Hardy-Ramanujan Journal 41 (2018), 157-171 submitted 02/11/2018, accepted 13/11/2018, revised 23/12/2018

The Barban-Vehov Theorem in Arithmetic Progressions

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To the memory of S. Srinivasan

Abstract. A result of Barban-Vehov (and independently Motohashi) gives an estimate for the mean square of a sequence related to Selberg's sieve. This upper bound was refined to an asymptotic formula by S. Graham in 1978. In 1992, I made the observation that Graham's method can be used to obtain an asymptotic formula when the sum is restricted to an arithmetic progression. This formula immediately gives a version of the Brun-Titchmarsh theorem. I am taking the occasion of a volume in honour of my friend S. Srinivasan to revisit and publish this observation in the hope that it might still be of interest.

Keywords. Selberg's sieve, Brun-Titchmarsh theorem, arithmetic progressions

2010 Mathematics Subject Classification. Primary 11N37, 11N13 ; Secondary 11B25, 11N35, 11N69

1. Introduction

Let $1 \le z_1 \le z_2$ and define for $i = 1, 2$,

$$\Lambda_i(n) \;=\; \begin{cases} \mu(n)\log\frac{z_i}{n} & \text{if } n \le z_i \\ 0 & \text{if } n > z_i \end{cases}$$

Also, set

$$\lambda_n \;=\; \frac{\Lambda_2(n) - \Lambda_1(n)}{\log z_2/z_1}$$

and

$$a(n) \;=\; \sum_{d|n} \lambda_d.$$

The λ_n are weights that are related to Selberg's sieve. Notice that we have

$$a(1) = 1$$

and $a(n) = 0$ for $1 < n \le z_1$. Moreover, for primes p, we have

$$a(p) \;=\; \begin{cases} \frac{\log p/z_1}{\log z_2/z_1} & \text{if } z_1 < p < z_2 \\ 0 & \text{otherwise.} \end{cases}$$

It was shown by Barban and Vehov [BaVe68] and Motohashi [Mo74] that

$$\sum_{n\le N} |a(n)|^2 \;\ll\; \frac{N}{\log z_2/z_1}.$$

Soon afterwards, S. Graham [Gr78] was able to prove the following asymptotic formulae: if $N \ge z_2$, then

$$\sum_{n\le N} |a(n)|^2 \;=\; \frac{N}{\log z_2/z_1} + \mathbf{O}\left(\frac{N}{(\log z_2/z_1)^2}\right)$$

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and if $z_1 \le N \le z_2$, we have

$$\sum_{n \le N} |a(n)|^2 = \frac{N \log N/z_1}{(\log z_2/z_1)^2} + \mathbf{O}\left(\frac{N}{(\log z_2/z_1)^2}\right).$$

This result has found significant applications to zero density theorems and to the estimation of Linnik's constant (see [Ju77] and [Gr81] for example).

The purpose of this note is to study the size of the sum when n is constrained to range through a fixed arithmetic progression. In my joint work with R. Balasubramanian [BaMu92], we observed (Proposition 1.2) that for $N \ge r$ and $(b, r) = 1$, we have

$$\sum_{\substack{n \le N \\ n \equiv b \pmod r}} |a(n)| \ll \frac{N}{\phi(r)^{\frac{1}{2}} (\log z_2/z_1)^{\frac{1}{2}}}.$$

This follows immediately from Graham's result by the Cauchy-Schwarz inequality.

Soon after [BaMu92] was written, I worked out an asymptotic formula for the sum on the left by adapting Graham's methods. The result is that for $(b, r) = 1$ and $N \ge rz_2^2$, we have

$$\sum_{\substack{n \le N \\ n \equiv b \bmod r}} |a(n)|^2 = \frac{N}{\phi(r) \log z_2/z_1} + \mathbf{O}\left(\frac{N\sigma(r)}{\phi(r)^2 (\log z_2/z_1)^2}\right).$$

If $rz_1z_2 \le N \le rz_2^2$, then we show that

$$\sum_{\substack{n \le N \\ n \equiv b \bmod r}} |a(n)|^2 = \frac{N}{\phi(r) \log z_2/z_1} + \mathbf{O}\left(\frac{N(\log rz_2^2/N)^5}{r(\log z_2/z_1)^2}\right) + \mathbf{O}\left(\frac{N(\log r)^2}{r(\log z_2/z_1)^2}\right) + \mathbf{O}\left(\frac{rz_2}{(\log z_2/z_1)^2}\right).$$

The last two terms on the right are not present if we have the additional condition $z_1 > r$. Also, It will be clear from the arguments that the same methods will actually allow us to get estimates for $N < rz_1z_2$ as well but we do not pursue that here.

An immediate consequence of the first formula is a version of the Brun-Titchmarsh theorem in the following form. Denote by $\pi(N, r, b)$ the number of primes $\le N$ which are $\equiv b \bmod r$. Then we have

$$\pi(N, r, b) \le \frac{2N}{\phi(r) \log N/r} + \mathbf{O}\left(\frac{N\sigma(r)}{\phi(r)^2 (\log N/r)^2}\right).$$

We could try to use the second formula in a similar manner.

Our method of proof for both formulae is elementary and uses only the usual prime number theorem. The proof of the first formula is a direct generalization of the work of Graham ([Gr78], §3) and represents the easy case.

Though these calculations were completed some years ago, I had not published them. However, the occasion of a volume honouring the memory of my friend S. Srinivasan caused me to look at them again. In particular, the consequence for the Brun-Titchmarsh theorem may still be of interest. When I was a Visiting Fellow at the Tata Institute for Fundamental Research in 1983-1984, Srinivasan was my office-mate and we shared many hours of mathematical conversation. We also enjoyed many social occasions together when we had a chance to discuss philosophical and even spiritual questions. Srinivasan was always a thorough and thoughtful individual and I look back on those occasions with many pleasant memories. Given that his main interest was in analytic number theory, I thought the topic of this article might have been of interest to him.

The note is organized as follows. In §2, we prove the first formula. In §3, we begin the proof of the second formula. In §4 and §5, we estimate certain error terms and in §6, we study the main term. Finally in §7, we complete the proof of the second formula.

I would like to thank the referees for helpful comments that helped to streamline the presentation.

2. The First Formula

The purpose of this section is to prove the following.

Theorem 2.1. *Suppose that $rz_2^2 \le N$ and $(b,r)=1$. Then,*

$$\sum_{\substack{n \le N \\ n \equiv b \pmod r}} |a(n)|^2 = \frac{N}{\phi(r)\log z_2/z_1} + \mathbf{O}\left(\frac{N\sigma(r)}{\phi(r)^2(\log z_2/z_1)^2}\right).$$

The proof follows closely the method of [Gr78], §3 but we give the details in order to orient the reader. First, we note an immediate consequence. Denote by $\psi(N,z_2,r,b)$ the number of integers $n \le N$ with $n \equiv b \pmod r$ all of whose prime factors are $\ge z_2$. Denote also by $\pi(N,r,b)$ the number of primes $p \le N$ with $p \equiv b \pmod r$.

Corollary 2.2. *If $rz_2^2 \le N$, then*

$$\psi(N,z_2,r,b) \le \frac{N}{\phi(r)\log z_2} + \mathbf{O}\left(\frac{N\sigma(r)}{\phi(r)^2(\log z_2)^2}\right).$$

Proof. If $n \le N$, and $n \equiv b \pmod r$ and it has all its prime divisors $\ge z_2$, then $a(n) = 1$.

Corollary 2.3. *We have*

$$\pi(N,r,b) \le \frac{2N}{\phi(r)\log N/r} + \mathbf{O}\left(\frac{N\sigma(r)}{\phi(r)^2(\log N/r)^2}\right).$$

Proof. This follows on noting that

$$\pi(N,r,b) \le \pi(z_2,r,b) + \psi(N,z_2,r,b),$$

and the trivial bound $\pi(z_2,r,b) \le z_2/r+1$ and choosing $z_2 = (N/r)^{\frac{1}{2}}$.

To prove the theorem, we notice that

$$(\log z_2/z_1)^2 \sum_{\substack{n \le N \\ n \equiv b \pmod r}} |a(n)|^2 = \sum_{\substack{n \le N \\ n \equiv b \pmod r}} \left(\sum_{d|n}\Lambda_1(d) - \sum_{e|n}\Lambda_2(e)\right)^2.$$

The right hand side is a sum of terms of the form

$$S_{i,j} = \sum_{\substack{n \le N \\ n \equiv b \pmod r}} \sum_{d,e|n} \Lambda_i(d)\Lambda_j(e).$$

where $i,j \in \{1,2\}$. The theorem will follow from the following.

Proposition 2.4. *We have*

$$S_{i,j} = \frac{N}{\phi(r)}\log\min(z_i,z_j) + \mathbf{O}(z_iz_j) + \mathbf{O}(N\sigma(r)/\phi(r)^2).$$

In particular, if $rz_iz_j \le N$, the first error term is $\mathbf{O}(N/r)$.

Proof. By definition

$$S_{i,j} \;=\; \sum \Lambda_i(d)\Lambda_j(e)\{\frac{N}{r[d,e]} + \mathbf{O}(1)\}.$$

The error term is

$$\ll \left(\sum_{d\le z_i} |\Lambda_i(d)|\right)\left(\sum_{e\le z_j} |\Lambda_j(e)|\right) \;\ll\; z_i z_j.$$

The main term is

$$\frac{N}{r} \sum_{\substack{d\le z_i \\ e\le z_j \\ (d,r)=(e,r)=1}} \frac{\Lambda_i(d)\Lambda_j(e)}{[d,e]}.$$

Without loss of generality, we may suppose that $z_i \le z_j$. We have

$$\sum_{\substack{d\le z_i \\ e\le z_j \\ (d,r)=(e,r)=1}} \frac{\Lambda_i(d)\Lambda_j(e)}{[d,e]} \;=\; \sum \frac{\Lambda_i(d)\Lambda_j(e)}{de} \sum_{m|(d,e)} \phi(m)$$

and inserting the definition of Λ_i and Λ_j, the right hand side is seen to be

$$\sum_{\substack{m\le z_i \\ (m,r)=1}} \frac{\mu(m)^2\phi(m)}{m^2}\left(\sum_{\substack{d_0\le z_i/m \\ (d_0,mr)=1}} \frac{\mu(d_0)}{d_0}\log\frac{z_i}{md_0}\right)\left(\sum_{\substack{e_0\le z_j/m \\ (e_0,mr)=1}} \frac{\mu(e_0)}{e_0}\log\frac{z_j}{me_0}\right).$$

We quote the following from [Gr78], §2:

Lemma 2.5. *For any integer a and any $c>0$, we have*

$$\sum_{\substack{n\le Q \\ (n,a)=1}} \frac{\mu(n)}{n}\log\frac{Q}{n} \;=\; \frac{a}{\phi(a)} \;+\; \mathbf{O}_c(\sigma_{-\frac{1}{2}}(a)(\log 2Q)^{-c}).$$

Using this on the terms in parentheses, we find that the above is

$$\sum_{\substack{m\le z_i \\ (m,r)=1}} \frac{\mu(m)^2\phi(m)}{m^2}\left\{\frac{mr}{\phi(mr)} + \mathbf{O}(\sigma_{-\frac{1}{2}}(mr)(\log 2z_i/m)^{-2}\right\}^2$$

The error terms are $\mathbf{O}(1)$ just as in [Gr78], pp. 89-90. The main term is

$$\frac{r^2}{\phi(r)^2}\sum_{\substack{m\le z_i \\ (m,r)=1}} \frac{\mu(m)^2}{\phi(m)}.$$

This is seen to be

$$\frac{r}{\phi(r)}\log z_i \;+\; \mathbf{O}(\frac{r\sigma(r)}{\phi(r)^2}).$$

3. The second formula

The aim of the remaining sections is to prove the following asymptotic formula which will suffice to deduce the second formula.

Theorem 3.1. *Suppose that $rz_1z_2 \le N \le rz_2^2$. Then for $(b,r)=1$, we have*

$$\sum_{\substack{n\le N\\ n\equiv b \bmod r}} \left(\sum_{d|n}\Lambda_2(d)\right)^2 = \frac{N}{\phi(r)}\log z_2 + \mathbf{O}\left(\frac{N}{r}(\log rz_2^2/N)^5\right) + \mathbf{O}\left(\frac{N}{r}(\log r)^2\right) + \mathbf{O}(rz_2).$$

Expanding the sum on the left, we have

$$\sum_{\substack{n\le N\\ n\equiv b \bmod r}} \left(\sum_{d|n}\Lambda_2(d)\right)\left(\sum_{e|n}\Lambda_2(e)\right)$$

and we begin the proof by splitting this sum into three components

$$S_A + S_B + \mathbf{O}(S_C)$$

where in S_A we restrict the sum to d,e satisfying $[d,e] > N/r$. The remaining terms may be rearranged to give

$$\sum_{[d,e]\le N/r} \Lambda_2(d)\Lambda_2(e) \sum_{\substack{n\le N\\ n\equiv b \bmod r\\ n\equiv 0 \bmod [d,e]}} 1$$

which is

$$= \frac{N}{r}\sum_{\substack{[d,e]\le N/r\\ (d,r)=(e,r)=1}} \frac{\Lambda_2(d)\Lambda_2(e)}{[d,e]} + \mathbf{O}\left(\sum_{[d,e]\le N/r}|\Lambda_2(d)\Lambda_2(e)|\right)$$

which we write as

$$S_B + \mathbf{O}(S_C).$$

In §4., we shall show that

$$S_A \ll \frac{N}{r}\left(\log rz_2^2/N\right)^5 + \frac{N}{r}(\log r)^2 + rz_2.$$

In §5., we shall show that

$$S_C \ll \frac{N}{r}\left(\log rz_2^2/N\right)^4.$$

Finally, in §6., we shall deal with the main term S_B, and in §7. we collect together the various pieces to complete the proof of Theorem 3.1.

We state explicitly the consequence of Theorem 3.1 for the $a(n)$.

Theorem 3.2. *Suppose that $rz_2^2 \ge N \ge rz_1z_2$. Then*

$$\sum_{\substack{n\le N\\ n\equiv b \pmod r}} |a(n)|^2 = \frac{N}{\phi(r)\log z_2/z_1} + \mathbf{O}\left(\frac{N(\log rz_2^2/N)^5}{r(\log z_2/z_1)^2}\right) + \mathbf{O}\left(\frac{N(\log r)^2}{r(\log z_2/z_1)^2}\right) + \mathbf{O}\left(\frac{rz_2}{(\log z_2/z_1)^2}\right).$$

4. Estimation of S_A

Proposition 4.1. *Suppose that $rz_2^2 \geq N > r$. Then*

$$\sum_{\substack{n\leq N\\ n\equiv b \pmod r}} \sum_{\substack{[d,e]|n\\ [d,e]>N/r}} |\Lambda_2(d)\Lambda_2(e)| \ll \frac{N}{r}\left(\log rz_2^2/N\right)^5 + \frac{N}{r}(\log r)^2 + rz_2.$$

Remark 4.2. *Our argument will show that in certain ranges of r, z_i, z_j and N, this estimate can be refined.*

.

Proof. Write $\rho = (d,e)$. Then, each n can be written as $n = \rho e_0 d_0 n_0$ with $d = \rho d_0$, $e = \rho e_0$ and $[d,e] = \rho d_0 e_0 > N/r$. This last condition implies that $n_0 < r$. The sum over e_0 then ranges over the interval

$$\frac{N}{r\rho d_0} < e_0 \leq \frac{z_2}{\rho}.$$

In order for this to be nonempty, we need

$$d_0 > N/rz_2.$$

But $d_0 \leq z_2/\rho$ and so $\rho < rz_2^2/N$. Thus, our sum is

$$\sum_{\substack{\rho<rz_2^2/N\\ (\rho,r)=1}} \sum_{\substack{n_0<r\\ (n_0,r)=1}} \sum_{\substack{\frac{N}{r\rho}<d_0e_0\leq\frac{N}{\rho n_0}\\ d_0e_0\equiv\overline{\rho n_0}b \pmod r\\ d_0<z_2/\rho, e_0<z_2/\rho}} |\Lambda_2(d_0\rho)\Lambda_2(e_0\rho)|.$$

Here, for any residue class $\beta \pmod r$ (with $(\beta,r)=1$), we are writing $\overline{\beta} \pmod r$ for the inverse class. The inner sum is

$$\sum_{\substack{\frac{N}{r\rho}<d_0e_0\leq\frac{N}{\rho n_0}\\ d_0e_0\equiv\overline{\rho n_0}b \pmod r}} |\mu(d_0\rho)\left(\log z_2/d_0\rho\right)\mu(e_0\rho)\left(\log z_2/e_0\rho\right)|.$$

Separating the d_0 and the e_0 sums, we find that this is

$$\sum_{d_0} |\mu(d_0\rho)\log\frac{z_2}{d_0\rho}| \sum_{e_0} |\mu(e_0\rho)\log\frac{z_2}{e_0\rho}|$$

where the sum over d_0 is in the range

$$\frac{N}{rz_2} < d_0 < \min\left(\frac{z_2}{\rho}, \frac{N}{n_0\rho}\right)$$

and the sum over e_0 is in the range

$$\frac{N}{rd_0\rho} < e_0 < \min\left(\frac{z_2}{\rho}, \frac{N}{n_0d_0\rho}\right)$$

with the additional condition

$$e_0 \equiv b\cdot \overline{d_0\rho n_0} \pmod r.$$

First, consider the contribution of the e_0 which satisfy $e_0 > r$. Writing $e_0 = * + e_1 r$, we see that the inner sum is

$$\leq \sum_{1\leq e_1\leq\min\left(\frac{N}{n_0\rho d_0 r}, \frac{z_2}{r\rho}\right)} \log\frac{z_2}{e_1 r\rho}. \tag{4.1}$$

4.A. Case 1

If $N/n_0 d_0 \le z_2$, then this is

$$\ll \frac{N}{n_0 \rho d_0 r}\left(\log \frac{z_2 n_0 d_0}{N} + 1\right).$$

Inserting this into the sum over d_0, we find that it is

$$\ll \frac{N}{n_0 \rho r} \sum_{\frac{N}{z_2 n_0} < d_0 \le \frac{z_2}{\rho}} \frac{1}{d_0}\left(\log \frac{z_2}{d_0 \rho}\right)\left(\log \frac{z_2 n_0 d_0}{N} + 1\right).$$

For this sum to be nonempty, we require

$$\rho < z_2^2 n_0/N.$$

Using the following consequence of the arithmetic mean - geometric mean inequality,

$$(\log A)(\log B) \ \le (\log AB)^2 \tag{4.2}$$

for $A, B \ge 1$, the above sum is

$$\ll \frac{N}{n_0 \rho r} \sum \frac{1}{d_0}\left(\log \frac{z_2^2 n_0}{N\rho}\right)^2.$$

The sum over d_0 is

$$\ll \log \frac{z_2^2 n_0}{N\rho}$$

and so, we have to estimate

$$\sum_{n_0 < r} \sum_{\rho < n_0 z_2^2/N} \frac{N}{r n_0 \rho}\left(\log \frac{z_2^2 n_0}{N\rho}\right)^3$$

and this is

$$\ll \frac{N}{r} \sum_{N/z_2^2 \le n_0 < r} \frac{1}{n_0}\left(\log \frac{z_2^2 n_0}{N}\right)^4$$

which in turn is

$$\ll \frac{N}{r}\left(\log \frac{r z_2^2}{N}\right)^5. \tag{4.3}$$

4.B. Case 2

If $N/n_0 d_0 \ge z_2$, the sum in (4.1) is

$$\ll \sum_{e_1 \le z_2/r\rho} \log \frac{z_2}{e_1 r \rho} \ \ll \ \frac{z_2}{r\rho}.$$

Notice that in order for such terms to exist, we need $\rho \le z_2/r$ and in particular, $r \le z_2$. Inserting this estimate into the sum over d_0, we get

$$\frac{z_2}{r\rho} \sum_{d_0 < \min(N/n_0 z_2, z_2/\rho)} \log \frac{z_2}{d_0 \rho}.$$

We distinguish two sub cases.

4.C. Case 2(a)

Suppose that

$$z_2/\rho < N/n_0 z_2.$$

Then, the sum over d_0 is $\mathbf{O}(z_2/\rho)$ and so the overall contribution is

$$\ll \sum_{\rho < r z_2^2/N} \sum_{\substack{n_0 < N\rho/z_2^2 \\ n_0 < r}} \frac{z_2^2}{r\rho^2}.$$

This simplifies to

$$\frac{z_2^2}{r} \sum_{\rho < r z_2^2/N} \frac{1}{\rho^2} \frac{N\rho}{z_2^2}$$

and this is

$$\ll \frac{N}{r} \log \frac{r z_2^2}{N}.$$

4.D. Case 2(b)

Consider the remaining case

$$z_2/\rho \geq N/n_0 z_2.$$

Then, the sum over d_0 is

$$\ll \frac{N}{n_0 z_2} \log \left(\frac{n_0 z_2^2}{\rho N} \right)$$

and so the overall contribution is

$$\ll \sum_{n_0 < r} \sum_{\rho < \min(n_0 z_2^2/N, z_2/r)} \frac{z_2}{\rho r} \frac{N}{n_0 z_2} \left(\log \frac{n_0 z_2^2}{\rho N} \right)$$

which is

$$= \frac{N}{r} \sum_{n_0 < r} \frac{1}{n_0} \sum_{\rho} \frac{1}{\rho} \left(\log \frac{n_0 z_2^2}{\rho N} \right).$$

This sum can be split into two subsums, the first of which is

$$\frac{N}{r} \sum_{n_0 < \min(r, N/r z_2)} \frac{1}{n_0} \sum_{\rho < n_0 z_2^2/N} \frac{1}{\rho} \left(\log \frac{n_0 z_2^2}{\rho N} \right)$$

and this is

$$\ll \frac{N}{r} \sum_{N/z_2^2 < n_0 < r} \frac{1}{n_0} \left(\log \frac{n_0 z_2^2}{N} \right)^2$$

which is

$$\ll \frac{N}{r} \left(\log \frac{r z_2^2}{N} \right)^3.$$

The second is

$$\frac{N}{r} \sum_{\rho < z_2/r} \frac{1}{\rho} \sum_{\substack{N/r z_2 < n_0 < r \\ \rho N/z_2^2 < n_0}} \frac{1}{n_0} \left(\log \frac{n_0 z_2^2}{\rho N} \right)$$

which is seen to be

$$\ll \frac{N}{r} \sum_{\rho < rz_2^2/N} \frac{1}{\rho} \left(\log \frac{rz_2^2}{N} \right)^2$$

and this is

$$\ll \frac{N}{r} \left(\log \frac{rz_2^2}{N} \right)^3 .$$

Note that this term is present only if $r^2 z_2 > N$.

To summarize, Case (2) occurs only if $r < z_2$ and in this case, it contributes

$$\frac{N}{r} \left(\log \frac{rz_2^2}{N} \right)^3 .$$

4.E. The contribution of terms with $e_0 < r$

By interchanging the roles of e_0 and d_0, we may also suppose that $d_0 < r$. We see that as $n_0 < r$, the congruence condition

$$e_0 d_0 \rho n_0 \equiv b \pmod r$$

implies that e_0, d_0, ρ uniquely determine n_0. Thus, our sum is

$$\le \sum_{d_0, e_0} \sum_{\rho} \left(\log \frac{z_2}{d_0 \rho} \right) \left(\log \frac{z_2}{e_0 \rho} \right) .$$

Here, the outer sum ranges over $d_0, e_0 < r$ satisfying $N/rz_2 < d_0 < z_2$ and $e_0 < z_2$ and the inner sum ranges over ρ satisfying

$$\frac{N}{rd_0e_0} \le \rho \le \min \left(\frac{z_2}{d_0}, \frac{z_2}{e_0}, \frac{rz_2^2}{N} \right) .$$

Since

$$\frac{N}{rz_2} < d_0$$

we see that

$$\frac{z_2}{d_0} \le \frac{rz_2^2}{N} .$$

Also,

$$e_0 \ge \frac{N}{r\rho d_0} > \frac{N}{rz_2}$$

and so

$$\frac{z_2}{e_0} \le \frac{rz_2^2}{N} .$$

Let us set

$$w = \min(\frac{z_2}{d_0}, \frac{z_2}{e_0}).$$

We will consider the case $w = z_2/d_0$, the other case being similar. In this case we must have

$$rz_2 \le N.$$

It forces the condition

$$\frac{N}{rz_2} \le e_0 \le d_0 .$$

Using the identity (4.2), the sum over ρ is

$$\ll \frac{N}{rd_0e_0}\log\left(\frac{N^2}{r^2z_2^2d_0e_0}\right) \;+\; \frac{z_2}{d_0}\log\frac{d_0}{e_0}.$$

Now we insert this into the sum over d_0 and e_0. For the e_0 sum to be nonempty, we must also have $r^2z_2 \geq N$ (since $r^2z_2 \geq rz_2d_0 \geq N$). In this case, the e_0 sum is

$$\ll \frac{N}{rd_0}\log\frac{N}{rz_2} + z_2.$$

Summing this over d_0, we get an estimate of

$$\ll \frac{N}{r}\left(\log\frac{N}{rz_2}\right)(\log r) + rz_2$$

and this is

$$\leq \frac{N}{r}(\log r)^2 + rz_2.$$

5. Estimation of S_C

Proposition 5.1. *Suppose that $rz_iz_j \geq N > r$. Then, we have*

$$\sum_{[d,e]\leq N/r}|\Lambda_i(d)\Lambda_j(e)| \;\ll\; \frac{N}{r}\left(\log\frac{rz_iz_j}{N}\right)^4.$$

Proof. Set $\rho = (d,e)$. Thus, the sum is

$$\sum_{de\leq N\rho/r}|\Lambda_i(d)\Lambda_j(e)| \;=\; \sum_{d}|\Lambda_i(d)|\sum_{\rho|d}\sum_{\substack{e\leq N\rho/rd\\ \rho|e\\ (e/\rho,d/\rho)=1}}|\Lambda_j(e)|.$$

Write $d = d_0\rho$ and $e = e_0\rho$. Then the above is

$$\sum_{\rho\leq z_i}\sum_{d_0\leq z_i/\rho}|\Lambda_i(d_0\rho)|\sum_{\substack{e_0\leq N/rd\\ (e_0,d_0)=1}}|\Lambda_j(e_0\rho)|.$$

In the inner sum, we need in fact that

$$e_0 \;\leq\; \min\left(\frac{N}{rd},\frac{z_j}{\rho}\right).$$

Consider the contribution of terms with

$$z_j \leq N/rd_0.$$

We have to estimate

$$\sum_{\rho\leq z_i}\sum_{d_0\leq\min(z_i/\rho,N/rz_j)}\left(\log\frac{z_i}{d_0\rho}\right)\sum_{\substack{e_0\leq z_j/\rho\\ (e_0,d_0)=1}}\left(\log\frac{z_j}{e_0\rho}\right).$$

We see that

$$\frac{z_i}{\rho} \;<\; \frac{N}{rz_j} \tag{5.4}$$

holds if and only if

$$\rho > \frac{rz_iz_j}{N}.$$

In this case, the sum is

$$\sum_{\rho > rz_iz_j/N} \sum_{d_0 \le z_i/\rho} \left(\log \frac{z_i}{d_0\rho}\right) \sum_{e_0 \le z_j/\rho} \left(\log \frac{z_j}{e_0\rho}\right)$$

and this is

$$\ll \sum_{\rho > rz_iz_j/N} \frac{z_iz_j}{\rho^2} \;\ll\; z_iz_j \frac{N}{rz_iz_j} \;=\; \frac{N}{r}.$$

The contribution of the remaining terms (the one that do not satisfy (5.4)) is

$$\sum_{\rho \le rz_iz_j/N} \sum_{d_0 \le N/rz_j} \left(\log \frac{z_i}{d_0\rho}\right) \sum_{e_0 \le z_j/\rho} \left(\log \frac{z_j}{e_0\rho}\right).$$

This is seen to be

$$\ll \sum_{\rho \le rz_iz_j/N} \frac{z_j}{\rho} \left(\frac{N}{rz_j} \left(\log \frac{z_i}{\rho}\frac{rz_j}{N}\right) \;+\; \mathbf{O}\left(\frac{N}{rz_j}\right)\right).$$

Simplifying, this is

$$\ll \frac{N}{r} \sum_{\rho < rz_iz_j/N} \frac{1}{\rho} \left(\log \frac{rz_iz_j}{N\rho}\right) + \mathbf{O}\left(\frac{N}{r} \sum_{\rho < rz_iz_j/N} \frac{1}{\rho}\right)$$

which is

$$\ll \frac{N}{r} \left(\log \frac{rz_iz_j}{N}\right)^2.$$

Consider now the case that

$$z_j > N/rd_0.$$

Note that

$$\frac{z_i}{\rho} \;\ge\; d_0 \;>\; \frac{N}{rz_j}.$$

Thus, $\rho < rz_iz_j/N$. We have

$$\sum_{\rho \le \min(z_i, rz_iz_j/N)} \sum_{\frac{N}{rz_j} < d_0 \le z_i/\rho} \left(\log \frac{z_i}{d_0\rho}\right) \sum_{e_0 \le N/rd_0\rho} \left(\log \frac{z_j}{e_0\rho}\right).$$

The sum over e_0 is

$$\frac{N}{rd_0\rho} \left(\log\left(\frac{z_j}{\rho}\frac{rd_0\rho}{N}\right) \;+\; \mathbf{O}(1)\right).$$

Inserting this, we get that our sum is

$$\ll \frac{N}{r} \sum_{\rho \le \min(z_i, rz_iz_j/N)} \frac{1}{\rho} \sum_{\frac{N}{rz_j} < d_0 \le z_i/\rho} \frac{1}{d_0} \left(\log \frac{z_i}{d_0\rho}\right)\left(\log \frac{rd_0z_j}{N}\right).$$

The sum over d_0 is easily seen to be

$$\sim \frac{1}{6}\left(\log \frac{rz_iz_j}{N\rho}\right)^3.$$

Inserting this, we find that our sum is

$$\ll \frac{N}{r} \sum_{\rho \le \min(z_i, rz_iz_j/N)} \frac{1}{\rho}\left(\log \frac{rz_iz_j}{N\rho}\right)^3 \ll \frac{N}{r}\left(\log \frac{rz_iz_j}{N}\right)^4.$$

6. The main term

Finally, we deal with the main term. We need the following technical result.

Proposition 6.1. *Suppose that $rz_iz_j \ge N > r$. Then, we have*

$$\sum_{\substack{[a_1,a_2]\le M \\ (a_1,r_1)=(a_2,r_2)=1}} \frac{\Lambda_i(d_1a_1)\Lambda_j(d_2a_2)}{a_1a_2} = \mu(d_1)\mu(d_2)\frac{d_1r_1}{\phi(d_1r_1)}\frac{d_2r_2}{\phi(d_2r_2)}\sum_{\gamma}\frac{\mu(\gamma)^2}{\phi(\gamma)^2} +$$

$$+\mathbf{O}\left(\left(\log \frac{z_iz_j}{d_1d_2M}\right)^4\right) + E$$

where γ ranges over

$$\gamma \le \min(\frac{z_i}{d_1}, \frac{z_j}{d_2})$$

and

$$(\gamma, r_1r_2d_1d_2) = 1.$$

Here,

$$\begin{aligned} E \ll\ & \frac{d_1r_1}{\phi(d_1r_1)}\sigma_{-\frac{1}{2}}(d_2r_2)(\log 2z_j/d_2)^{-4} \\ & + \frac{d_2r_2}{\phi(d_2r_2)}\sigma_{-\frac{1}{2}}(d_1r_1)(\log 2z_i/d_1)^{-4} \\ & + \sigma_{-\frac{1}{2}}(d_1r_1)\sigma_{-\frac{1}{2}}(d_2r_2)(\log 2z_i/d_1)^{-4}(\log 2z_j/d_2)^{-4}. \end{aligned}$$

Proof. Write $\gamma = (a_1, a_2)$. The condition $[a_1, a_2] \le M$ is then $a_1a_2 \le M\gamma$. Write

$$a_1 = \gamma a_1', \quad a_2 = \gamma a_2'.$$

Then, our sum is

$$\sum_{(\gamma, r_1r_2)=1} \sum_{(a_1', r_1)=1} \frac{\Lambda_i(d_1\gamma a_1')}{\gamma a_1'} \sum_{\substack{(a_2', r_2)=1 \\ a_2' \le M/\gamma a_1'}} \frac{\Lambda_j(d_2\gamma a_2')}{\gamma a_2'}$$

since $a_1a_2 \le M\gamma$ means that $\gamma a_1'a_2' \le M$. Moreover, we may as well assume that

$$(a_1, d_1) = (a_2, d_2) = 1.$$

Thus, we can rewrite our sum as

$$\sum_{(\gamma, r_1r_2d_1d_2)=1} \frac{\mu(d_1\gamma)\mu(d_2\gamma)}{\gamma^2} \sum_{\substack{(a_1', d_1\gamma r_1)=1 \\ a_1' \le \min(M/\gamma, z_i/d_1\gamma)}} \frac{\mu(a_1')\log z_i/d_1\gamma a_1'}{a_1'} \times \sum_{\substack{(a_2', d_2\gamma r_2)=1 \\ a_2' \le \min(M/\gamma a_1', z_j/d_2\gamma)}} \frac{\mu(a_2')\log z_j/d_2\gamma a_2'}{a_2'}.$$

Suppose that $a_1' \geq Md_2/z_j$. Then, the innermost sum is (using Lemma 2.5 quoted in §2)

$$\frac{d_2\gamma r_2}{\phi(d_2\gamma r_2)} \;+\; \mathbf{O}(\sigma_{-\frac{1}{2}}(d_2\gamma r_2)(\log 2z_j/d_2\gamma)^{-4}) \;+\; \mathbf{O}\left(\left(\log\frac{z_j a_1'}{d_2 M}\right)^2\right)$$

where the last term is present only if

$$\frac{M}{a_1'} \;<\; \frac{z_j}{d_2}.$$

The contribution of the last term above, when inserted into the a_1' sum is

$$\ll \sum_{a_1'} \frac{1}{a_1'}\left(\log\frac{z_i}{d_1\gamma a_1'}\right)\left(\log\frac{z_j a_1'}{d_2 M}\right)^2$$

where the range of the sum is

$$\frac{Md_2}{z_j} \;\leq\; a_1' \;\leq\; \frac{z_i}{d_1\gamma}.$$

We see that it is

$$\ll \left(\log\frac{z_i/d_1\gamma}{d_2M/z_j}\right)^4 \;=\; \left(\log\frac{z_iz_j}{d_1d_2\gamma M}\right)^4.$$

Inserting this into the γ sum yields an error term of

$$\ll \sum_{\gamma} \frac{1}{\gamma^2}\left(\log\frac{z_iz_j}{d_1d_2\gamma M}\right)^4 \;\ll\; \left(\log\frac{z_iz_j}{d_1d_2M}\right)^4.$$

We are left with the problem of estimating

$$\sum_{(\gamma,r_1r_2)=1} \frac{\mu(d_1\gamma)\mu(d_2\gamma)}{\gamma^2}\left(\sum_{(a_1',d_1\gamma r_1)=1}\frac{\mu(a_1')\log z_i/d_1\gamma a_1'}{a_1'}\left(\frac{d_2\gamma r_2}{\phi(d_2\gamma r_2)} \;+\; \mathbf{O}(\sigma_{-\frac{1}{2}}(d_2\gamma r_2)(\log 2z_j/d_2\gamma)^{-4})\right)\right).$$

The sum over a_1' can also be estimated using the Lemma 2.5. It is equal to

$$\frac{d_1\gamma r_1}{\phi(d_1\gamma r_1)} \;+\; \mathbf{O}(\sigma_{-\frac{1}{2}}(d_1\gamma r_1)(\log 2z_i/d_1\gamma)^{-4}).$$

Inserting this, we find that the main terms give

$$\sum_{(\gamma,r_1r_2d_1d_2)=1} \frac{\mu(d_1\gamma)\mu(d_2\gamma)}{\gamma^2}\frac{d_1d_2r_1r_2\gamma^2}{\phi(d_1\gamma r_1)\phi(d_2\gamma r_2)}$$

which is equal to

$$\mu(d_1)\mu(d_2)\frac{d_1r_1}{\phi(d_1r_1)}\frac{d_2r_2}{\phi(d_2r_2)}\sum_{(\gamma,r_1r_2d_1d_2)=1}\frac{\mu(\gamma)^2}{\phi(\gamma)^2}.$$

The sum over γ extends to

$$\gamma \leq \min(\frac{z_i}{d_1}, \frac{z_j}{d_2}).$$

Now we consider the cross terms. There are three of them. The first is

$$\ll \sum_{(\gamma,r_1r_2d_1d_2)=1} \frac{1}{\gamma^2}\frac{d_1\gamma r_1}{\phi(d_1\gamma r_1)}\sigma_{-\frac{1}{2}}(d_2\gamma r_2)(\log 2z_j/d_2\gamma)^{-4}$$

which is

$$\ll \frac{d_1 r_1}{\phi(d_1 r_1)} \sigma_{-\frac{1}{2}}(d_2 r_2)(\log 2z_j/d_2)^{-4}.$$

Similarly, the second is

$$\ll \frac{d_2 r_2}{\phi(d_2 r_2)} \sigma_{-\frac{1}{2}}(d_1 r_1)(\log 2z_i/d_1)^{-4}$$

and the third is

$$\ll \sigma_{-\frac{1}{2}}(d_1 r_1)\sigma_{-\frac{1}{2}}(d_2 r_2)(\log 2z_i/d_1)^{-4}(\log 2z_j/d_2)^{-4}.$$

This proves the result.

We only need to apply this result in the following case.

Proposition 6.2. *Suppose that $rz_2^2 \geq N > r$. Then, we have*

$$\sum_{\substack{[d,e]\leq N/r\\(d,r)=(e,r)=1}} \frac{\Lambda_2(d)\Lambda_2(e)}{[d,e]} \;=\; \frac{r}{\phi(r)}\log z_2 \;+\; \mathbf{O}\left(\left(\log\frac{rz_2^2}{N}\right)^5\right) \;+\; \mathbf{O}\left(\frac{r}{\phi(r)}\sigma_{-\frac{1}{2}}(r)(\log 2z_2)^{-4}\right).$$

Proof. Let us set $\rho = (d,e)$. Then the sum in question may be written as

$$\sum_{\substack{[d,e]\leq N/r\\(d,r)=(e,r)=1}} \frac{\Lambda_2(d)\Lambda_2(e)}{de}\sum_{u|\rho}\phi(u) \;=\; \sum_{\substack{u\leq z_2\\(u,r)=1}}\frac{\phi(u)}{u^2}\sum_{\substack{[d_1,e_1]\leq N/ru\\(d_1,r)=(e_1,r)=1}}\frac{\Lambda_2(ud_1)\Lambda_2(ue_1)}{d_1e_1}.$$

Applying Proposition 5.1 to the inner sum, we find that the above is

$$\sum_{\substack{u\leq z_2\\(u,r)=1}}\frac{\phi(u)}{u^2}\left(\mu(u)^2\left(\frac{ur}{\phi(ur)}\right)^2\sum_{\substack{(\gamma_1,ur)=1\\\gamma_1\leq z_2/u}}\frac{\mu(\gamma_1)^2}{\phi(\gamma_1)^2} \;+\; E\right)$$

where

$$E = \mathbf{O}\left(\left(\frac{ur}{\phi(ur)}\right)^2\frac{(\log\log z_2)^2}{z_2}\right) \;+\; \mathbf{O}\left(\frac{ur}{\phi(ur)}\sigma_{-\frac{1}{2}}(ur)(\log 2z_2/u)^{-4}\right)$$
$$+\;\mathbf{O}(\sigma_{-\frac{1}{2}}(ur)^2(\log 2z_2/u)^{-8}) \;+\; \mathbf{O}\left(\left(\log\frac{rz_2^2}{uN}\right)^4\right).$$

This is

$$\sum_{\substack{u\leq z_2\\(u,r)=1}}\frac{\mu(u)^2}{\phi(u)}\frac{r^2}{\phi(r)^2}\sum_{\substack{(\gamma_1,ur)=1\\\gamma_1\leq z_2/u}}\frac{\mu(\gamma_1)^2}{\phi(\gamma_1)^2} \;+\; \mathbf{O}\left(\frac{r}{\phi(r)}\sigma_{-\frac{1}{2}}(r)(\log 2z_2)^{-4}\right) \;+\; \mathbf{O}\left(\left(\log\frac{rz_2^2}{N}\right)^5\right).$$

The main term is

$$\frac{r^2}{\phi(r)^2}\left(\sum_{\gamma=1}^{\infty}\frac{\mu(\gamma)^2}{\phi(\gamma)^2}\right)\sum_{\substack{u\leq z_2\\(u,r)=1}}\frac{\mu(u)^2}{\phi(u)}\prod_{p|ur}\left(1+\frac{1}{(p-1)^2}\right)^{-1} \;+\; \mathbf{O}\left(\frac{r^2}{\phi(r)^2}\log\log z_2\right).$$

7. Proof of Theorems 3.1 and 3.2

As described in §3, we have written

$$\sum_{\substack{n\le N\\ n\equiv b \pmod r}} \left(\sum_{d|n}\Lambda_2(d)\right)\left(\sum_{e|n}\Lambda_2(e)\right) = S_A + S_B + \mathbf{O}(S_C),$$

where in S_A, we require that $[d,e] > N/r$.

By Proposition 4.1, we deduce that

$$S_A \ll \frac{N}{r}\left(\log\frac{rz_2^2}{N}\right)^5 + \frac{N}{r}(\log r)^2 + rz_2.$$

As for S_C, we have by Proposition 5.1 that

$$S_C \ll \frac{N}{r}\left(\log\frac{rz_2^2}{N}\right)^4.$$

By Proposition 6.2, we have

$$S_B = \frac{N}{r}\left(\frac{r}{\phi(r)}\log z_2 + \mathbf{O}(\frac{r}{\phi(r)}\sigma_{-\frac{1}{2}}(r)(\log 2z_2)^{-4}) + \mathbf{O}((\log rz_2^2/N)^5)\right).$$

This proves Theorem 3.1.

For Theorem 3.2, as in §2, we have to estimate three sums of the form

$$S_{i,j} = \sum_{\substack{n\le N\\ n\equiv b \pmod r}} \left(\sum_{d|n}\Lambda_i(d)\right)\left(\sum_{e|n}\Lambda_j(e)\right).$$

For the cases $i=j=1$ and $i=1, j=2$, the condition $N \ge rz_iz_j$ is satisfied and so we get the desired estimate from Proposition 1. The only remaining case is $i=j=2$ where this condition is *not* satisfied. This case follows from Theorem 3.1.

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