n}{X},\frac{m}{Y}\right)g\left(\frac{q}{Q},\frac{n-m+rM}{PQ^2}\right).\end{aligned}$$

For each q, applying lemma 4.2 first in the n-sum and then in the m-sum results in

$$\frac{1}{P^2q^2}\sum_n \lambda_{f_1^*}(n)e\left(\frac{-n\overline{\gamma}}{Pq}\right)\sum_m \lambda_{f_2^*}(m)e\left(\frac{m\overline{\gamma}}{Pq}\right)J\left(\frac{\sqrt{n}}{Pq},\frac{\sqrt{m}}{Pq};1\right),$$

where J is again the function given in lemma 4.3. This produces a Kloosterman sum modulo Pq:

$$\frac{1}{Q^2P^3}\sum_{\substack{q=1\\(q,P)=1}}^{Q}\frac{1}{q^2}\sum_n \lambda_{f_1^*}(n)\sum_m \lambda_{f_2^*}(m)\ J\left(\frac{\sqrt{n}}{Pq},\frac{\sqrt{m}}{Pq};1\right)S(rM,m-n;Pq). \tag{6.9}$$

Using the first bound of the J-function in lemma 4.3 allows us to truncate the n- and m-sums to the ranges

$$n\le T_1 := \frac{P^2q^2}{X}\left(Z_x+\frac{X}{qQP}\right)^2,\quad m\le T_2 := \frac{P^2q^2}{Y}\left(Z_y+\frac{Y}{qQP}\right)^2.$$

Applying the Weil bound and the second bound in lemma 4.3, we see that

$$\begin{aligned}\mathcal{S}_1 \ll& \frac{Q^\varepsilon}{Q^2P^3}\sum_{\substack{q=1\\(q,P)=1}}^{Q}\frac{1}{q^2}\sum_{n\le T_1}\sum_{m\le T_2}\frac{Z}{\frac{\sqrt{n}}{Pq}\frac{\sqrt{m}}{Pq}\left(1+\frac{\sqrt{nX}}{Pq}\right)^{\frac12}\left(1+\frac{\sqrt{mY}}{Pq}\right)^{\frac12}}\\ &\times\frac{Q}{q}\min\left\{Z_x\frac{\sqrt{mY}}{Pq},Z_y\frac{\sqrt{nX}}{Pq}\right\}(rM,m-n,Pq)^{\frac12}(Pq)^{\frac12}.\end{aligned}$$

We bound the minimum by taking the geometric mean and simplify to

$$\begin{aligned}\mathcal{S}_1 &\ll Q^\varepsilon Z(Z_xZ_y)^{\frac12}\frac{1}{QP^{\frac12}}\sum_{\substack{q=1\\(q,P)=1}}^{Q}\frac{1}{q^{\frac12}}\sum_{n\le T_1}\sum_{m\le T_2}\frac{1}{(nm)^{\frac12}}(rM,m-n,Pq)^{\frac12}\\ &\ll Q^\varepsilon Z(Z_xZ_y)^{\frac12}\frac{1}{QP^{\frac12}}\sum_{\substack{q=1\\(q,P)=1}}^{Q}\frac{1}{q^{\frac12}}(T_1T_2)^{\frac12}(rM,m-n,Pq)^{\frac12}\\ &\ll Q^\varepsilon Z(Z_xZ_y)^{\frac12}\frac{1}{(XY)^{\frac12}}\frac{P^{\frac32}}{Q}\sum_{\substack{q=1\\(q,P)=1}}^{Q}q^{\frac32}\left(Z_x+\frac{X}{qQP}\right)\left(Z_y+\frac{Y}{qQP}\right)(rM,m-n,Pq)^{\frac12}.\end{aligned}$$

Notice that $(rM, m-n, Pq) \leq (rM, Pq) \leq (rM, q)$, since r and P are coprime, and P does not divide M. Rewriting q as qd with $d = (rM, q)$ gives

$$\begin{aligned}
\mathcal{S}_1 &\ll \mathcal{Q}^{\varepsilon} Z (Z_x Z_y)^{\frac{1}{2}} \frac{1}{(XY)^{\frac{1}{2}}} \frac{P^{\frac{3}{2}}}{Q} \sum_{d \mid rM} d^{\frac{1}{2}} \sum_{q \leq Q/d} (qd)^{\frac{3}{2}} \left(Z_x + \frac{X}{qdQP} \right) \left(Z_y + \frac{Y}{qdQP} \right) \\
&\ll \mathcal{Q}^{\varepsilon} Z (Z_x Z_y)^{\frac{1}{2}} \frac{1}{(XY)^{\frac{1}{2}}} P^{\frac{3}{2}} Q^{\frac{3}{2}} \sum_{d \mid rM} \frac{1}{d^{\frac{1}{2}}} \left(Z_x + \frac{X}{Q^2 P} \right) \left(Z_y + \frac{Y}{Q^2 P} \right) \\
&\ll \mathcal{Q}^{\varepsilon} r^{\varepsilon} Z (Z_x Z_y)^{\frac{1}{2}} \frac{1}{(XY)^{\frac{1}{2}}} P^{\frac{3}{2}} Q^{\frac{3}{2}} \left(Z_x + \frac{X}{Q^2 P} \right) \left(Z_y + \frac{Y}{Q^2 P} \right) \\
&\ll \mathcal{Q}^{\varepsilon} r^{\varepsilon} Z (Z_x Z_y)^{\frac{1}{2}} \frac{1}{(XY)^{\frac{1}{2}}} P^{\frac{3}{2}} Q^{\frac{3}{2}} \left(\max\{Z_x, Z_y\} + \frac{\max\{X, Y\}}{Q^2 P} \right)^2 .
\end{aligned}$$

With $Q^2 = 8 \frac{\max\{X,Y\}}{P}$, we obtain

$$\mathcal{S}_1 \ll \mathcal{Q}^{\varepsilon} r^{\varepsilon} Z (Z_x Z_y)^{\frac{1}{2}} \max\{Z_x, Z_y\}^2 P^{\frac{3}{4}} \frac{\max\{X, Y\}^{\frac{3}{4}}}{(XY)^{\frac{1}{2}}}.$$

This matches the bound in the statement of theorem 2.2.

Next we deal with the sum where $P \mid \gamma$.

$$\begin{aligned}
\mathcal{S}_2 :=& \frac{c_Q}{PQ^2} \sum_{\substack{q=1 \\ (q,P)=1}}^{Q} \sum_{\substack{\gamma \mod qP \\ (\gamma,q)=1 \\ P \mid \gamma}} e\left(\frac{rM\gamma}{qP} \right) \sum_n \frac{\lambda_{f_1}(n)}{\sqrt{n}} e\left(\frac{n\gamma}{qP} \right) \\
& \times \sum_m \frac{\lambda_{f_2}(m)}{\sqrt{m}} e\left(-\frac{m\gamma}{qP} \right) F\left(\frac{n}{X}, \frac{m}{Y} \right) g\left(\frac{q}{Q}, \frac{n-m+rM}{PQ^2} \right) \\
=& \frac{c_Q}{PQ^2} \sum_{\substack{q=1 \\ (q,P)=1}}^{Q} \sideset{}{^\star}\sum_{\gamma \mod q} e\left(\frac{rM\gamma}{q} \right) \sum_n \frac{\lambda_{f_1}(n)}{\sqrt{n}} e\left(\frac{n\gamma}{q} \right) \\
& \times \sum_m \frac{\lambda_{f_2}(m)}{\sqrt{m}} e\left(-\frac{m\gamma}{q} \right) F\left(\frac{n}{X}, \frac{m}{Y} \right) g\left(\frac{q}{Q}, \frac{n-m+rM}{PQ^2} \right).
\end{aligned}$$

Applying Voronoi summation in the m- and n-sums gives

$$\frac{1}{P^2 Q^2} \sum_{q=1}^{Q} \frac{1}{q^2} \sum_n \lambda_{f_1^*}(n) \sum_m \lambda_{f_2^*}(m) J\left(\frac{\sqrt{n}}{q\sqrt{P}}, \frac{\sqrt{m}}{q\sqrt{P}}; 1 \right) S(rM, (m-n)\bar{P}; q). \tag{6.10}$$

The n- and m-sums can be truncated to

$$n \leq T_1 := \frac{Pq^2}{X} \left(Z_x + \frac{X}{qQP} \right)^2, \quad m \leq T_2 := \frac{Pq^2}{Y} \left(Z_y + \frac{Y}{qQP} \right)^2.$$

Repeating the estimation process used for $\mathcal{S}_1$ with these parameters, one is led to

$$\mathcal{S}_2 \ll \mathcal{Q}^{\varepsilon} r^{\varepsilon} Z (Z_x Z_y)^{\frac{1}{2}} \max\{Z_x, Z_y\}^2 \frac{1}{P^{\frac{3}{4}}} \frac{\max\{X, Y\}^{\frac{3}{4}}}{(XY)^{\frac{1}{2}}}.$$

This is better than the bound of theorem 2.2. We note that we get Kloosterman sums attached to cups 0 and ∞ in (6.10), but the contribution of the case when $P \mid \gamma$ comes with a smaller weight. However we obtain Kloosterman sums attached to cups 0 and 0 in (6.9) as it is stated in section 3.

6.B.b. $P \mid q$ case

Finally, observing that γ is coprime with Pq, the remaining sum over q, when $P \mid q$ is

$$\mathcal{T} := \frac{c_Q}{PQ^2} \sum_{\substack{q=1\\P|q}}^{Q} \sideset{}{^\star}\sum_{\gamma \bmod Pq} e\left(\frac{rM\gamma}{Pq}\right) \sum_n \frac{\lambda_{f_1}(n)}{\sqrt{n}} e\left(\frac{n\gamma}{Pq}\right)$$
$$\times \sum_m \frac{\lambda_{f_2}(m)}{\sqrt{m}} e\left(-\frac{m\gamma}{Pq}\right) F\left(\frac{n}{X}, \frac{m}{Y}\right) g\left(\frac{q}{Q}, \frac{n-m+rM}{PQ^2}\right).$$

After we rewrite q as qP, Voronoi summation in the m-sum then the n-sum gives

$$\frac{1}{Q^2P^5} \sum_{q \le Q/P} \frac{1}{q^2} \sum_n \sum_m \lambda_{f_1^*}(n) \lambda_{f_2^*}(m) J\left(\frac{\sqrt{n}}{P^2q}, \frac{\sqrt{m}}{P^2q}; P\right) S(rM, m-n; P^2q).$$

Lemma 4.3 implies that we can truncate the n- and m-sums to the ranges

$$n \le T_1 := \frac{P^4q^2}{X}\left(Z_x + \frac{X}{qQP^2}\right)^2, \quad m \le T_2 := \frac{P^4q^2}{Y}\left(Z_y + \frac{Y}{qQP^2}\right)^2.$$

Using the Weil bound for Kloosterman sums as in the previous case, we see that

$$\mathcal{T} \ll \mathcal{Q}^\varepsilon r^\varepsilon Z (Z_x Z_y)^{\frac{1}{2}} \max\{Z_x, Z_y\}^2 \frac{1}{P^{\frac{1}{4}}} \frac{\max\{X,Y\}^{\frac{3}{4}}}{(XY)^{\frac{1}{2}}}.$$

This bound on $\mathcal{T}$ is better than the bound on $\mathcal{S}_1$. This completes the proof of theorem 2.2.

Acknowledgments. This work was completed as an extension of a topics course given by Roman Holowinsky at The Ohio State University in Spring 2015. We are very grateful to Roman Holowinsky for a careful reading of the manuscript. We thank the reviewer for their thorough review and highly appreciate the comments and suggestions, which contributed to improving the quality of the publication. The authors would also like to thank Fei Hou for identifying an error in an earlier version of this paper.

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Keshav Aggarwal
Department of Mathematics
The Ohio State University, Columbus
Ohio, USA 43210-1174
e-mail: aggarwal.78@osu.edu

Yeongseong Jo
Department of Mathematics
The Ohio State University, Columbus
Ohio, USA 43210-117
e-mail: jo.59@osu.edu
and
Department of Mathematics
The University of Iowa, Iowa City
Iowa, USA 52242-1419
e-mail: yeongseong-jo@uiowa.edu

Kevin Nowland
Department of Mathematics
The Ohio State University, Columbus
Ohio, USA 43210-117
e-mail: nowland.159@osu.edu