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Integral points on circles

A. Schinzel and M. Skalba

In memory of S. Srinivasan

Abstract. Sixty years ago the first named author gave an example [Sch58] of a circle passing through an arbitrary number of integral points. Now we shall prove: The number \(N\) of integral points on the circle \((x-a)^2 + (y-b)^2 = r^2\) with radius \(r = \frac{1}{n} \sqrt{m}\), where \(m, n \in \mathbb{Z}\), \(m, n > 0\), \(\text{gcd}(m, n^2)\) squarefree and \(a, b \in \mathbb{Q}\) does not exceed \(r(m)/4\), where \(r(m)\) is the number of representations of \(m\) as the sum of two squares, unless \(n|2\) and \(n \cdot (a, b) \in \mathbb{Z}^2\); then \(N \leq r(m)\).

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Sixty years ago the first named author gave an example [Sch58] of a circle passing through an arbitrary number of integral points. If the center of a circle is not a rational point (i.e. not both coordinates are rational numbers) then it passes through no more than 2 rational points. In fact, the equation of the perpendicular bisector of a segment joining two rational points has rational coefficients, hence the circumcenter of a triangle with rational vertices has to be rational as well. From now on we will consider only circles

\[(x-a)^2 + (y-b)^2 = r^2, \tag{0.1}\]

with \(a, b \in \mathbb{Q}\) and we shall prove

**Theorem 0.1.** The number \(N\) of integral points on the circle (0.1) with radius \(r = \frac{1}{n} \sqrt{m}\), where \(m, n \in \mathbb{Z}\), \(m, n > 0\), \(\text{gcd}(m, n^2)\) squarefree does not exceed \(r(m)/4\), where \(r(m)\) is the number of representations of \(m\) as the sum of two squares, unless \(n|2\) and \(n \cdot (a, b) \in \mathbb{Z}^2\); then \(N \leq r(m)\).

**Lemma 0.2.** Assume that \(\beta, \gamma_1, \gamma_2 \in \mathbb{Z}[i]\) and \(c \in \mathbb{N}\) satisfy

\[N(\gamma_1) = N(\gamma_2) = c^2, \tag{0.2}\]

\[\beta \gamma_1 \equiv \beta \gamma_2 \pmod{c}, \tag{0.3}\]

if a rational prime \(t\) divides \(c\) then \(t \not| \beta \gamma_1\) and \(t \not| \beta \gamma_2\). \(\tag{0.4}\)

Then \(\gamma_1 \sim \gamma_2\) in \(\mathbb{Z}[i]\).

**Proof.** We assume from the beginning that \(\gamma_1 \neq \gamma_2\).

1. **Case** \(\text{gcd}(\beta, c) = 1\): We can divide the congruence (0.3) by \(\beta\) and obtain

\[\gamma_1 - \gamma_2 = c \delta \quad \text{with} \quad \delta \in \mathbb{Z}[i], \delta \neq 0.\]

Further

\[N(\gamma_1) + N(\gamma_2) = \gamma_1 \overline{\gamma_2} - \gamma_2 \overline{\gamma_1} = c^2 N(\delta).\]

If we put \(\gamma_1 \overline{\gamma_2} = f + gi\) with \(f, g \in \mathbb{Z}\) then by equation (0.2) we obtain

\[2f = (2 - N(\delta))c^2.\]
Hence
\[ f = \frac{u}{2} \cdot c^2 \quad \text{with } u \in \mathbb{Z}, \ u \leq 1. \]

Because
\[ f^2 + g^2 = N(\gamma_1 \overline{\gamma}_2) = c^4 \quad \text{by } (0.2) \]
one obtains
\[ g^2 = c^4 - f^2 = c^4(1 - \frac{u^2}{4}). \]

It follows \( u \in \{-2, -1, 0, 1\} \) but \( u \in \{-1, 1\} \) would lead to \( g \notin \mathbb{Q} \). Hence \( u \in \{0, -2\} \).

If \( u = 0 \) then \( f = 0 \), \( g = \pm c^2 \) hence
\[ \gamma_1 \overline{\gamma}_2 = \pm c^2 i \quad \text{what gives } \gamma_1 c^2 = \pm c^2 i \gamma_2, \]
and finally \( \gamma_1 = \pm i \gamma_2 \).

If \( u = -2 \) then \( f = -c^2, g = 0 \) hence \( \gamma_1 \overline{\gamma}_2 = -c^2 \) and \( \gamma_1 = -\gamma_2 \).

2. Case \( N(\gcd(\beta, c)) = d > 1 \): We adopt inductive method and assume that the assertion of lemma holds for \( N(\gcd(\beta, c)) < d \). Let \( \pi \) be a prime element of the ring \( \mathbb{Z}[i] \) satisfying \( \pi | \beta \) and \( \pi | c \). By condition \((0.4)\) and \((0.2)\) \( N(\pi) = p \) is a rational prime of the form \( 4k + 1 \).

By \((0.2)\) \( \pi | \gamma_1 \) or \( \pi | \gamma_2 \), but the latter is excluded by \((0.4)\), hence \( \pi^{2l} | \gamma_1 \) where \( p^l | c \). In the same way \( \pi^{2l} | \gamma_2 \). Rewrite the initial equality
\[ \beta \gamma_1 - \beta \gamma_2 = \delta c \quad \text{with } \delta \in \mathbb{Z}[i], \delta \neq 0 \]
in the form
\[ \frac{\beta \gamma_1}{\pi^{2l}} - \frac{\beta \gamma_2}{\pi^{2l}} = \frac{\delta \pi^{2l}}{p^l} \cdot \frac{c}{p^l} \]
where all fractions are algebraic integers. Using the inductive assumption finishes the proof of lemma.

**Proof of Theorem.** The considered circle \((0.1)\) is given by the equation
\[ (x - a)^2 + (y - b)^2 = \frac{m}{n^2}. \quad (0.5) \]

Put \( a = A/C, b = B/C \), where \( A, B, C \in \mathbb{Z}, \ C > 0, \ (A, B, C) = 1 \). It follows that \( n | C \) and hence \( C = nc \) with \( c \in \mathbb{N} \). The number \( N \) of integral points on the circle \((0.5)\) satisfies
\[ N = \text{card}\{(x, y) \in \mathbb{Z}^2 | (Cx - A)^2 + (Cy - B)^2 = c^2 m\}. \]

Each solution \((x, y) \in \mathbb{Z}^2\) to the equation
\[ (Cx - A)^2 + (Cy - B)^2 = c^2 m \quad (0.6) \]
is encoded by the equality
\[ (Cx - A) + (Cy - B)i = \beta \cdot \gamma \quad (0.7) \]
with \( \beta, \gamma \in \mathbb{Z}[i] \) and \( N(\beta) = m, \ N(\gamma) = c^2 \).

Assume now to the contrary that the number of solutions \((x, y) \in \mathbb{Z}^2\) to the equation \((0.6)\) exceeds \( r(m)/4 \). It follows that there exist \( \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{Z}[i] \) satisfying
\[ \beta_1 \sim \beta_2, \ N(\beta_1) = N(\beta_2) = m, \ N(\gamma_1) = N(\gamma_2) = c^2, \]
\[ \gamma_1 \beta_1 \equiv \gamma_2 \beta_2 \pmod{c} \quad \text{and } \gamma_1 \beta_1 \neq \gamma_2 \beta_2. \]

Adjusting \( \gamma_1, \gamma_2 \) for a unit (if necessary) we may assume that there are \( \beta, \gamma_1, \gamma_2 \in \mathbb{Z}[i] \) satisfying
\[ N(\beta) = m, \ N(\gamma_1) = N(\gamma_2) = c^2, \ \beta_1 \equiv \beta \gamma_2 \pmod{c}, \ \gamma_1 \neq \gamma_2. \]
Now we infer by Lemma that
\[ \gamma_2 \in \{ -\gamma_1, i\gamma_1, -i\gamma_1 \}. \]
((0.4) is fulfilled by the assumption \((A, B, C) = 1\).) In all above cases we get
\[ 2\beta \gamma_1 \equiv 0 \pmod{c}. \]

For \(c > 2\) this contradicts the condition \((A, B, C) = 1\). In case \(c = 2\), for any integers \(A, B\) and \(C \equiv 0 \pmod{2}\) the conditions \((A, B, C) = 1\) and
\[ (Cx - A)^2 + (Cy - B)^2 = 4m \]
are incompatible. Concluding: \(N > r(m)/4\) is possible only for \(c = 1\). In case \(c = 1\), \(C = n\) and by (0.6) one gets \(N \leq r(m)\). It remains to deduce \(n|2\) from \(N > r(m)/4\). It follows from the last inequality that there exist integers \(x_1, x_2, y_1, y_2\) and \(k \in \{1, 2, 3\}\) satisfying
\[ (nx_2 - A) + (ny_2 - B)i = i^k[(nx_1 - A) + (ny_1 - B)i] \]
hence
\[ (1 - i^k)(A + Bi) \equiv 0 \pmod{n}. \]
It follows \(n|(2A, 2B)\) and since \((A, B, n) = 1\) we infer that \(n|2\).

Remark. The number 1/4 in our theorem is optimal and here is an example. Let \(m\) be of the form \(3k + 2\) and satisfying \(r(m) > 0\). The equality \(m = x^2 + y^2\) implies \(x \equiv \pm 1 \pmod{3}, y \equiv \pm 1 \pmod{3}\). It follows that \((x - 1/3)^2 + (y - 1/3)^2 = m/9\) has \(r(m)/4\) integer solutions.

References


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