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The Barban-Vehov Theorem in Arithmetic Progressions

V. Kumar Murty

To the memory of S. Srinivasan

Abstract. A result of Barban-Vehov (and independently Motohashi) gives an estimate for the mean square of a sequence related to Selberg's sieve. This upper bound was refined to an asymptotic formula by S. Graham in 1978. In 1992, I made the observation that Graham’s method can be used to obtain an asymptotic formula when the sum is restricted to an arithmetic progression. This formula immediately gives a version of the Brun-Titchmarsh theorem. I am taking the occasion of a volume in honour of my friend S. Srinivasan to revisit and publish this observation in the hope that it might still be of interest.

Keywords. Selberg’s sieve, Brun-Titchmarsh theorem, arithmetic progressions

2010 Mathematics Subject Classification. Primary 11N37, 11N13 ; Secondary 11B25, 11N35, 11N69

1. Introduction

Let \( 1 \leq z_1 \leq z_2 \) and define for \( i = 1, 2 \),

\[
\Lambda_i(n) = \begin{cases} 
\mu(n) \log \frac{z_i}{n} & \text{if } n \leq z_i \\
0 & \text{if } n > z_i
\end{cases}
\]

Also, set

\[
\lambda_n = \frac{\Lambda_2(n) - \Lambda_1(n)}{\log z_2/z_1}
\]

and

\[
a(n) = \sum_{d|n} \lambda_d.
\]

The \( \lambda_n \) are weights that are related to Selberg’s sieve. Notice that we have

\[
a(1) = 1
\]

and \( a(n) = 0 \) for \( 1 < n \leq z_1 \). Moreover, for primes \( p \), we have

\[
a(p) = \begin{cases} 
\frac{\log p/z_1}{\log z_2/z_1} & \text{if } z_1 < p < z_2 \\
0 & \text{otherwise}.
\end{cases}
\]

It was shown by Barban and Vehov [BaVe68] and Motohashi [Mo74] that

\[
\sum_{n \leq N} |a(n)|^2 \ll \frac{N}{\log z_2/z_1}.
\]

Soon afterwards, S. Graham [Gr78] was able to prove the following asymptotic formulae: if \( N \geq z_2 \), then

\[
\sum_{n \leq N} |a(n)|^2 = \frac{N}{\log z_2/z_1} + O\left( \frac{N}{(\log z_2/z_1)^2} \right)
\]

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and if $z_1 \leq N \leq z_2$, we have

$$
\sum_{n \leq N} |a(n)|^2 = \frac{N \log N/z_1}{(\log z_2/z_1)^2} + O\left(\frac{N}{(\log z_2/z_1)^2}\right).
$$

This result has found significant applications to zero density theorems and to the estimation of Linnik’s constant (see [Ju77] and [Gr81] for example).

The purpose of this note is to study the size of the sum when $n$ is constrained to range through a fixed arithmetic progression. In my joint work with R. Balasubramanian [BaMu92], we observed (Proposition 1.2) that for $N \geq r$ and $(b, r) = 1$, we have

$$
\sum_{n \leq N, \ (n \equiv b \pmod{r})} |a(n)| \ll \frac{N}{\varphi(r)^2 (\log \frac{z_2}{z_1})^2}.
$$

This follows immediately from Graham’s result by the Cauchy-Schwarz inequality.

Soon after [BaMu92] was written, I worked out an asymptotic formula for the sum on the left by adapting Graham’s methods. The result is that for $(b, r) = 1$ and $N \geq r z_2^2$, we have

$$
\sum_{n \leq N, \ (n \equiv b \pmod{r})} |a(n)|^2 = \frac{N}{\varphi(r) \log \frac{z_2}{z_1}} + O\left(\frac{N \sigma(r)}{\varphi(r)^2 (\log \frac{z_2}{z_1})^2}\right).
$$

If $r z_1 z_2 \leq N \leq r z_2^2$, then we show that

$$
\sum_{n \leq N, \ (n \equiv b \pmod{r})} |a(n)|^2 = \frac{N}{\varphi(r) \log \frac{z_2}{z_1}} + O\left(\frac{N (\log r z_2/N)^5}{r (\log \frac{z_2}{z_1})^2}\right) + O\left(\frac{N (\log r)^2}{r (\log \frac{z_2}{z_1})^2}\right) + O\left(\frac{r z_2}{(\log \frac{z_2}{z_1})^2}\right).
$$

The last two terms on the right are not present if we have the additional condition $z_1 > r$. Also, it will be clear from the arguments that the same methods will actually allow us to get estimates for $N < r z_1 z_2$ as well but we do not pursue that here.

An immediate consequence of the first formula is a version of the Brun-Titchmarsh theorem in the following form. Denote by $\pi(N, r, b)$ the number of primes $\leq N$ which are $\equiv b \pmod{r}$. Then we have

$$
\pi(N, r, b) \leq \frac{2N}{\varphi(r) \log N/r} + O\left(\frac{N \sigma(r)}{\varphi(r)^2 (\log N/r)^2}\right).
$$

We could try to use the second formula in a similar manner.

Our method of proof for both formulae is elementary and uses only the usual prime number theorem. The proof of the first formula is a direct generalization of the work of Graham ([Gr78], §3) and represents the easy case.

Though these calculations were completed some years ago, I had not published them. However, the occasion of a volume honouring the memory of my friend S. Srinivasan caused me to look at them again. In particular, the consequence for the Brun-Titchmarsh theorem may still be of interest. When I was a Visiting Fellow at the Tata Institute for Fundamental Research in 1983-1984, Srinivasan was my office-mate and we shared many hours of mathematical conversation. We also enjoyed many social occasions together when we had a chance to discuss philosophical and even spiritual questions. Srinivasan was always a thorough and thoughtful individual and I look back on those occasions with many pleasant memories. Given that his main interest was in analytic number theory, I thought the topic of this article might have been of interest to him.

The note is organized as follows. In §2, we prove the first formula. In §3, we begin the proof of the second formula. In §4 and §5, we estimate certain error terms and in §6, we study the main term. Finally in §7, we complete the proof of the second formula.

I would like to thank the referees for helpful comments that helped to streamline the presentation.
2. The First Formula

The purpose of this section is to prove the following.

**Theorem 2.1.** Suppose that \( rz_2^2 \leq N \) and \((b, r) = 1\). Then, 
\[
\sum_{n \equiv b \pmod{r}} |a(n)|^2 = \frac{N}{\phi(r) \log z_2 / z_1} + O\left( \frac{N \sigma(r)}{\phi(r)^2 (\log z_2 / z_1)^2} \right).
\]

The proof follows closely the method of [Gr78], §3 but we give the details in order to orient the reader. First, we note an immediate consequence. Denote by \( \psi(N, z_2, r, b) \) the number of integers \( n \leq N \) with \( n \equiv b \pmod{r} \) all of whose prime factors are \( \geq z_2 \). Denote also by \( \pi(N, r, b) \) the number of primes \( p \leq N \) with \( p \equiv b \pmod{r} \).

**Corollary 2.2.** If \( rz_2^2 \leq N \), then 
\[
\psi(N, z_2, r, b) \leq \frac{N}{\phi(r) \log z_2} + O\left( \frac{N \sigma(r)}{\phi(r)^2 (\log z_2)^2} \right).
\]

*Proof.* If \( n \leq N \), and \( n \equiv b \pmod{r} \) and it has all its prime divisors \( \geq z_2 \), then \( a(n) = 1 \).

**Corollary 2.3.** We have 
\[
\pi(N, r, b) \leq \frac{2N}{\phi(r) \log N/r} + O\left( \frac{N \sigma(r)}{\phi(r)^2 (\log N/r)^2} \right).
\]

*Proof.* This follows on noting that 
\[
\pi(N, r, b) \leq \pi(z_2, r, b) + \psi(N, z_2, r, b),
\]
and the trivial bound \( \pi(z_2, r, b) \leq z_2/r + 1 \) and choosing \( z_2 = (N/r)^{1/2} \).

To prove the theorem, we notice that 
\[
(\log z_2 / z_1)^2 \sum_{n \equiv b \pmod{r}} |a(n)|^2 = \sum_{n \leq N} \left( \sum_{d | n} \Lambda_1(d) - \sum_{e | n} \Lambda_2(e) \right)^2.
\]

The right hand side is a sum of terms of the form 
\[
S_{i,j} = \sum_{n \equiv b \pmod{r}} \sum_{d | n} \Lambda_i(d) \Lambda_j(e),
\]
where \( i, j \in \{1, 2\} \). The theorem will follow from the following.

**Proposition 2.4.** We have 
\[
S_{i,j} = \frac{N}{\phi(r)} \log \min(z_i, z_j) + O(z_i z_j) + O(N \sigma(r) / \phi(r)^2).
\]

In particular, if \( rz_i z_j \leq N \), the first error term is \( O(N/r) \).
Proof. By definition
\[ S_{i,j} = \sum \Lambda_i(d) \Lambda_j(e) \left\{ \frac{N}{r[d,e]} + O(1) \right\}. \]
The error term is
\[ \ll \left( \sum_{d \leq z_i} |\Lambda_i(d)| \right) \left( \sum_{e \leq z_j} |\Lambda_j(e)| \right) \ll z_i z_j. \]

The main term is
\[ \frac{N}{r} \sum_{\substack{d \leq z_i \\ e \leq z_j \\ (d,r) = (e,r) = 1}} \frac{\Lambda_i(d) \Lambda_j(e)}{[d,e]}. \]

Without loss of generality, we may suppose that \( z_i \leq z_j \). We have
\[ \sum_{\substack{d \leq z_i \\ e \leq z_j \\ (d,r) = (e,r) = 1}} \frac{\Lambda_i(d) \Lambda_j(e)}{[d,e]} = \sum_{d \leq z_i} \frac{\Lambda_i(d)}{d} \sum_{e \leq z_j} \phi(m) \]
and inserting the definition of \( \Lambda_i \) and \( \Lambda_j \), the right hand side is seen to be
\[ \sum_{m \leq z_i} \frac{\mu(m)^2 \phi(m)}{m^2} \left( \sum_{d_0 \leq z_i/m} \frac{\mu(d_0)}{d_0} \log \frac{z_i}{md_0} \right) \left( \sum_{e_0 \leq z_j/m} \frac{\mu(e_0)}{e_0} \log \frac{z_j}{me_0} \right). \]

We quote the following from [Gr78], §2:

**Lemma 2.5.** For any integer \( a \) and any \( c > 0 \), we have
\[ \sum_{n \leq Q} \frac{\mu(n)}{n} \log \frac{Q}{n} = \frac{a}{\phi(a)} + O_c(\sigma_{-\frac{1}{2}}(a)(\log 2Q)^{-c}). \]

Using this on the terms in parentheses, we find that the above is
\[ \sum_{m \leq z_i} \frac{\mu(m)^2 \phi(m)}{m^2} \left\{ \frac{mr}{\phi(mr)} + O(\sigma_{-\frac{1}{2}}(mr)(\log 2z_i/m)^{-2}) \right\}^2. \]

The error terms are \( O(1) \) just as in [Gr78], pp. 89-90. The main term is
\[ \frac{r^2}{\phi(r)^2} \sum_{m \leq z_i} \frac{\mu(m)^2}{\phi(m)}. \]

This is seen to be
\[ \frac{r}{\phi(r)} \log z_i + O\left( \frac{r\sigma(r)}{\phi(r)^2} \right). \]
3. The second formula

The aim of the remaining sections is to prove the following asymptotic formula which will suffice to deduce the second formula.

**Theorem 3.1.** Suppose that \( r z_1 z_2 \leq N \leq r z_2^2 \). Then for \((b, r) = 1\), we have

\[
\sum_{n \leq N \atop n \equiv b \mod r} \left( \sum_{d \mid n} \Lambda_2(d) \right)^2 = \frac{N}{\phi(r)} \log z_2 + O\left( \frac{N}{r} \left( \log r z_2^2 / N \right)^5 \right) + O\left( \frac{N}{r} \left( \log r \right)^2 \right) + O(r z_2).
\]

Expanding the sum on the left, we have

\[
\sum_{n \leq N \atop n \equiv b \mod r} \left( \sum_{d \mid n} \Lambda_2(d) \right) \left( \sum_{e \mid n} \Lambda_2(e) \right)
\]

and we begin the proof by splitting this sum into three components

\[ S_A + S_B + O(S_C) \]

where in \( S_A \) we restrict the sum to \( d, e \) satisfying \([d, e] > N/r\). The remaining terms may be rearranged to give

\[
\sum_{[d, e] \leq N/r} \Lambda_2(d) \Lambda_2(e) \sum_{n \leq N \atop n \equiv b \mod r \atop n \equiv 0 \mod [d, e]} 1
\]

which is

\[
= \frac{N}{r} \sum_{[d, e] \leq N/r \atop (d, r) = (e, r) = 1} \Lambda_2(d) \Lambda_2(e) \left[ \frac{1}{[d, e]} \right] + O \left( \sum_{[d, e] \leq N/r} |\Lambda_2(d) \Lambda_2(e)| \right)
\]

which we write as

\[ S_B + O(S_C). \]

In §4., we shall show that

\[ S_A \ll \frac{N}{r} \left( \log r z_2^2 / N \right)^5 + \frac{N}{r} \left( \log r \right)^2 + r z_2. \]

In §5., we shall show that

\[ S_C \ll \frac{N}{r} \left( \log r z_2^2 / N \right)^4. \]

Finally, in §6., we shall deal with the main term \( S_B \), and in §7. we collect together the various pieces to complete the proof of Theorem 3.1.

We state explicitly the consequence of Theorem 3.1 for the \( a(n) \).

**Theorem 3.2.** Suppose that \( r z_2^2 \geq N \geq r z_1 z_2 \). Then

\[
\sum_{n \leq N \atop n \equiv b \mod r} |a(n)|^2 = \frac{N}{\phi(r) \log z_2 / z_1} + O\left( \frac{N \left( \log r z_2^2 / N \right)^5}{r \log (z_2 / z_1)} \right) + O\left( \frac{N \left( \log r \right)^2}{r \log (z_2 / z_1)^2} \right) + O\left( \frac{r z_2}{(\log z_2 / z_1)^2} \right).
\]
4. Estimation of $S_A$

Proposition 4.1. Suppose that $rz_2^2 \geq N > r$. Then

$$\sum_{n \equiv k \pmod{r}} \sum_{[d,e] > N/r} |\Lambda_2(d)\Lambda_2(e)| \ll \frac{N}{r} \left(\frac{\log rz_2^2}{N}\right)^5 + \frac{N}{r} \left(\log r\right)^2 + rz_2.$$ 

Remark 4.2. Our argument will show that in certain ranges of $r, z_i, z_j$ and $N$, this estimate can be refined.

Proof. Write $\rho = (d,e)$. Then, each $n$ can be written as $n = \rho d_0 e_0$ with $d = \rho d_0$, $e = \rho e_0$ and $[d,e] = \rho d_0 e_0 > N/r$. This last condition implies that $n_0 < r$. The sum over $e_0$ then ranges over the interval

$$\frac{N}{r d_0} < e_0 \leq \frac{z_2}{\rho}.$$ 

In order for this to be nonempty, we need $d_0 > N/rz_2$.

But $d_0 \leq z_2/\rho$ and so $\rho < rz_2^2/N$. Thus, our sum is

$$\sum_{\rho < rz_2^2/N} \sum_{(\rho, r)=1} \sum_{n_0 \leq \rho \mid \rho d_0 e_0} |\Lambda_2(d_0\rho)\Lambda_2(e_0\rho)|.$$

Here, for any residue class $\beta \pmod{r}$ (with $(\beta, r) = 1$), we are writing $\overline{\beta} \pmod{r}$ for the inverse class. The inner sum is

$$\sum_{d_0 \rho \mid d_0 e_0} |\mu(d_0\rho) (\log z_2/d_0\rho) \mu(e_0\rho) (\log z_2/e_0\rho)|.$$

Separating the $d_0$ and the $e_0$ sums, we find that this is

$$\sum_{d_0} |\mu(d_0\rho) \log \frac{z_2}{d_0\rho}| \sum_{e_0} |\mu(e_0\rho) \log \frac{z_2}{e_0\rho}|$$

where the sum over $d_0$ is in the range

$$\frac{N}{rz_2} < d_0 < \min \left(\frac{z_2}{\rho}, \frac{N}{n_0\rho}\right)$$

and the sum over $e_0$ is in the range

$$\frac{N}{r d_0 \rho} < e_0 < \min \left(\frac{z_2}{\rho}, \frac{N}{n_0 d_0 \rho}\right)$$

with the additional condition

$$e_0 \equiv b \cdot \frac{d_0 \rho n_0}{(d_0 \rho)} \pmod{r}.$$ 

First, consider the contribution of the $e_0$ which satisfy $e_0 > r$. Writing $e_0 = \ast + e_1 r$, we see that the inner sum is

$$\leq \sum_{1 \leq e_1 \leq \min \left(\frac{N}{n_0 d_0 \rho}, \frac{z_2}{r\rho}\right)} \log \frac{z_2}{e_1 r \rho} \quad (4.1)$$
4.A. Case 1
If \(N/n_0d_0 \leq z_2\), then this is
\[
\ll \frac{N}{n_0d_0r} \left( \log \frac{z_2n_0d_0}{N} + 1 \right).
\]
Inserting this into the sum over \(d_0\), we find that it is
\[
\ll \frac{N}{n_0r} \sum_{N/z_2n_0 < d_0 \leq z_2} \frac{1}{d_0} \left( \log \frac{z_2}{d_0r} \right) \left( \log \frac{z_2n_0d_0}{N} + 1 \right).
\]
For this sum to be nonempty, we require
\[
\rho < z_2^2n_0/N.
\]
Using the following consequence of the arithmetic mean - geometric mean inequality,
\[
(\log A)(\log B) \leq (\log AB)^2
\] (4.2)
for \(A, B \geq 1\), the above sum is
\[
\ll \frac{N}{n_0r} \sum_{N/z_2n_0 < d_0 \leq z_2} \frac{1}{d_0} \left( \log \frac{z_2^2n_0}{N\rho} \right)^2.
\]
The sum over \(d_0\) is
\[
\ll \log \frac{z_2^2n_0}{N\rho}
\] and so, we have to estimate
\[
\sum_{n_0 < r \rho < n_0z_2^2/N} \frac{N}{r n_0\rho} \left( \log \frac{z_2^2n_0}{N\rho} \right)^3
\]
and this is
\[
\ll \frac{N}{r} \sum_{N/z_2^2 \leq n_0 < r} \frac{1}{n_0} \left( \log \frac{z_2^2n_0}{N} \right)^4
\]
which in turn is
\[
\ll \frac{N}{r} \left( \log \frac{r z_2^2}{N} \right)^5. \tag{4.3}
\]
4.B. Case 2
If \(N/n_0d_0 \geq z_2\), the sum in (4.1) is
\[
\ll \sum_{e_1 \leq z_2/r\rho} \log \frac{z_2}{e_1r\rho} \ll \frac{z_2}{r\rho},
\]
Notice that in order for such terms to exist, we need \(\rho \leq z_2/r\) and in particular, \(r \leq z_2\). Inserting this estimate into the sum over \(d_0\), we get
\[
\frac{z_2}{r\rho} \sum_{d_0 < \min(N/n_0z_2, z_2/\rho)} \log \frac{z_2}{d_0\rho}.
\]
We distinguish two sub cases.
4.C. Case 2(a)

Suppose that

\[ \frac{z_2}{\rho} < \frac{N}{n_0z_2}. \]

Then, the sum over \( d_0 \) is \( O(z_2/\rho) \) and so the overall contribution is

\[ \ll \sum_{\rho<r^2z_2/N} \sum_{n_0<\min(n_0z_2^2/N,z_2^2/\rho)} \frac{z_2^2}{\rho^2 n_0}. \]

This simplifies to

\[ \frac{z_2^2}{r} \sum_{\rho<r^2z_2/N} \frac{1}{\rho^2} N \rho \]

and this is

\[ \ll \frac{N}{r} \log \frac{rz_2^2}{N}. \]

4.D. Case 2(b)

Consider the remaining case

\[ \frac{z_2}{\rho} \geq \frac{N}{n_0z_2}. \]

Then, the sum over \( d_0 \) is

\[ \ll \frac{N}{n_0z_2} \log \left( \frac{n_0z_2^2}{\rho N} \right) \]

and so the overall contribution is

\[ \ll \sum_{n_0<r} \sum_{\rho<n_0z_2^2/N} \frac{z_2}{\rho} n_0 \frac{N}{\rho n_0z_2} \left( \log \frac{n_0z_2^2}{\rho N} \right) \]

which is

\[ = \frac{N}{r} \sum_{n_0<r} \frac{1}{n_0} \sum_{\rho<n_0z_2^2/N} \frac{1}{\rho} \left( \log \frac{n_0z_2^2}{\rho N} \right). \]

This sum can be split into two subsums, the first of which is

\[ \frac{N}{r} \sum_{n_0<\min(r,N/rz_2)} \frac{1}{n_0} \sum_{\rho<n_0z_2^2/N} \frac{1}{\rho} \left( \log \frac{n_0z_2^2}{\rho N} \right) \]

and this is

\[ \ll \frac{N}{r} \sum_{N/z_2^2<n_0<r} \frac{1}{n_0} \left( \log \frac{n_0z_2^2}{N} \right)^2 \]

which is

\[ \ll \frac{N}{r} \left( \log \frac{rz_2^2}{N} \right)^3. \]

The second is

\[ \frac{N}{r} \sum_{\rho<z_2/r} \frac{1}{\rho} \sum_{n_0/z_2^2<n_0<r} \frac{1}{n_0} \left( \log \frac{n_0z_2^2}{\rho N} \right) \]
which is seen to be
\[ \ll \frac{N}{r} \sum_{\rho < r z_2^2/N} \frac{1}{\rho} \left( \log \frac{r z_2^2}{N} \right)^2 \]
and this is
\[ \ll \frac{N}{r} \left( \log \frac{r z_2^2}{N} \right)^3. \]
Note that this term is present only if \( r^2 z_2^2 > N \).

To summarize, Case (2) occurs only if \( r < z_2 \) and in this case, it contributes
\[ \frac{N}{r} \left( \log \frac{r z_2^2}{N} \right)^3. \]

4.E. The contribution of terms with \( e_0 < r \)

By interchanging the roles of \( e_0 \) and \( d_0 \), we may also suppose that \( d_0 < r \). We see that as \( n_0 < r \), the congruence condition
\[ e_0 d_0 \rho n_0 \equiv b \pmod{r} \]
implies that \( e_0, d_0, \rho \) uniquely determine \( n_0 \). Thus, our sum is
\[ \leq \sum_{d_0, e_0} \sum_{\rho} \left( \log \frac{z_2}{d_0 \rho} \right) \left( \log \frac{z_2}{e_0 \rho} \right). \]
Here, the outer sum ranges over \( d_0, e_0 < r \) satisfying \( N/rz_2 < d_0 < z_2 \) and \( e_0 < z_2 \) and the inner sum ranges over \( \rho \) satisfying
\[ \frac{N}{rd_0 e_0} \leq \rho \leq \min \left( \frac{z_2}{d_0}, \frac{z_2}{e_0}, \frac{rz_2^2}{N} \right). \]
Since
\[ \frac{N}{rz_2} < d_0 \]
we see that
\[ \frac{z_2}{d_0} \leq \frac{r z_2^2}{N}. \]
Also,
\[ e_0 \geq \frac{N}{r \rho d_0} > \frac{N}{rz_2} \]
and so
\[ \frac{z_2}{e_0} \leq \frac{r z_2^2}{N}. \]
Let us set
\[ w = \min \left( \frac{z_2}{d_0}, \frac{z_2}{e_0} \right). \]
We will consider the case \( w = z_2/d_0 \), the other case being similar. In this case we must have
\[ rz_2 \leq N. \]
It forces the condition
\[ \frac{N}{rz_2} \leq e_0 \leq d_0. \]
Using the identity (4.2), the sum over $\rho$ is
\[ \ll \frac{N}{rd_0e_0} \log \left( \frac{N^2}{r^2z_2d_0e_0} \right) + \frac{z_2}{d_0} \log \frac{d_0}{e_0}. \]

Now we insert this into the sum over $d_0$ and $e_0$. For the $e_0$ sum to be nonempty, we must also have $r^2z_2 \geq N$ (since $r^2z_2 \geq rz_2d_0 \geq N$). In this case, the $e_0$ sum is
\[ \ll \frac{N}{r} \log \frac{N}{r^2} + z_2. \]
Summing this over $d_0$, we get an estimate of
\[ \ll \frac{N}{r} \left( \log \frac{N}{r^2} \right) (\log r) + r^2z_2 \]
and this is
\[ \leq \frac{N}{r} (\log r)^2 + r^2z_2. \]

5. Estimation of $S_C$

Proposition 5.1. Suppose that $rz_iz_j \geq N > r$. Then, we have
\[ \sum_{[d,e] \leq N/r} |\Lambda_i(d)\Lambda_j(e)| \ll \frac{N}{r} \left( \log \frac{r^2z_j}{N} \right)^4. \]

Proof. Set $\rho = (d,e)$. Thus, the sum is
\[ \sum_{d,e \leq N/r} |\Lambda_i(d)\Lambda_j(e)| = \sum_{d} |\Lambda_i(d)| \sum_{\rho|d} \sum_{e \leq N/rd \atop (\rho|e,d/\rho)=1} |\Lambda_j(e)|. \]
Write $d = d_0^\rho$ and $e = e_0^\rho$. Then the above is
\[ \sum_{\rho \leq z_i} \sum_{d_0 \leq z_i/\rho} |\Lambda_i(d_0^\rho)| \sum_{e_0 \leq N/rd \atop (e_0,d_0)=1} |\Lambda_j(e_0^\rho)|. \]
In the inner sum, we need in fact that
\[ e_0 \leq \min \left( \frac{N}{rd}, \frac{z_j}{\rho} \right). \]
Consider the contribution of terms with
\[ z_j \leq N/rd_0. \]
We have to estimate
\[ \sum_{\rho \leq z_i} \sum_{d_0 \leq \min(z_i/\rho,N/rd_0)} \left( \log \frac{z_i}{d_0^\rho} \right) \sum_{e_0 \leq z_j/\rho \atop (e_0,d_0)=1} \left( \log \frac{z_j}{e_0^\rho} \right). \]
We see that
\[
\frac{z_i}{\rho} < \frac{N}{r z_j}
\]  \tag{5.4}
holds if and only if
\[
\rho > \frac{r z_i z_j}{N}.
\]
In this case, the sum is
\[
\sum_{\rho > r z_i z_j/N} \sum_{d_0 \leq z_i/\rho} \sum_{e_0 \leq z_j/\rho} \left( \log \frac{z_i}{d_0 \rho} \right) \left( \log \frac{z_j}{e_0 \rho} \right)
\]
and this is
\[
\ll \sum_{\rho > r z_i z_j/N} \frac{z_i z_j}{\rho^2} \ll \frac{z_i z_j}{r z_i z_j} = \frac{N}{r}.
\]
The contribution of the remaining terms (the one that do not satisfy (5.4)) is
\[
\sum_{\rho \leq r z_i z_j/N} \sum_{d_0 \leq N/r z_j} \left( \log \frac{z_i}{d_0 \rho} \right) \sum_{e_0 \leq z_j/\rho} \left( \log \frac{z_j}{e_0 \rho} \right).
\]
This is seen to be
\[
\ll \sum_{\rho \leq r z_i z_j/N} \frac{z_j}{\rho} \left( \frac{N}{r z_j} \log \frac{r z_i z_j}{\rho N} \right) + O\left( \frac{N}{r z_i z_j} \right).
\]
Simplifying, this is
\[
\ll \frac{N}{r} \sum_{\rho \leq r z_i z_j/N} \frac{1}{\rho} \left( \log \frac{r z_i z_j}{N \rho} \right) + O\left( \frac{N}{r} \sum_{\rho \leq r z_i z_j/N} \frac{1}{\rho} \right)
\]
which is
\[
\ll \frac{N}{r} \left( \log \frac{r z_i z_j}{N} \right)^2.
\]
Consider now the case that
\[
z_j > \frac{N}{r d_0}.
\]
Note that
\[
\frac{z_i}{\rho} \geq d_0 > \frac{N}{r z_j}.
\]
Thus, \(\rho < r z_i z_j/N\). We have
\[
\sum_{\rho \leq \min (z_i, r z_i z_j/N)} \sum_{\frac{N}{r z_j} < d_0 \leq z_i/\rho} \sum_{e_0 \leq N/r d_0 \rho} \left( \log \frac{z_i}{d_0 \rho} \right) \left( \log \frac{z_j}{e_0 \rho} \right).
\]
The sum over \(e_0\) is
\[
\frac{N}{r d_0 \rho} \left( \log \frac{z_i r d_0 \rho}{N} \right) + O(1).
\]
Inserting this, we get that our sum is
\[
\ll \frac{N}{r} \sum_{\rho \leq \min (z_i, r z_i z_j/N)} \frac{1}{\rho} \sum_{\frac{N}{r z_j} < d_0 \leq z_i/\rho} \frac{1}{d_0} \left( \log \frac{z_i}{d_0 \rho} \right) \left( \log \frac{r d_0 z_j}{N} \right).
\]
The sum over \(d_0\) is easily seen to be
\[
\sim \frac{1}{6} \left( \log \frac{rz_i z_j}{N \rho} \right)^3.
\]
Inserting this, we find that our sum is
\[
\ll \frac{N}{r} \sum_{\rho \leq \min(z_i, rz_i, z_j/N)} \frac{1}{\rho} \left( \log \frac{rz_i z_j}{N \rho} \right)^3 \ll \frac{N}{r} \left( \log \frac{rz_i z_j}{N} \right)^4.
\]

6. The main term

Finally, we deal with the main term. We need the following technical result.

**Proposition 6.1.** Suppose that \(rz_i z_j \geq N > r\). Then, we have
\[
\sum_{\substack{|a_1 \cdot a_2| \leq M \quad (a_1, r_1) = (a_2, r_2) = 1}} \frac{\Lambda_i(d_1 a_1) \Lambda_j(d_2 a_2)}{a_1 a_2} = \mu(d_1) \mu(d_2) \frac{d_1 r_1}{\phi(d_1 r_1)} \frac{d_2 r_2}{\phi(d_2 r_2)} \sum_{\gamma} \frac{\mu(\gamma)^2}{\phi(\gamma)^2} + O \left( \frac{d_1 r_1}{\phi(d_1 r_1)} (\log d_1)^4 \right) + E
\]
where \(\gamma\) ranges over
\[
\gamma \leq \min\left(\frac{z_i}{d_1}, \frac{z_j}{d_2}\right)
\]
and
\[
(\gamma, r_1 r_2 d_1 d_2) = 1.
\]
Here,
\[
E \ll \frac{d_1 r_1}{\phi(d_1 r_1)} \sigma_1(d_2 r_2)(\log 2 z_j/d_2)^{-4}
+ \frac{d_2 r_2}{\phi(d_2 r_2)} \sigma_1(d_1 r_1)(\log 2 z_i/d_1)^{-4}
+ \sigma_1(d_1 r_1) \sigma_1(d_2 r_2)(\log 2 z_i/d_1)^{-4}(\log 2 z_j/d_2)^{-4}.
\]

**Proof.** Write \(\gamma = (a_1, a_2)\). The condition \([a_1, a_2] \leq M\) is then \(a_1 a_2 \leq M \gamma\). Write
\[
a_1 = \gamma a_1', \quad a_2 = \gamma a_2'.
\]
Then, our sum is
\[
\sum_{(\gamma, r_1 r_2 d_1 d_2) = 1} \sum_{(a_1', r_1) = 1} \frac{\Lambda_i(d_1 a_1')}{\gamma a_1'} \sum_{(a_2', r_2) = 1} \frac{\Lambda_j(d_2 a_2')}{\gamma a_2'}
\]
since \(a_1 a_2 \leq M \gamma\) means that \(\gamma a_1' a_2' \leq M\). Moreover, we may as well assume that
\[
(a_1, d_1) = (a_2, d_2) = 1.
\]
Thus, we can rewrite our sum as
\[
\sum_{(\gamma, r_1 r_2 d_1 d_2) = 1} \frac{\mu(d_1) \mu(d_2)}{\gamma^2} \sum_{(a_1', d_1, \gamma_1) = 1} \frac{\mu(a_1') \log z_i/d_1 \gamma a_1'}{a_1'} \times \sum_{(a_2', d_2, \gamma_2) = 1} \frac{\mu(a_2') \log z_j/d_2 \gamma a_2'}{a_2'}
\]
Suppose that $a'_1 \geq Md_2/z_j$. Then, the innermost sum is (using Lemma 2.5 quoted in §2)

$$\frac{d_2 \gamma r_2}{\phi(d_2 \gamma r_2)} + O\left(\sigma_{-\frac{1}{2}}(d_2 \gamma r_2)(\log 2z_j/d_2 \gamma)^{-4}\right) + O\left(\left(\log \frac{z_j a'_1}{d_2 M}\right)^2\right)$$

where the last term is present only if

$$\frac{M}{a'_1} < \frac{z_j}{d_2}.$$ 

The contribution of the last term above, when inserted into the $a'_1$ sum is

$$\ll \sum_{a'_1} \frac{1}{a'_1} \left(\log \frac{z_i}{d_1 \gamma a'_1}\right) \left(\log \frac{z_j a'_1}{d_2 M}\right)^2$$

where the range of the sum is

$$\frac{Md_2}{z_j} \leq a'_1 \leq \frac{z_i}{d_1 \gamma}.$$ 

We see that it is

$$\ll \left(\log \frac{z_i/d_1 \gamma}{d_2 M/z_j}\right)^4 = \left(\log \frac{z_i z_j}{d_1 d_2 \gamma M}\right)^4.$$ 

Inserting this into the $\gamma$ sum yields an error term of

$$\ll \sum_{\gamma} \frac{1}{\gamma^2} \left(\log \frac{z_i z_j}{d_1 d_2 \gamma M}\right)^4 \ll \left(\log \frac{z_i z_j}{d_1 d_2 \gamma M}\right)^4.$$ 

We are left with the problem of estimating

$$\sum_{(\gamma, r_2) = 1} \frac{\mu(d_1 \gamma) \mu(d_2 \gamma)}{\gamma^2} \left(\sum_{(a'_1, d_1 \gamma r_1) = 1} \frac{\mu(a'_1) \log z_i/d_1 \gamma a'_1}{a'_1} \left(\frac{d_2 \gamma r_2}{\phi(d_2 \gamma r_2)} + O\left(\sigma_{-\frac{1}{2}}(d_2 \gamma r_2)(\log 2z_j/d_2 \gamma)^{-4}\right)\right)\right).$$

The sum over $a'_1$ can also be estimated using the Lemma 2.5. It is equal to

$$\frac{d_1 \gamma r_1}{\phi(d_1 \gamma r_1)} + O\left(\sigma_{-\frac{1}{2}}(d_1 \gamma r_1)(\log 2z_i/d_1 \gamma)^{-4}\right).$$

Inserting this, we find that the main terms give

$$\sum_{(\gamma, r_2) = 1} \frac{\mu(d_1 \gamma) \mu(d_2 \gamma)}{\gamma^2} \frac{d_1 d_2 r_1 r_2 \gamma^2}{\phi(d_1 \gamma r_1) \phi(d_2 \gamma r_2)}$$

which is equal to

$$\mu(d_1) \mu(d_2) \frac{d_1 r_1}{\phi(d_1 r_1)} \frac{d_2 r_2}{\phi(d_2 r_2)} \sum_{(\gamma, r_2) = 1} \frac{\mu(\gamma)}{\phi(\gamma)^2}.$$ 

The sum over $\gamma$ extends to

$$\gamma \leq \min\left(\frac{z_i}{d_1}, \frac{z_j}{d_2}\right).$$ 

Now we consider the cross terms. There are three of them. The first is

$$\ll \sum_{(\gamma, r_2) = 1} \frac{1}{\gamma^2} \frac{d_1 \gamma r_1}{\phi(d_1 \gamma r_1)} \sigma_{-\frac{1}{2}}(d_2 \gamma r_2)(\log 2z_j/d_2 \gamma)^{-4}$$

...
which is
\[ \ll \frac{d_1 r_1}{\phi(d_1 r_1)} \sigma_{-\frac{1}{2}}(d_2 r_2)(\log 2z_j/d_2)^{-4}. \]
Similarly, the second is
\[ \ll \frac{d_2 r_2}{\phi(d_2 r_2)} \sigma_{-\frac{1}{2}}(d_1 r_1)(\log 2z_i/d_1)^{-4} \]
and the third is
\[ \ll \sigma_{-\frac{1}{2}}(d_1 r_1) \sigma_{-\frac{1}{2}}(d_2 r_2)(\log 2z_i/d_1)^{-4}(\log 2z_j/d_2)^{-4}. \]
This proves the result.

We only need to apply this result in the following case.

**Proposition 6.2.** Suppose that \( r \geq 2 \geq N > r. \) Then, we have
\[ \sum_{[d,e] \leq N/r \atop (d,e)=1} \frac{\Lambda_2(d)\Lambda_2(e)}{|d,e|} = \frac{r}{\phi(r)} \log z_2 + O \left( \left( \log \frac{r z_2^2}{N} \right)^5 \right) + O \left( \frac{r}{\phi(r)} \sigma_{-\frac{1}{2}}(r)(\log 2z_2)^{-4} \right). \]

**Proof.** Let us set \( \rho = (d,e). \) Then the sum in question may be written as
\[ \sum_{[d,e] \leq N/r \atop (d,e)=1} \frac{\Lambda_2(d)\Lambda_2(e)}{|d,e|} \sum_{u \rho} \phi(u) = \sum_{u \leq z_2} \frac{\phi(u)}{u^2} \sum_{u \leq z_2} \frac{\phi(u)}{u^2} \sum_{[d_1,e_1] \leq N/ru \atop (d_1,e_1)=1} \frac{\Lambda_2(u d_1)\Lambda_2(u e_1)}{d_1 e_1}. \]

Applying Proposition 5.1 to the inner sum, we find that the above is
\[ \sum_{u \leq z_2} \frac{\phi(u)}{u^2} \left( \frac{\mu(u)^2}{\phi(u^2)} \sum_{\gamma_1 \leq z_2/u} \frac{\mu(\gamma_1)^2}{\phi(\gamma_1)^2} + E \right) \]
where
\[ E = O \left( \left( \frac{ur}{\phi(ur)} \right)^2 \left( \log \log z_2 \right)^2 \right) + O \left( \frac{ur}{\phi(ur)} \sigma_{-\frac{1}{2}}(ur)(\log 2z_2/u)^{-4} \right) \]
\[ + O(\sigma_{-\frac{1}{2}}(ur)^2(\log 2z_2/u)^{-8}) + O \left( \left( \log \frac{r z_2^2}{uN} \right)^4 \right). \]
This is
\[ \sum_{u \leq z_2} \frac{\mu(u)^2}{\phi(u)} \frac{r^2}{\phi(r)^2} \sum_{\gamma_1 \leq z_2/u} \frac{\mu(\gamma_1)^2}{\phi(\gamma_1)^2} + O \left( \frac{r}{\phi(r)} \sigma_{-\frac{1}{2}}(r)(\log 2z_2)^{-4} \right) + O \left( \left( \log \frac{r z_2^2}{N} \right)^5 \right). \]
The main term is
\[ \frac{r^2}{\phi(r)^2} \left( \sum_{\gamma_1} \frac{\mu(\gamma_1)^2}{\phi(\gamma_1)^2} \right) \sum_{u \leq z_2} \frac{\mu(u)^2}{\phi(u)} \prod_{p | ur} \left( 1 + \frac{1}{(p-1)^2} \right)^{-1} + O \left( \frac{r^2}{\phi(r)^2} \log \log z_2 \right). \]
7. Proof of Theorems 3.1 and 3.2

As described in §3, we have written

\[
\sum_{n \equiv b \pmod{r}} \left( \sum_{d|n} \Lambda_2(d) \right) \left( \sum_{e|n} \Lambda_2(e) \right) = S_A + S_B + O(S_C),
\]

where in $S_A$, we require that $[d, e] > N/r$.

By Proposition 4.1, we deduce that

\[
S_A \ll \frac{N}{r} \left( \log \frac{rz_2}{N} \right)^5 + \frac{N}{r} (\log r)^2 + rz_2.
\]

As for $S_C$, we have by Proposition 5.1 that

\[
S_C \ll \frac{N}{r} \left( \log \frac{rz_2}{N} \right)^4.
\]

By Proposition 6.2, we have

\[
S_B = \frac{N}{r} \left( \frac{r}{\phi(r)} \log z_2 + O\left( \frac{r}{\phi(r)} \frac{\sigma_1}{2} (\log 2z_2)^{-4} \right) \right).
\]

This proves Theorem 3.1.

For Theorem 3.2, as in §2, we have to estimate three sums of the form

\[
S_{i,j} = \sum_{n \equiv b \pmod{r}} \left( \sum_{d|n} \Lambda_i(d) \right) \left( \sum_{e|n} \Lambda_j(e) \right).
\]

For the cases $i = j = 1$ and $i = 1, j = 2$, the condition $N \geq rz_1z_2$ is satisfied and so we get the desired estimate from Proposition 1. The only remaining case is $i = j = 2$ where this condition is not satisfied. This case follows from Theorem 3.1.

References


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