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Jean-Marc Deshouillers

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A remark on cube-free numbers in Segal-Piatestki-Shapiro sequences

Jean-Marc Deshouillers

To the memory of S. Srinivasan

Abstract. Using a method due to G. J. Rieger, we show that for $1 < c < 2$ one has, as x tends to infinity

$$\mathrm{Card}\{n \le x \colon \lfloor n^c \rfloor \text{ is cube-free } \} = \frac{x}{\zeta(3)} + O\left(x^{(c+1)/3}\log x\right),$$

thus improving on a recent result by Zhang Min and Li Jinjiang.

Keywords. Segal-Piatetski-Shapiro sequences, cube-free numbers, estimation of trigonometric sums, discrepancy

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1. Introduction

The Segal-Piatetski-Shapiro sequence of parameter c (in short SPS_c) is the sequence $(\lfloor n^c \rfloor)_n$, where $\lfloor u \rfloor$ denotes the integral part of the real number u. Those sequences have been introduced by B. I. Segal in 1933 [Se33] who studied their additive properties. In 1953, I. I. Piatestki-Shapiro [Pi53] proved that for $1 < c < 12/11$, the sequence SPS_c contains infinitely many primes, with the expected density.

Of interest for us, here, we recall some steps in the study of squarefree and cube-free numbers in the sequence SPS_c. To our knowledge, I. E. Stux [St75] was the first to study the squarefree numbers in SPS sequences; in 1975, he showed a result which among others implies that, as x tends to infinity, one has

$$\mathrm{Card}\{n \le x \colon \lfloor n^c \rfloor \text{ is squarefree}\} = (\zeta(2)^{-1} + o(1))x \quad \text{for} \quad 1 < c < 4/3. \tag{1.1}$$

Shortly after, G. J. Rieger [Ri78] proved that (1.1) holds true for $1 < c < 3/2$ with an explicit error term implying a polynomial saving and mentioned that his approach could be extended to cube-free values. In 1998, X-D. Cao and W-G. Zhai [CaZh98] improved the range of validity of (1.1) to $1 < c < 61/36$. In 2008, [CaZh08], they showed that for $1 < c < 149/87$, the sequence SPS_c contains infinitely many squarefree numbers.

Building on the method of Cao and Zhai, M. Zhang and J-J. Li proved a very general result which implies that for any ϵ less that 10^{-10}, one has

$$\mathrm{Card}\{n \le x \colon \lfloor n^c \rfloor \text{ is cube-free}\} = (\zeta(3)^{-1} + o(x^{-\epsilon}))x \text{ for } 1 < c < 11/6. \tag{1.2}$$

Using the approach of Rieger, we prove the following

Theorem 1. *For $1 < c < 2$ one has, as x tends to infinity*

$$\mathrm{Card}\{n \le x \colon \lfloor n^c \rfloor \textit{ is cube-free}\} = \frac{x}{\zeta(3)} + O\left(x^{(c+1)/3}\log x\right). \tag{1.3}$$

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The key ingredient in the proof is a good estimate for the number of integers n up to x such that $\lfloor n^c \rfloor$ belongs to an arithmetic progression. Such a result has been announced in [Des73, Théorème 1] which is equivalent to the following statement

Claim 1. *Let c be in $(1,2)$, x be a real number and q and a be two integers such that $0 \le a < q \le x^c$. One has*

$$\left| N_c(x;q,a) - \frac{x}{q} \right| \ll_c \min\left(\frac{x^c}{q}, \frac{x^{(c+1)/3}}{q^{1/3}}, \frac{x^{(c+4)/7}}{q^{1/7}} \right), \tag{1.4}$$

where

$$N_c(x;q,a) = \mathrm{Card}\{n \le x\colon\ \lfloor n^c \rfloor \equiv a \pmod q\}. \tag{1.5}$$

For proving Theorem 1, we simply need the weaker form

Theorem 2. *Let c be in $(1,2)$, x be a real number and q and a be two integers such that $0 \le a < q \le x^c$. One has*

$$\left| N_c(x;q,a) - \frac{x}{q} \right| \ll_c \min\left(\frac{x^c}{q}, \frac{x^{(c+1)/3}}{q^{1/3}} \right). \tag{1.6}$$

Since the *note* [Des73] does not give complete proofs, we shall prove Theorem 2 in Section 3. and give a hint on how to prove Claim 1. In the next Section, we show how to deduce Theorem 1 from Theorem 2.

We end this introduction with a general remark concerning the study of the intersection of SPS_c with a given sequence $\mathcal{A}$: one can use a *direct* approach, namely starting with the elements of SPS_c and asking whether they are in $\mathcal{A}$, or an *inverse* one: starting with the elements of $\mathcal{A}$ and asking whether they are in SPS_c. The popularity of the inverse approach may be due to the fact that it is - up to now - the only one to be successful for detecting primes in SPS_c: indeed, Piatetski-Shapiro was looking at elements of SPS_c in the sequence of primes! In the above-mentioned results, Stux uses the inverse approach, whereas Rieger uses the direct one; Cao and Zhai, as well as Zhang and Li, use the inverse one and we use Rieger's, i.e. the direct one. By the way, in the study of the distribution of SPS_c in arithmetic progressions, [Des73] uses the direct method and improves on previous results by Rieger [Ri67] and Somayajulu [So71].

2. Proof of Theorem 1

Let χ_3 denote the indicator function of cube-free numbers. It is readily seen that one has

$$\chi_3(n) = \sum_{d^3 \mid n} \mu(d);$$

indeed both terms are mutiplicative functions of n which coincide on prime powers. Let us denote by $A_c(x)$ the number of integers n at most equal to x such that $\lfloor n^c \rfloor$ is cube-free. We have

$$A_c(x) = \sum_{n \le x} \sum_{d^3 \mid \lfloor n^c \rfloor} \mu(d) = \sum_{d \le x^{c/3}} \mu(d) N_c(x; d^3, 0).$$

Using Theorem 2, we get

$$A_c(x) = x \sum_{d \le x^{c/3}} \mu(d)/d^3 + O\left(\sum_{d \le x^{c/3}} \min\left(\frac{x^c}{d^3}, \frac{x^{(c+1)/3}}{d} \right) \right). \tag{2.7}$$

For the first sum, we have

$$x \sum_{d \le x^{c/3}} \mu(d)/d^3 - \zeta(3)^{-1} x = O\left(x \sum_{d > x^{c/3}} d^{-3} \right) = O\left(x^{1-2c/3} \right). \tag{2.8}$$

We break the sum in the error term of (2.7) at $x^{c/3-1/6}$ and get

$$\sum_{d \le x^{c/3}} \min\left(\frac{x^c}{d^3}, \frac{x^{(c+1)/3}}{d} \right) \le \sum_{d \le x^{c/3-1/6}} \frac{x^{(c+1)/3}}{d} + \sum_{d > x^{c/3-1/6}} \frac{x^c}{d^3} \ll x^{(c+1)/3} \log x. \tag{2.9}$$

Theorem 1 simply comes from (2.7), (2.8) and (2.9).

3. Proof of Theorem 2

3.A. The case of a dyadic interval

The key ingredient in the proof of Theorem 2 is the following.

Proposition 1. *Let c be in $(1,2)$. There exists a real K_c such that for any real number M and integers a and q satisfying*

$$0 \le a < q < M^{c-1/2}, \tag{3.10}$$

one has

$$\left| \sum_{\substack{M < m \le 2M \\ \lfloor m^c \rfloor \equiv a \bmod q}} 1 - \frac{M}{q} \right| \le K_c \frac{M^{(c+1)/3}}{q^{1/3}}. \tag{3.11}$$

Proof. One has the equivalence

$$\lfloor n^c \rfloor \equiv a \pmod q \;\Leftrightarrow\; \left\{ \frac{n^c}{q} \right\} \in \left[\frac{a}{q}, \frac{a+1}{q} \right),$$

where $\{u\}$ denotes the fractional part of the real number u. Thanks to the Erdős-Turán inequality (in the form given in [Mon94]), we have for any positive integer H

$$\left| \sum_{\substack{M < m \le 2M \\ \lfloor m^c \rfloor \equiv a \bmod q}} 1 - \frac{M}{q} \right| \le \frac{M}{H+1} + 2 \sum_{h=1}^{H} \left(\frac{1}{H+1} + \min\left(\frac{1}{q}, \frac{1}{\pi h} \right) \right) \left| \sum_{M < m \le 2M} e\left(\frac{h n^c}{q} \right) \right|, \tag{3.12}$$

where $e(u) = \exp(2\pi i u)$. For the estimation of the trignometric sum occuring in (3.12), we use one of the first results of van der Corput (cf. [GrKo91], p. 8, Theorem 2.2). Since the second derivative of the function $t \mapsto (h/q)t^c$ is $t \mapsto c(c-1)(h/q)t^{c-2}$, van der Corput's theorem leads to

$$\sum_{M < m \le 2M} e\left(\frac{h n^c}{q} \right) \ll_c (h/q)^{1/2} M^{c/2} + (q/h)^{1/2} M^{1-c/2}. \tag{3.13}$$

We let

$$H = \lfloor q^{1/3} M^{(2-c)/3} \rfloor. \tag{3.14}$$

We have

$$M/(H+1) \le M^{(c+1)/3}/q^{1/3},$$

which leads to an admissible contribution to (3.11).

For the contribution of the second part of (3.12), we consider two cases according to the values of q.

First case, $q < M^{1-c/2}$. This inequality is equivalent to $q < H$. We break the sum over h in (3.12) at q.
For the contribution of the terms $h \le q$, we use

$$\frac{1}{H+1} + \min\left(\frac{1}{q}, \frac{1}{\pi h}\right) \le \frac{2}{q}.$$

By (3.13), we have

$$\sum_{h=1}^{q} \frac{1}{q} \left| \sum_{M<m\le 2M} e\left(\frac{hn^c}{q}\right) \right| \ll_c \sum_{h=1}^{q} h^{1/2} q^{-3/2} M^{c/2} + \sum_{h=1}^{q} (qh)^{-1/2} M^{1-c/2}.$$

The first sum from the RHS has the order $M^{c/2}$ which is admissible since $q < M^{1-c/2}$. The second sum has the order $M^{1-c/2}$, which is admissible since $c > 1$. For the contribution of the terms with $q < h \le H$, we use

$$\frac{1}{H+1} + \min\left(\frac{1}{q}, \frac{1}{\pi h}\right) \le \frac{2}{h}.$$

By (3.13), we have

$$\sum_{h=q+1}^{H} \frac{1}{h} \left| \sum_{M<m\le 2M} e\left(\frac{hn^c}{q}\right) \right| \ll_c \sum_{h=q+1}^{H} (qh)^{-1/2} M^{c/2} + \sum_{h=q+1}^{H} q^{1/2} h^{-3/2} M^{1-c/2}.$$

The first sum from the RHS has the order $H^{1/2} q^{-1/2} M^{c/2} = M^{(c+1)/3}/q^{1/3}$ which is admissible. By the way, the contribution of this term explains the choice of H. The second sum has the order $M^{1-c/2}$, which is admissible, as we have seen earlier.

Thus, Proposition 1 is proved in the case $q < M^{1-c/2}$.

Second case, $q \ge M^{1-c/2}$. This inequality is equivalent to $q \ge H$. In this case, we have

$$\frac{1}{H+1} + \min\left(\frac{1}{q}, \frac{1}{\pi h}\right) \le \frac{2}{H}.$$

By (3.13), we have

$$\sum_{h=q+1}^{H} \frac{1}{H} \left| \sum_{M<m\le 2M} e\left(\frac{hn^c}{q}\right) \right| \ll_c \sum_{h=1}^{H} H^{-1} (h/q)^{1/2} M^{c/2} + \sum_{h=1}^{H} H^{-1} (q/h)^{1/2} M^{1-c/2}.$$

The first sum from the RHS has the order $H^{1/2} q^{-1/2} M^{c/2} = M^{(c+1)/3}/q^{1/3}$, which is admissible. The second sum has the order $H^{-1/2} q^{1/2} M^{1-c/2} = q^{1/3} M^{(2-c)/3}$ which is admissible since $q \le M^{c-1/2}$.

Thus Proposition 1 is also proved in the second case.

3.B. Proof of Theorem 2

The number $N_c(x;q,a)$ does not exceed the number of natural integers which are at most equal to x^c and which are congruent to a modulo q; we thus have $0 \le N_c(x;q,a) \le x^c/q$ which implies $|N_c(x;q,a) - x/q| \le 2x^c/q$. Since we have $x^{(c+1)/3}/q^{1/3} \ge 4x^c/q$ when $q \ge (1/8)x^{c-1/2}$, in order to prove Theorem 2, it is enough to prove the inequality

$$|N_c(x;q,a) - x/q| \ll_c \frac{x^{(c+1)/3}}{q^{1/3}} \tag{3.15}$$

under the condition

$$q < (1/8)x^{c-1/2} \tag{3.16}$$

which we now assume to hold.

We cover the interval $[2,x]$ with the union of intervals $(x/2^\ell, 2x/2^\ell]$, for ℓ from 1 to $L = \lfloor \log x/\log 2 \rfloor$. We define κ by the relation $x/2^\kappa = q^{2/(2c-1)}$ and check that our assumption on q implies $\kappa > 2$.

For each positive integer ℓ which is at most equal to κ, we use Proposition 1 and obtain

$$\left| \sum_{\substack{x/2^\ell < m \le 2x/2^\ell \\ \lfloor m^c \rfloor \equiv a \bmod q}} 1 - \frac{x}{2^\ell q} \right| \le K_c \frac{x^{(c+1)/3}}{(2^{\ell(c+1)}q)^{1/3}}. \tag{3.17}$$

Since the series $\sum 2^{-\ell(c+1)}$ converges, there exists a constant K'_c depending only on c such that

$$\left| \sum_{\substack{x/2^{\lfloor\kappa\rfloor} < m \le x \\ \lfloor m^c \rfloor \equiv a \bmod q}} 1 - \frac{x - x/2^{\lfloor\kappa\rfloor}}{q} \right| \le K'_c \frac{x^{(c+1)/3}}{q^{1/3}}. \tag{3.18}$$

By the definition of κ, we have

$$q = \left(\frac{x}{2^\kappa}\right)^{c-1/2} \ge \left(\frac{x}{2 \times 2^{\lfloor\kappa\rfloor}}\right)^{c-1/2} > \frac{1}{8}\left(\frac{x}{2^{\lfloor\kappa\rfloor}}\right)^{c-1/2} \tag{3.19}$$

Using the trivial estimate mentioned at the beginning of this subsection, we have

$$\left| \sum_{\substack{1 \le m \le x/2^{\lfloor\kappa\rfloor} \\ \lfloor m^c \rfloor \equiv a \bmod q}} 1 - \frac{x/2^{\lfloor\kappa\rfloor}}{q} \right| \le 2\frac{(x/2^{\lfloor\kappa\rfloor})^c}{q} \le 8\frac{(x/2^\kappa)^c}{q} = \frac{8}{q^{1/3}}\frac{(x/2^\kappa)^c}{q^{2/3}}.$$

By the definition of κ and relation (3.16), we have

$$\frac{(x/2^\kappa)^c}{q^{2/3}} \le q^{(2c+2)/(3(2c-1))} \le x^{(c+1)/3}$$

which leads to

$$\left| \sum_{\substack{1 \le m \le x/2^{\lfloor\kappa\rfloor} \\ \lfloor m^c \rfloor \equiv a \bmod q}} 1 - \frac{x/2^{\lfloor\kappa\rfloor}}{q} \right| \le 8\frac{x^{(c+1)/3}}{q^{1/3}}.$$

From the last upper bound and relation (3.18) we obtained the expected relation (3.15) under the condition (3.16), thus ending the proof of Theorem 2.

3.C. Towards Claim 1

After having proved Theorem 2, in order to prove Claim 1 it is enough to prove that for c in $(1,2)$ and $0 \le a < q \le x^{c-5/4}$, one has

$$\left| N_c(x;q,a) - \frac{x}{q} \right| \ll_c \frac{x^{(c+4)/7}}{q^{1/7}}.$$

The proof is similar to that of Theorem 2 except that we use another result of van der Corput, based on the theorem we used and the Weyl - van der Corput inequality, namely Theorem 2.6 of [GrKo91].

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Jean-Marc Deshouillers
Institut de Mathématiques de Bordeaux
Université de Bordeaux, Bordeaux INP and CNRS
33405 Talence, France
e-mail: jean-marc.deshouillers@math.u-bordeaux.fr