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Explicit *abc*-conjecture and its applications

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Dedicated to the memory of Professor S. Srinivasan.

Abstract. We state well-known *abc*-conjecture of Masser-Oesterlé and its explicit version, popularly known as the explicit *abc*-conjecture, due to Baker. Laishram and Shorey derived from the explicit *abc*-conjecture that (1.1) implies that $c < N^{1.75}$. We give a survey on improvements of this result and its consequences. Finally we prove that $c < N^{1.7}$ and apply this estimate on an equation related to a conjecture of Hickerson that a factorial is not a product of factorials non-trivially.

Keywords. Primes, factorials, *abc*-conjecture, explicit conjecture, Diophantine equations.

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1. Introduction

For a positive integer ν, we define the radical $N(\nu)$ of ν by the product of primes dividing ν and $\omega(\nu)$ for the number of distinct prime divisors of ν. The letter p always denote a prime number in this paper. We denote the radical of abc by

$$N = N(abc) = \prod_{p|abc} p$$

unless otherwise specified. Further we write $\omega = \omega(N)$ for the number of distinct prime divisors of N.

The well known *abc*-conjecture was formulated by Joseph Oesterlé [Oe88-89] and David Masser [Ma90] in 1988. It states that for any given $\epsilon > 0$ there exists a computable constant κ_ϵ depending only on ϵ such that if

$$a + b = c, \tag{1.1}$$

where a, b and c are coprime positive integers, then

$$c \leq \kappa_\epsilon N^{1+\epsilon}.$$

We see when $\omega \in \{0, 1\}$ or N is odd then (1.1) does not hold. Therefore we always have N even and $\omega \geq 2$ unless $(a, b, c) = (1, 1, 2)$. We understand that $\log_2 x = \log\log x$ for $x \geq 2$ and $\log_3 x = \log\log\log x$ for $x \geq 3$. The number κ_ϵ need not be explicit which is not desirable if, for example, we wish to solve an equation completely using *abc*-conjecture. We state the following explicit version of *abc*-conjecture due to Baker [Ba04].

The explicit *abc*-conjecture: The explicit *abc*-conjecture states that (1.1) implies that

$$c < \frac{6}{5}\frac{N(\log N)^\omega}{\omega!} \text{ for } N > 2. \tag{1.2}$$

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It is convenient for applications to derive from (1.2) that

$$c < KN^{1+\theta}$$

for some $\theta > 0$ and $K = K(\theta)$, a computable constant. We observe that $N > \frac{(\log N)^{\omega}}{\omega!} + \frac{(\log N)^{\omega+1}}{(\omega+1)!} > \frac{6(\log N)^{\omega}}{5\omega!}$ since $\log N \geq \frac{\omega+1}{5}$ and thus (1.2) implies that

$$c < N^2 \text{ for } N \geq 1 \tag{1.3}$$

which was conjectured in Granville and Tucker [GrTu02]. Replacing the exponent 2 by a smaller exponent is always good for applications. We give a survey on improvements in the exponent of N in (1.3) in Section 2 and in Section 3 we give a short survey on consequences of explicit *abc*-conjecture. In Section 4, we give our improvement on (1.3) and in Section 5, we consider an equation on product of consecutive positive odd integers and improve the bounds for the solution of the equation under the explicit *abc*-conjecture using our improved estimate on (1.3).

2. Survey on improvements in (1.3)

We begin this section with a result of Laishram and Shorey [LaSh12].

Theorem 2.1. *Assume the explicit abc-conjecture and* (1.1) *holds. Then*

$$c < N^{\frac{7}{4}} \quad \textit{for} \quad N \geq 1.$$

Further for every $\epsilon > 0$, there exists ω_ϵ depending only on ϵ such that when $N = N(abc) \geq N_\epsilon = \prod_{p \leq p_\omega} p$, we have

$$c < \kappa_\epsilon N^{1+\epsilon}$$

where $\kappa_\epsilon \leq \frac{6}{5\sqrt{2\pi\omega_\epsilon}}$. Here are some values of $\epsilon, \omega_\epsilon$ and N_ϵ. [1]

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$	$\frac{5}{12}$	$\frac{1}{3}$
ω_ϵ	14	49	72	128	175	548	6016
N_ϵ	$e^{37.1101}$	$e^{204.75}$	$e^{335.71}$	$e^{686.163}$	$e^{1004.763}$	$e^{3894.57}$	$e^{59365.671}$

Further Chim, Shorey and Sinha [ChShSi] proved the following result.

Theorem 2.2. *Assume the explicit abc-conjecture. Then* (1.1) *implies that for $N \geq 1$,*

$$c < N^{1.72}. \tag{2.4}$$

Further

$$c < 10N^{1.62991}$$

and

$$c < 32N^{1.6}.$$

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[1] The values of ω_ϵ and N_ϵ for $\epsilon = \frac{1}{2}$ and $\frac{1}{3}$ given in [LaSh12] have been amended.

The bound $c < 10N^{1.62991}$ compares with the following example given by E. Reyssat [Rey18]. Consider $a = 2, b = 3^{10} \times 109$ and $c = 23^5$. Then $a + b = c$ with $N = N(abc) = 15042$ and $c > N^{1.62991}$. The exponents in the above inequalities of Theorem 2.2 can be sharpened if N is sufficiently large. For this, we introduce functions $G(N)$ and $G_1(N)$ as follows:

For integer $N > 2$, let

$$A(N) = \log_2 N - \log_3 N, A_1(N) = A(N) + \log A(N) - 1.076869$$

and

$$G(N) = \frac{1 + \log A(N)}{A(N)}.$$

Further for integer $N \geq 40$, let

$$G_1(N) = \frac{1 + \log A_1(N)}{A_1(N)}.$$

We observe the following for $G(N)$ and $G_1(N)$.

(i) $G(N)$ is decreasing for $N \geq 16$

(ii) $G_1(N)$ is decreasing whenever $N \geq 297856$

(iii) $G(N)$ is positive valued function that tends to zero as N tends to infinity

(iv) $G_1(N)$ tends to zero as N tends to infinity

(v) $G(N) \geq G_1(N)$ for $N \geq 1.5 \times 10^{36}$

(vi) $G(N) \leq G_1(N)$ for $297856 \leq N \leq 10^{36}$.

Further Chim, Shorey and Sinha [ChShSi] proved that

Theorem 2.3. *Assume the explicit abc-conjecture. Then* (1.1) *implies that*

$$c < \frac{6}{5} N^{1+G(N)} \text{ for } N > 2$$

and

$$c < \frac{6}{5} N^{1+G_1(N)} \text{ for } N \geq 297856.$$

On the other hand, Stewart and Tijdeman [StTi86] showed that $G(N)$ and $G_1(N)$ cannot be replaced by a function $F(N)$ such that $\lim\limits_{N \longrightarrow \infty} \dfrac{F(N)}{\frac{1}{\sqrt{(\log N)\log_2(N)}}} = 0.$

3. Some Consequences of explicit *abc*-conjecture

We give a short survey on applications on explicit *abc*-conjecture in Section 2.

3.A. A conjecture of Hickerson and Erdős

We consider

$$a_1!a_2! \cdots a_t! = n! \text{ in integers } n > a_1 \geq a_2 \cdots \geq a_t > 1, \ t > 1. \tag{3.5}$$

We always assume that $n \geq a_1+2$ otherwise (3.5) is satisfied for any positive integers $a_2, a_3, \ldots, a_t, a_1 = a_2! \ldots a_t! - 1$ and $n = a_1 + 1$. This equation, which we call the equation of Hickerson and Erdős, has solutions given by

$$7!3!^2 2! = 9!, \ 7!6! = 10!, \ 7!5!3! = 10!, \ 14!5!2! = 16!.$$

Hickerson (see [ErGr80]) conjectured that the largest solution of (3.5) is given by $n = 16$. This is a difficult problem and even the case $a_1 = n-2$ and $t = 2$ remains open. Luca [Lu07] proved that (3.5) has only finitely many solutions whenever *abc*-conjecture holds. The proof depends on the theory of linear forms in logarithms and it does not allow to determine all the solutions of (3.5). Nair and Shorey [NaSh16] confirmed the conjecture for $n \leq e^{80}$. Further, under Baker's explicit *abc*-conjecture, they confirmed the conjecture of Hickerson completely. We delete $a_1!$ on both sides of (3.5) and let $y = a_1 + 1, m = n - a_1 \geq 2$. Then (3.5) can be re-written as

$$a_2! \cdots a_t! = y(y+1)\cdots(y+m-1).$$

Since $y > a_1 \geq a_2$, we see that all the terms $y, y+1, \dots, y+m-1$ are composite. The proof also uses the following sharpening of a theorem of Sylvester due to Nair and Shorey [NaSh16].

Theorem 3.1. *Assume that $x > 100$ and $x, x+1, \cdots, x+k-1$ are all composite integers. Then*

$$P(x(x+1)\cdots(x+k-1)) > 4.42k$$

unless $x = 125, 224, 2400, 4374$ if $k = 2$ and $x = 350$ if $k = 3$.

The first result in this direction is due to Sylvester [Sy1912] that a product of k consecutive positive integers each exceeding k is divisible by a prime greater than k.

3.B. Triples of consecutive powerful integers

An integer ν is called powerful if $\nu > 0$ and $p^2|\nu$ whenever $p|\nu$ for every prime p. Golomb [Go70] proved in 1970 that there are infinitely many pairs of consecutive powerful integers and there exists no four (or more) consecutive powerful integers. Erdős conjectured that there is no three consecutive powerful integers. Trudgian [Tr16] proved, under explicit *abc*-conjecture, that $t < 10^{20000}$ whenever $(t-1, t, t+1)$ is a triple of consecutive powerful integers. [2] We recall the result of Mollin and Walsh [MoWa86]. Assume $t-1, t, t+1$ are powerful. Put

$$P = t, \quad Q = (t-1)(t+1) = my^2$$

where m is squarefree. Then $t \equiv 0 \pmod 4$ which implies that $m \equiv 7 \pmod 8$ and (t, y) is a solution of $x^2 - my^2 = 1$. Let $m = 7$. Then Mollin and Walsh [MoWa86] proved that

$$t > 10^{10^8}. \tag{3.6}$$

Hence, together with the result by Trudgian [Tr16], under explicit *abc*-conjecture, there is no triple $(t-1, t, t+1)$ of consecutive powerful integers such that $t^2 - 7y^2 = 1$. In [ChShSi], Chim, Shorey and Sinha checked that when $m \in \{15, 23, 31, 39, 47, 55, 87\}$, then (3.6) can be replaced by

$$t > 10^{3\times 10^{13}}.$$

Therefore, combining with the result by Trudgian [Tr16] and explicit *abc*-conjecture, there is no triple $(t-1, t, t+1)$ of consecutive powerful integers such that $t^2 - my^2 = 1$ with $m \in \{7, 15, 23, 31, 39, 47, 55, 87\}$. If $(t-1, t, t+1)$ is a triple of powerful integers, then $N(t(t^2-1)) < t^{3/2}$. It was also proved in [ChShSi], that the above inequality does not hold for all sufficiently large t whenever explicit *abc*-conjecture holds. More precisely, they proved

Theorem 3.2. *If $t > 10^{51075}$, then explicit abc-conjecture implies that*

$$N(t(t^2-1)) > t^{1.52}$$

where N is the square free part of $t(t^2-1)$.

This is obtained by using $c < 32N^{1.6}$ from Theorem 2.2 and $c < N^{1+G_1(N)}$ from Theorem 2.3 with $N = 10^{77544}$ and $N = 10^{77785}$.

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[2] It should be noted that the bound $t < 10^{20000}$ can be strengthened to $t < 10^{14000}$ if the same deduction as in [Tr16] with $\epsilon = \frac{1}{3}$ and $\omega_\epsilon = 6016$ from Theorem 2.1 are applied.

3.C. Generalised Fermat's equation

Let p, q, r be positive integers ≥ 2 with $(p,q,r) \neq (2,2,2)$. The equation

$$x^p + y^q = z^r, \quad (x,y,z) = 1 \ \text{ with integers } x > 0, y > 0, z > 0 \tag{3.7}$$

is called the *generalized Fermat equation*. We consider (3.7) with $p \geq 3, q \geq 3, r \geq 3$. For solving (3.7), there is no loss of generality in assuming $x > 1$, $y > 1$ and $z > 1$ since otherwise (3.7) is completely solved by Mihăilescu [Mi04].

Let $[p,q,r]$ denote all permutations of the ordered triple (p,q,r). Let

$$Q = \{[3,5,p] : 7 \leq p \leq 23, p \text{ prime}\} \cup \{[3,4,p] : p \text{ prime}\}.$$

Then Laishram and Shorey [LaSh12] proved that (3.7) with $x > 1, y > 1, z > 1, p \geq 3, q \geq 3, r \geq 3$ implies that $[p,q,r] \in Q$ such that

$$\max\left(x^p, y^q, z^r\right) < e^{1758.3353}$$

whenever explicit *abc*-conjecture holds. Chim, Shorey and Sinha [ChShSi] sharpen the above result using Theorem 2.2 as follows.

Theorem 3.3. *Assume explicit abc-conjecture. Let*

$$Q_1 = \{[3,5,p] : 7 \leq p \leq 19\} \cup \{[3,4,p] : p \geq 11\}$$

where p is a prime number. Then (3.7) *with $x > 1$, $y > 1$, $z > 1$, $p \geq 3, q \geq 3$ and $r \geq 3$ implies that $[p,q,r] \in Q_1$.*

Further for each $[p,q,r] \in Q_1$, they gave the following upper bound for $\max(x^p, y^q, z^r)$.

$[p,q,r]$	$\max(x^p, y^q, z^r) <$
$[3,4,p], p \geq 37$	8.1×10^{75}
$[3,4,31]$	1.3×10^{123}
$[3,4,29]$	4.3×10^{130}
$[3,4,23]$	1.2×10^{167}
$[3,4,19]$	9.8×10^{217}
$[3,4,17]$	1.2×10^{263}
$[3,4,13]$	1.5×10^{481}
$[3,4,11]$	2.2×10^{599}

$[p,q,r]$	$\max(x^p, y^q, z^r) <$
$[3,5,19]$	1.6×10^{61}
$[3,5,17]$	6.7×10^{69}
$[3,5,13]$	3.9×10^{107}
$[3,5,11]$	3.9×10^{155}
$[3,5,7]$	6.6×10^{645}

3.D. Conjecture of Erdős and Woods

Under explicit *abc*-conjecture, Shorey and Tijdeman [ShTi16] proved the conjecture of Erdős and Woods [Er80] which states that there are no positive integers $m < n$ such that for $i = 0,1,2$ the numbers $m+i$ and $n+i$ have the same prime factors. On the other hand, there are infinitely many pairs (m,n) with $m \neq n$ such that m, n and $m+1, n+1$ have the same prime factors. For example, for $h \geq 2$, if we take $(m,n) = (2^h - 2, 2^h(2^h - 2))$, then $(m+1, n+1) = (2^h - 1, (2^h - 1)^2)$. Thus m, n and $m+1, n+1$ have the same prime factors. We are not aware of any other infinite family contradicting the above conjecture of Erdős and Woods. But there is an isolated example given by $(m,n) = (75, 1215)$. Then $(m,n) = (3 \cdot 5^2, 3^5 \cdot 5)$ and $(m+1, n+1) = (2^2 \cdot 19, 2^6 \cdot 19)$. It is proved in [BLSW96, Proposition 1 with $d = d' = 1$] that there are only finitely many possibilities of pairs (m,n) of positive integers with $m < n$ such that $N(m+i) = N(n+i)$ for $i = 0,1,2$.

We give a short description on how explicit *abc*-conjecture is used in the proof of [ShTi16]. Assume that for $i = 0, 1, 2$ the numbers $m + i$ and $n + i$ have the same prime factors. We have

$$(n + 1)^2 = n(n + 2) + 1.$$

Using Theorem 2.1 with $a = n(n + 2), b = 1$ and $c = (n + 1)^2$, we get

$$n^2 < c < \left(\prod_{p|(n-m)} p\right)^{\frac{7}{4}} \leq (n - m)^{\frac{7}{4}} < n^{\frac{7}{4}},$$

which is a contradiction.

3.E. Equation of Nagell and Ljunggren

Nagell-Ljunggren equation is the equation

$$y^q = \frac{x^n - 1}{x - 1} \tag{3.8}$$

in integers $x > 1, y > 1, n > 2, q > 1$. This equation has solutions given by

$$\frac{3^5 - 1}{3 - 1} = 11^2, \ \frac{7^4 - 1}{7 - 1} = 20^2, \ \frac{18^3 - 1}{18 - 1} = 7^3.$$

These are called exceptional solutions and any other solution is termed as non-exceptional solution. For an account of results on (3.8), see Shorey [Sh99] and Bugeaud and Mignotte [BuMi02]. It is conjectured that there are no non-exceptional solution and Laishram and Shorey [LaSh12] confirmed this under explicit *abc*-conjecture.

3.F. Ideal Waring's Conjecture

For each integer $k \geq 2$, denote by $g(k)$ the smallest integer g such that any positive integer is the sum of at most g integers of the form x^k. A result of J. A. Euler implies that a lower bound for $g(k)$ is $2^k + \lfloor\left(\frac{3}{2}\right)^k\rfloor - 2$. The Ideal Waring's conjecture, dating back to 1853 states that, for any $k \geq 2$, the equality $g(k) = 2^k + \lfloor\left(\frac{3}{2}\right)^k\rfloor - 2$ holds. Dickson and Pillai proved independently in 1936 that the Ideal Waring's conjecture holds if $k > 6$ and if $\left(3^k + 1\right) / \left(2^k - 1\right) \leq \lfloor\left(\frac{3}{2}\right)^k\rfloor + 1$. (See [HaWr54], end of Chapter XXI.) In 1957, Mahler [Ma57] used the Ridout's extension of the Thue-Siegel-Roth theorem to show that $g(k) = 2k + \lfloor\left(\frac{3}{2}\right)^k\rfloor - 2$ except possibly for a finite number of values of k. It has been verified by several mathematicians that Ideal Waring's conjecture holds for $3 \leq k \leq 471600000$. Laishram [La15] proved in 2015 that under explicit *abc*-conjecture, Ideal Waring's conjecture is true.

4. New improvement on (1.3)

Now we give a sharpening to (2.4) as follows.

Theorem 4.1. *Assume the explicit abc-conjecture. Then* (1.1) *implies that for* $N \geq 1$,

$$c < N^{1.7}. \tag{4.9}$$

The improvement depends crucially on the records of ABC-triples in [Rey18], and on the recent work of Matschke and von Känel [MaKä18a, MaKä18b, MaKä18c] for solving S-unit equations via Shimura-Taniyama conjecture which is confirmed in [BCDT01].

4.A. Lemmas

For any real number $x > 0$, let $\Theta(x) = \prod_{p \le x} p$ and $\theta(x) = \log(\Theta(x))$. In 1983, G. Robin [Ro83] proved the following lemma for $\theta(x)$.

Lemma 4.2. *Let p_n be the nth prime. Then*

$$\theta(p_n) \ge n\Big(\log n + \log_2 n - 1.076869\Big) \textit{ for } n > 1.$$

For given $0 < \theta < 1$, $m \ge 2$ and $K > 0$, let

$$f(x) = \frac{(\log x)^m}{m!} - Kx^\theta.$$

Then

$$g(x) = x^{1-\theta}(m-1)!f'(x) = \frac{(\log x)^{m-1}}{x^\theta} - K\theta(m-1)!$$

and

$$g'(x) = \frac{(\log x)^{m-2}}{x^{1+\theta}}\big(m - 1 - \theta \log x\big).$$

Then we have the following Lemma.

Lemma 4.3. *Assume that there exist positive numbers x_0 and x_1 with $1 < x_1 \le x_0$ such that*

$$f(x_0) < 0,\ g(x_0) < 0 \textit{ and } g'(x_1) < 0. \tag{4.10}$$

Then $f(x) < 0$ for $x \ge x_0$.

Proof. The proof is in [ChShSi, Lemma 2.8].

4.B. Proof of Theorem 4.1

First, by following the same proof as in [LaSh12, Theorem 1], we have $\omega_1 = 20$ and $\omega_\epsilon = 19$ for $\epsilon = 0.7$ such that

$$\epsilon \ge \frac{1 + \log X_0(i)}{X_0(i)} \text{ for } i \ge \omega_1 \text{ and } \frac{i!\Theta(p_i)^\epsilon}{\theta(p_i)^i} > \sqrt{2\pi i} \text{ for } i \ge \omega_\epsilon$$

holds. Here we have $X_0(i) = \log i + \log_2 i - 1.076869$, then $\theta(p_i) \ge iX_0(i)$ by Lemma 4.2 and $\frac{i!N^\epsilon}{(\log N)^i} > \frac{i!\Theta(p_i)^\epsilon}{\theta(p_i)^i}$. Therefore, we have (4.9) for $\omega \ge 19$.

Next, we check that for $13 \le \omega < 19$, we have

$$\frac{\omega!\Theta(p_\omega)^\epsilon}{\theta(p_\omega)^\omega} > \frac{6}{5}.$$

Thus we get

$$\frac{(\log N)^\omega}{\omega!} < \frac{5}{6}N^{0.7} \text{ for } N > 2,\ 13 \le \omega < 19.$$

Therefore, for $13 \le \omega < 19$, we also have (4.9).

Now we consider $\omega \le 12$. We apply Lemma 4.3 with $x_1 = x_0, K = 5/6$ and $\theta = 0.7$. Then N's lies in the range $\Big[\prod_{p \le p_\omega} p, x_0\Big)$.

(i). We observe that for $2 \le \omega \le 3$, we may choose $x_1 = x_0 = \prod_{p \le p_\omega} p$ so that (4.10) is satisfied.

Table 1:

ω	$L=\prod_{p\leq p_\omega} p$	$U=x_0$	No. of N with $N\in[L,U)$
4	210	270	0
5	2310	13500	39
6	30030	278000	148
7	510510	5250000	331
8	9699690	96800000	480
9	223092870	1773000000	456
10	6469693230	32600000000	270
11	200560490130	600000000000	81
12	7420738134810	11050000000000	9

Then (4.9) follows by Lemma 4.3 with $K=5/6$.

(ii). For $4\leq\omega\leq 12$, we choose $x_1=x_0$ as given in Table 1 so that (4.10) is satisfied and we perform SAGE computation to extract all square free N with $\omega(N)=\omega$ that lie in the range $\left[\prod_{p\leq p_\omega} p, x_0\right)$. Hence we obtain Table 1.

By (1.2), for each $N=Q_1Q_2\cdots Q_\omega$ where $Q_1,Q_2,\ldots,Q_\omega$ are distinct primes and $4\leq\omega\leq 12$, it suffices to restrict $c\in\left[N^{1.7},\frac{6}{5}N\frac{(\log N)^\omega}{\omega!}\right)$ otherwise (4.9) holds. We observe that $c<10^{20}$ in order to have $c\in\left[N^{1.7},\frac{6}{5}N\frac{(\log N)^\omega}{\omega!}\right)$ for those $N\in[L,U)$ for $4\leq\omega\leq 10$ in Table 1. We refer to the website [Rey18] maintained by de Smit in which a complete list of (a,b,c) with $q=\frac{\log c}{\log N}>1.4$ and $c<10^{20}$ extracted by various mathematicians are recorded. It is found that all have $q<1.7$ and hence satisfy $c<N^{1.7}$. Therefore, (4.9) holds for $4\leq\omega\leq 10$.

Besides referring to the results from [Rey18], we adopt the results from the work of Matschke and von Känel [MaKä18a], in connection to their work [MaKä18b], to tackle the cases in Table 1 with $11\leq\omega\leq 12$. They have a record of

$$\begin{gathered} a+b=c,\quad 0<a\leq b<c,\quad \gcd(a,b,c)=1,\\ \operatorname{rad}(abc)|2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot29\cdot31\cdot37\cdot41\cdot43\cdot47\cdot53. \end{gathered}\tag{4.11}$$

For all the (a,b,c) recorded in [MaKä18a], all satisfy $c<N^{1.7}$. For the case when $\omega=12$, the 9 values of $N\in[L,U)$ extracted are 7420738134810, 8222980095330, 8624101075590, 9426343036110, 9814524629910, 10293281928930, 10491388397490, 10629705976890 and 11003163441270. It is observed that they all have prime factors not exceeding 53. Therefore according to the results from [MaKä18a], (4.9) is fulfilled.

For the case when $\omega=11$, it is checked that among all the 81 values of $N\in[L,U)$ extracted, 55 of them have all prime factors not exceeding 53 so that (4.9) is fulfilled by the results from [MaKä18a] again. The list of 26 remaining N's and their prime factorization is shown in Section 6. (Appendix) for readers' reference. For these 26 values of N, 23 of them yield $c<10^{20}$ when only those c's in $\left[N^{1.7},\frac{6}{5}N\frac{(\log N)^\omega}{\omega!}\right)$ are considered. Therefore (4.9) is fulfilled according to the results from [Rey18]. The remaining three N's for consideration are listed in Table 2.

Finally, we make use of the SAGE program supplied by Matschke and von Känel [MaKä18a] in [MaKä18c] to obtain all coprime (a,b,c) satisfying $a+b=c$ and $0<a\leq b<c$ for the three remaining cases of N in Table 2. They all give $q=\frac{\log c}{\log N}<1.7$. Therefore, (4.9) is fulfilled for $\omega=11$ as well and hence (4.9) holds. The SAGE Program of [MaKä18c] depends on new algorithms so that the running time is reduced greatly compared to that of the algorithm applied in the proof of (2.4) in [ChShSi, Section 4]. The executing time for each case of N in Table 2 is less than 2 hours. $\square$

Table 2:

N	Prime factors	$N^{1.7} >$	$\frac{6}{5}N\frac{(\log N)^{\omega}}{\omega!} <$
584241427770	$2,3,5,7,11,13,17,19,23,37,71$	1.0074×10^{20}	1.0143×10^{20}
585172598010	$2,3,5,7,11,13,17,19,23,43,61$	1.01×10^{20}	1.0166×10^{20}
586064969490	$2,3,5,7,11,13,17,19,29,31,67$	1.012×10^{20}	1.188×10^{20}

5. Application of Theorem 4.1

We consider the following analogue of the equation of Hickerson and Erdős given in Section 3.1. For each non negative integer j, define u_j as the product of the odd numbers $\leq j$. Thus if j is odd,

$$u_j = 1\cdot 3\cdot 5\cdots j = \frac{1\cdot 2\cdot 3\cdot 4\cdots (j-1)\cdot j}{2\cdot 4\cdot 6\cdots (j-1)} = \frac{j!}{2^{\frac{j-1}{2}}\left(\frac{j-1}{2}\right)!}.$$

We consider the following equation

$$u_{a_1}u_{a_2}\cdots u_{a_t} = u_n \text{ in odd integers } n > a_1 \geq a_2 \geq \cdots \geq a_t \geq 3,\ t > 1. \tag{5.12}$$

If $n - a_1 = 2$, (5.12) has infinitely many solutions by choosing $a_2, a_3, \ldots, a_t$ arbitrary and $a_1 = u_{a_2}\cdot u_{a_3}\cdots u_{a_t} - 2$. Therefore we always assume that $n - a_1 \geq 4$ since $n - a_1$ is even. We observe that

$$u_{23}\cdot u_5^2\cdot u_3 = u_{27}$$

and this may be the only solution of (5.12) when $n - a_1 \geq 4$. We write x and k for integers satisfying $x > 0$ and $k \geq 2$,

$$\Delta(x, 2, k) = x(x+2)\cdots(x+2(k-1))$$

and

$$x = a_1 + 2,\ k = \frac{n-a_1}{2} \geq 2. \tag{5.13}$$

We re-write (5.12) as $u_{a_2}u_{a_3}\cdots u_{a_t} = \Delta(x,2,k)$. We observe that $x > 2$ is odd since $a_1 > 0$ is odd. Further $P(u_{a_2}u_{a_3}\cdots u_{a_t}) = P(\Delta(x,2,k)) \leq a_2$. Since $x = a_1 + 2 > a_2$, we have $x, x+2, \ldots, x+2(k-1)$ are all composite. Since x is odd, $x+1, x+3, \ldots, x+2k-3, x+2k-1$ are all even and therefore the interval $[x, x+2k)$ contains no prime. Therefore we consider equation

$$u_{a_2}u_{a_3}\cdots u_{a_t} = \Delta(x,2,k) \tag{5.14}$$

where x is odd and there is no prime in $\{x, x+2, \ldots, x+2(k-1)\}$. We observe that $(x,k) = (25,2)$ is a solution of (5.14). In [NaSh18], Nair and Shorey proved that (5.14) implies $k \leq 23$ under the assumptions of explicit *abc*-conjecture. Further, they gave the following upper bounds for x when $2 \leq k \leq 23$ where x and k are given by (5.13).

Table 3:

k	$\log x <$	k	$\log x <$	k	$\log x <$	k	$\log x <$
2	4042	8	2739	14	1150	20	143
3	594	9	2168	15	1051	21	115
4	2766	10	1987	16	443	22	98
5	587	11	1683	17	362	23	86
6	1350	12	1458	18	360		
7	3661	13	1286	19	199		

In this Section, we considerably improve the bounds for $\log x$ for $13 \leq k \leq 23$ given in Table 3 as follows. The new bounds are given in Table 4. We recall the inequalities from [NaSh18] which we shall use. For more details, we refer to [NaSh18, Section 2]. We count the power of 3 on both sides of (5.14). The power of 3 on the left hand side is at least the power of 3 in u_{a_2}. In the product on the right hand side of (5.14), we delete a term in which 3 appears to the highest power. The power of 3 in this term cannot exceed $\frac{\log(x+2(k-1))}{\log 3}$. Moreover, the power of 3 in the remaining terms does not exceed the power of 3 in $(k-1)!$ which is at most $\frac{k-1}{2}$. Thus,

Table 4:

k	$\log x <$	k	$\log x <$	k	$\log x <$
13	574	17	110	21	68
14	351	18	91	22	60
15	220	19	85	23	57
16	143	20	71		

$$\frac{a_2+1}{4} - \frac{\log(a_2+1)}{\log 3} < \frac{k-1}{2} + \frac{\log(2x)}{\log 3}.$$

which implies

$$a_2\left(\frac{1}{4} - \frac{\log(a_2+1)}{a_2 \log 3}\right) < \frac{k}{2} + \frac{\log x}{\log 3} - 0.119. \tag{5.15}$$

Choose distinct $x+2j_1$ and $x+2j_2$ such that $N(x+2j_1) \leq N(x+2j_2)$ are the smallest among $N(x+2i)$ for $0 \leq i < k$. Then

$$\begin{aligned} N(x+2j_2) &\leq \left(\prod_{i=0, i\neq j_1}^{k-1} N(x+2i)\right)^{\frac{1}{k-1}} \leq \left(\prod_{i=0}^{k-1} N(x+2i)\right)^{\frac{1}{k-1}} \\ &\leq \frac{1}{2}\exp\left(\frac{1.00008a_2}{k-1} + \frac{k\log k}{k-1} - \frac{\log 2}{2}\right). \end{aligned}$$

Consider

$$\frac{x+2j_1}{d} - \frac{x+2j_2}{d} = \frac{2(j_1-j_2)}{d}, \quad \text{where} \quad d = \gcd(x+2j_1, (j_1-j_2)). \tag{5.16}$$

We take $c = \frac{x+2j_1}{d}, a = \frac{x+2j_2}{d}, b = \frac{2(j_1-j_2)}{d}$ if $j_1 > j_2$ and $c = \frac{x+2j_2}{d}, a = \frac{x+2j_1}{d}, b = \frac{2(j_2-j_1)}{d}$ if $j_2 > j_1$ so that (1.1) is satisfied such that a, b, c are relatively prime positive integers. Applying (4.9), we get

$$\frac{x}{d} < \left(N\left(x+2j_1\right)N\left(x+2j_2\right)\left(\left|\frac{2(j_1-j_2)}{d}\right|\right)\right)^{1.7}.$$

Hence

$$\log x < 1.7\left(\frac{2.00016a_2}{k-1} + \frac{2k\log k}{k-1} + \log k - 2\log 2\right). \tag{5.17}$$

The bounds for $\log x$ in [NaSh18] were obtained using $P = P(\Delta(x,2,k)) > 4.7k$ whenever $x > 4.5k$ and $(x,k) \notin \{(25,2),(243,2)\}$. We consider the cases when $P(\Delta(x,2,k)) > Ck$ and $P(\Delta(x,2,k)) \leq Ck$ where C is a constant. This is the crucial step and we choose the values for C appropriately depending on k.

Let $k = 23$. Consider the case when $P = P(\Delta(x,2,k)) > 12k$. Then $a_2 \geq P > 12k$ implies $a_2 \geq 277$. Consider the function

$$F(a_2) = \frac{\log(a_2+1)}{a_2\log 3}.$$

This is a decreasing function and thus $F(a_2) \leq F(277) \leq 0.0185$ which we use in (5.15), to get

$$a_2(0.25 - 0.0185) < \frac{k}{2} + \frac{\log x}{\log 3} - 0.119. \tag{5.18}$$

We use the bound for a_2 given by (5.18) in (5.17) to get $\log x < 56$. Now we have to consider the case when $P \leq 12k$. This will imply either $a_2 \leq 12k$ or $a_2 > 12k$. If $a_2 > 12k$, this will reduce to the earlier case. Therefore, we can always assume that $a_2 \leq 12k$. We apply this bound for a_2 in (5.17) to get $\log x < 57$. Thus combining both the cases, we have $\log x < 57$ when $k = 23$. Similarly for $15 \leq k \leq 22$, we get the following bounds for $\log x$ with a suitable choice for C which determines the cases according as $P > Ck$ and $P \leq Ck$.

k	C	$\log x <$	k	C	$\log x <$
22	12	60	18	20	91
21	15	68	17	25	110
20	15	71	16	35	143
19	20	85	15	55	220

Let $k = 14$. Here we need to consider first when $N(abc) < e^{204.75}$. Applying (4.9) in (5.16), we get

$$\log x < 1.7 \times 204.75 + \log k < 351.$$

Therefore we may assume that $N(abc) \geq e^{204.75}$. Applying Theorem 2.1 with $\epsilon = \frac{7}{12}$ in (5.16), we get

$$\frac{x}{d} < \frac{6}{5\sqrt{98}\pi}\left(N\left(x+2j_1\right)N\left(x+2j_2\right)\left(\left|\frac{2(j_1-j_2)}{d}\right|\right)\right)^{\frac{19}{12}}.$$

This implies as in (5.17) that

$$\log x < \frac{19}{12}\left(\frac{2.00016a_2}{k-1} + \frac{2k\log k}{k-1} + \log k - 2\log 2\right) + \log\left(\frac{6}{5\sqrt{98}\pi}\right). \tag{5.19}$$

As in the earlier cases of $15 \leq k \leq 23$, now we consider the cases according as $P > 50k$ and $P \leq 50k$ along with (5.19) and (5.15) to get $\log x < 187$ and 179 respectively. Thus combining all the cases, we get $\log x < 351$ when $k = 14$.

Let $k = 13$. Assume that $N(abc) < e^{335.71}$. Applying (4.9) in (5.16), we get

$$\log x < 1.7 \times 335.71 + \log k < 574.$$

Therefore we may assume that $N(abc) \geq e^{335.71}$. Applying Theorem 2.1 with $\epsilon = \frac{6}{11}$ in (5.16), we get

$$\log x < \frac{17}{11}\left(\frac{2.00016a_2}{k-1} + \frac{2k\log k}{k-1} + \log k - 2\log 2\right) + \log\left(\frac{6}{5\sqrt{254\pi}}\right). \tag{5.20}$$

Now we consider the cases according as $P > 100k$ and $P \leq 100k$ along with (5.20) and (5.15) to get $\log x < 326$ and 343, respectively. Thus combining all the cases, we get $\log x < 574$ when $k = 13$. $\square$

6. Appendix

The following provides supplementary information to the proof of Theorem 4.1 in Section 4.B. for readers' reference. For $\omega = 11$, the list of 26 cases of N with prime factors exceeding 53 and their prime factorization is as follows:

$$\begin{aligned}
381711900570 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 59,\\
394651287030 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 61,\\
408036859230 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 59,\\
421868617170 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 61,\\
433469446410 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 67,\\
459348219330 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 71,\\
463363890990 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 67,\\
472287605790 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 73,\\
487011735210 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 37 \times 59,\\
491027406870 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 71,\\
503520607590 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 37 \times 61,\\
504859164810 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 73,\\
511105765170 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 79,\\
514481257290 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 29 \times 31 \times 59,\\
531921299910 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 29 \times 31 \times 61,\\
536984538090 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 83,\\
539661652530 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 41 \times 59,\\
546354438630 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 79,\\
553047224730 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 37 \times 67,\\
557955267870 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 41 \times 61,\\
565986611190 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 43 \times 59,\\
574017954510 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 83,\\
575802697470 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 89,\\
584241427770 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 29 \times 31 \times 67,\\
585172598010 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 43 \times 61,\\
586064969490 &= 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 37 \times 71.
\end{aligned}$$

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