Density modulo 1 of a sequence associated to some multiplicative functions evaluated at polynomial arguments

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Dedicated to the memory of Alan Baker

Abstract. We study density modulo 1 of the sequence with general term $\sum_{m\le n} f(G(m))$ where f is the strongly multiplicative function of the form $f(n)=\prod_{p|n}\left(1-\frac{\nu(p)}{p}\right)$ and ν is a multiplicative function for which there exists a real number $0<r\le 1$ such that $1\le|\nu(p)|\le p^{1-r}$ for all primes p, and G is a non constant polynomial with integral coefficients and taking positive values at positive arguments.

Keywords. Distribution modulo 1, Multiplicative function, Sieve.

2010 Mathematics Subject Classification. 11K06, 11N36, 11N37.

1. Introduction

At the Czech-Slovak Number Theory Conference in Smolenice in August 2007, F. Luca asked whether the sequences of arithmetic and geometric means of the first n values of the Euler totient function are uniformly distributed modulo 1. A. Schinzel modified these questions by asking whether these sequences are dense modulo 1. This question was positively answered in [DL08], as well as similar questions for different types of mean values of the Euler function taken on the integers. Shortly later, two extensions were introduced. The first one [DL11] was to consider the mean of the Euler function evaluated at Fibonacci arguments and the second one was to notice that since the sequences have a linear growth with an irrational leading coefficient, it could be uniformly distributed modulo 1; this was shown in [DI08]. Then the quadratic polynomial values was considered in [DH10] and [DI16] for some linear sequences associated to some multiplicative functions. Also, the distribution modulo one of some sequences associated to the arithmetical functions has been investigated in [Has12] and [Has10].

Most recently, J.-M. Deshouillers and author proved that the sequence

$$a_n=\sum_{m\le n}\frac{\varphi(G(m))}{G(m)}$$

is dense modulo 1, where G is a non constant polynomial with integral coefficients and taking positive values at positive arguments [DN18]. Since $\varphi(m)/m$ is a strongly multiplicative function, in the present paper, a strongly multiplicative function f is considered instead of $\varphi(m)/m$ and an extension of result for a family of strongly multiplicative functions is given as follows. We recall that a multiplicative function is called strongly multiplicative if $f(p^\nu)=f(p)$ for all $\nu\ge 1$.

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Theorem 1.1. *Let f be the strongly multiplicative function defined by*

$$f(n) = \prod_{p|n} \left(1 - \frac{\nu(p)}{p}\right), \tag{1.1}$$

for some multiplicative function ν and let G be a non constant polynomial with integral coefficients and taking positive values at positive arguments. Suppose there exists a real number $0 < r \le 1$ such that

$$\forall p : 1 \le |\nu(p)| \le p^{1-r}, \tag{1.2}$$

and suppose there exists a family $\mathcal{P}_G$ of primes such that for any $p \in \mathcal{P}_G$ the congruence $G(x) \equiv 0 \pmod{p}$ has a solution and

$$\sum_{p \in \mathcal{P}_G} \frac{\nu(p)}{p} = +\infty. \tag{1.3}$$

Then the sequence $(a_n)_{n \ge 1}$ with general term

$$a_n = \sum_{m \le n} f(G(m)) \tag{1.4}$$

is dense modulo 1.

Because of strongly multiplicativity of f, without loss of generality, we may assume that the polynomial G has no quadratic irreducible rational factor and that $\nu(p^k) = 0$ for any prime p and any integer $k \ge 2$. We keep these assumptions throughout the paper, as more as we set the following notations.

For a polynomial F with integral coefficients, we let

$$\omega(F, p) = \mathrm{Card}\{x \in \mathbb{Z} \cap [0, p) : \ F(x) \equiv 0 \pmod{p}\}, \tag{1.5}$$

and we say that a prime p is a fixed divisor of F if $\omega(F, p) = p$.

We denote by G a non constant polynomial with integral coefficients which takes positive values at positive arguments; we moreover assume that the polynomial G has no square irreducible rational factor; its degree is denoted by g and we denote by h an integer which is at least equal to the minimum of the degree of G and the maximal fixed prime divisor of G; in other words we have

$$h \ge \deg(G) = g \quad \text{and} \quad p > h \Rightarrow \exists x \in \mathbb{Z} \colon p \nmid G(x). \tag{1.6}$$

2. Auxiliary lemmas

In this section, we state some lammas which are needed to prove the main theorem. First we notice that the introduced function has approximately linear growth.

Lemma 2.1. *Let ν be a multiplicative function satisfying $1 \le |\nu(p)| \le p^{1-r}$ for some $0 < r \le 1$ and for all primes p. If*

$$f(n) = \prod_{p|n} \left(1 - \frac{\nu(p)}{p}\right),$$

then there exists $\alpha \ne 0$ such that

$$\sum_{n \le x} f(n) = \alpha x + O(x^{1-r} \log x) \quad \text{as} \quad x \to +\infty \tag{2.7}$$

Proof. Since ν is multiplicative, we have

$$\begin{aligned}\sum_{n\le x} f(n) &= \sum_{n\le x}\sum_{d|n}\frac{\mu(d)\nu(d)}{d} = \sum_{d\le x}\left(\frac{x}{d}+O(1)\right)\frac{\mu(d)\nu(d)}{d}\\ &= x\sum_{d=1}^{\infty}\frac{\mu(d)\nu(d)}{d^2} - x\sum_{d>x}\frac{\mu(d)\nu(d)}{d^2} + O\left(\sum_{d\le x}\frac{|\nu(d)|}{d}\right)\\ &= x\sum_{d=1}^{\infty}\frac{\mu(d)\nu(d)}{d^2} + O\left(x\sum_{d>x}\frac{|\mu(d)\nu(d)|}{d^2} + (\sum_{d\le x}\frac{|\mu(d)\nu(d)|}{d}\right).\end{aligned}$$

By (1.2), we have $|\mu(d)\nu(d)| \le d^{1-r}$, and so we have

$$x\sum_{d>x}\frac{|\mu(d)\nu(d)|}{d^2} \le x\sum_{d>x}\frac{1}{d^{1+r}} = O\left(x^{1-r}\right)$$

and

$$\sum_{d\le x}\frac{|\mu(d)\nu(d)|}{d} \le \sum_{d>x}\frac{1}{d^r} = O\left(x^{1-r}\log x\right),$$

which implies

$$\sum_{n\le x} f(n) = \alpha x + O(x^{1-r}\log x),$$

where

$$\alpha = \sum_{d=1}^{\infty}\frac{|\mu(d)\nu(d)|}{d^2} = \prod_p\left(1-\frac{\nu(p)}{p^2}\right).$$

Notice that the log factor can be dispensed with if $r < 1$.

Lemma 2.2. *Assume that there exists a family $\mathcal{P}_G$ of primes such that for any p in $\mathcal{P}_G$, $\omega(G,p) > 0$ and*

$$\sum_{p\in\mathcal{P}_G}\frac{\nu(p)}{p} = +\infty. \tag{2.8}$$

For M large enough, one can find a finite set of primes $\mathcal{Q}$ and a positive integer n_0 such that any $q\in\mathcal{Q}$ is larger than $2(h+1)M$ and

$$\forall m\in[1,M]:\ M^{-1/2} \le \prod_{\substack{q\in\mathcal{Q}\\ q|G(n_0+m)}}\left(1-\frac{\nu(q)}{q}\right) \le M^{-1/4g}. \tag{2.9}$$

Proof. According to (2.8), infinitly many primes are available for which $\nu(p) > 0$ and $\omega(G,p)\ne 0$ such that over those $\sum\frac{\nu(p)}{p} = +\infty$ and then the infinite prodoct $\prod\left(1-\frac{\nu(p)}{p}\right)$ tends to zero over such primes. This allows us to repeat the proof of Lemma 1 of [DN18] to construct $\mathcal{Q}$ and n_0.

Lemma 2.3. *Let a,b,c,k be positive integers with $bk > 4a$ and $u(n,j)$ be real numbers with*

$$0\le u(n,j)\le 1 \quad \textit{and} \quad \forall n\ge 2:\ \sum_{j\le bk} u(n,j) \le \frac{ck}{n}. \tag{2.10}$$

We have

$$\sum_{a\le j\le bk}\prod_{2\le n}\left(1-\frac{1}{n^r}\right)^{u(n,j)} \ge \frac{bk}{4}\exp\left(-\frac{4\eta c}{b}\right). \tag{2.11}$$

where $\eta = \frac{1}{r}\left(\zeta(1+r)-1\right)$.

Proof. Let us call S the sum which appears in (2.11). We denote as "bad guys" those integers j for which the product is at most $\lambda = \exp(-\frac{4\eta c}{b})$ and "good guys" the other ones. A lower bound for S is λ times the number of good guys, and so it is enough to find an upper bound for the number of bad guys. It is possible because we have an upper bound for the total number of the $u(n, j)$.

It is implicit that the integers j are always limited to $a \le j \le bk$ and the integers n to $n \ge 2$; let us define $\mathcal{B} = \{j\colon \sum_n u(n, j) \log(1 - \frac{1}{n^r}) \le \log \lambda\}$ and $\mathcal{G} = [a, bk] \backslash \mathcal{B}$. We have

$$\sum_j \left(\sum_n u(n, j) \log \left(1 - \frac{1}{n^r}\right)\right) \le \sum_{j \in \mathcal{B}} \left(\sum_n u(n, j) \log \left(1 - \frac{1}{n^r}\right)\right) \le \mathrm{Card}(\mathcal{B}) \log \lambda.$$

In the other direction, we have

$$\sum_j \left(\sum_n u(n, j) \log \left(1 - \frac{1}{n^r}\right)\right) = \sum_n \log \left(1 - \frac{1}{n^r}\right) \sum_j u(n, j) \ge ck \sum_n \frac{1}{n} \log \left(1 - \frac{1}{n^r}\right).$$

Using the fact that $\log\left(1 - \frac{1}{n^r}\right) \ge \frac{-2}{rn^r}$, we get

$$\frac{4\eta c}{b} \mathrm{Card}(\mathcal{B}) \le \frac{2ck}{r} \sum_n \frac{1}{n^{1+r}} = 2ck\eta,$$

whence

$$\mathrm{Card}(\mathcal{B}) \le \frac{bk}{2}$$

and so, using (2.10)

$$\mathrm{Card}(\mathcal{G}) \ge \frac{bk}{2} - a \ge \frac{bk}{4}$$

which implies

$$S \ge \mathrm{Card}(\mathcal{G})\, \lambda \ge \frac{bk}{4} \exp\left(-\frac{4\eta c}{b}\right).$$

The working engine of the present paper is the following sieve result which is Proposition 1 in [DN18].

Lemma 2.4. *Let M, $\mathcal{Q}$ and n_0 satisfy Lemma 2.2 and let H be the polynomial defined by*

$$H(x) = \prod_{m=1}^{M} G(Qx + n_0 + m), \qquad \textit{where} \qquad Q = \prod_{q \in \mathcal{Q}} q. \tag{2.12}$$

Then, there exist infinitely many integers x such that $H(x)$ has no other prime factor in the interval $[2(h+1)M, x^{1/7gM}]$ than those from $\mathcal{Q}$.

3. Proof of the main result

In order to prove Theorem 1.1, it is enough to prove that for all $m \in [1, M]$, $f(G(Qx+n_0+m))$ tends to zero with M, and

$$\sum_{m=1}^{M} f(G(Qx + n_0 + m)) \ge 1. \tag{3.13}$$

Using the Proposition 2.4 we can find sufficiently large integer x such that $H(x) = \prod_{m=1}^{M} G(Qx + n_0 + m)$ has no prime factor in the interval $[2(h+1)M, x^{1/7gM}]$, except those from $\mathcal{Q}$. So we have only three type of prime divisors:

1) Belong to $\mathcal{Q}$;

2) Smaller than $2(h+1)M$;

3) Larger than $x^{1/7gM}$.

Since $G(Qx+n_0+m)=O(x^g)$, number of large divisors is at most $7g^2M$. Thus for sufficiently large x,

$$\prod_{\substack{q|G(Qx+n_0+m)\\ q>x^{1/7gM}}}\left(1-\frac{\nu(q)}{q}\right)\leq\prod_{\substack{q|G(Qx+n_0+m)\\ q>x^{1/7gM}\\ \nu(q)<0}}\left(1+\frac{1}{q^r}\right)\leq\left(1+x^{-r/7gM}\right)^{7g^2M}\leq\frac{3}{2},\tag{3.14}$$

and

$$\prod_{\substack{q|G(Qx+n_0+m)\\ q>x^{1/7gM}}}\left(1-\frac{\nu(q)}{q}\right)\geq\prod_{\substack{q|G(Qx+n_0+m)\\ q>x^{1/7gM}\\ \nu(q)>0}}\left(1-\frac{1}{q^r}\right)\geq\left(1-x^{-r/7gM}\right)^{7g^2M}\geq\frac{1}{2}.\tag{3.15}$$

Considering (2.9) and (3.14) we write

$$f(G(Qx+n_0+m))\leq\frac{3}{2}M^{-1/4g}\prod_{q\leq 2(h+1)M}\left(1+\frac{1}{q^r}\right)\ll_r M^{-1/4g}\log M\tag{3.16}$$

an expression which tends to zero with M.

Also we have

$$\begin{aligned}\sum_{m=1}^{M}f(G(Qx+n_0+m))&\geq\sum_{m=1}^{M}\prod_{\substack{q|G(Qx+n_0+m)\\ \nu(q)>0}}\left(1-\frac{\nu(q)}{q}\right)\\ &\geq\frac{1}{2}M^{-1/2}\sum_{m=1}^{M}\prod_{\substack{q|G(Qx+n_0+m)\\ q\leq 2(h+1)M}}\left(1-\frac{1}{q^r}\right).\end{aligned}$$

So its enough to show that

$$\sum_{m=1}^{M}\prod_{\substack{q|G(Qx+n_0+m)\\ q\leq 2(h+1)M}}\left(1-\frac{1}{q^r}\right)\geq 2M^{1/2}.$$

We are going to apply Lemma 2.3. We let $u(n,j)=1$ when $n\leq 2(h+1)M$ is a prime which divides $G(Qx+n_0+j)$ and 0 otherwise; we thus have

$$\sum_{m=1}^{M}\prod_{\substack{q|G(Qx+n_0+m)\\ q\leq 2(h+1)M}}\left(1-\frac{1}{q^r}\right)=\sum_{j=1}^{M}\prod_{n\geq 2}\left(1-\frac{1}{n^r}\right)^{u(n,j)}.$$

In order to apply Lemma 2.3, we take $a=b=1$, $k=M$ and we have to show that there exists c such that $\sum_{j\leq M}u(n,j)\leq\frac{cM}{n}$. If n is a fixed divisor of G, we recall that $n\leq h$ and we have

$$\sum_{j\leq M}u(n,j)=M\leq\frac{hM}{n}.$$

If n is not a fixed divisor of G, for some $j \in [1, M]$, $n \mid G(Qx+n_0+j)$. Indeed the number of such j's is $\omega(G, n)$ in any interval of length n, and hence the total number of those in $[1, M]$ is $\left(\frac{M}{n}+1\right)\omega(G, n)$. Hence we have

$$\sum_{j \leq M} u(n, j) \leq \left(\frac{M}{n}+1\right)\omega(G, n) \leq g\left(\frac{M}{n}+1\right) \leq \frac{(2h+3)gM}{n}.$$

Thus, the second part of (2.10) is satisfied and we can apply Lemma 2.3 with $c = g(2h+3)$, which leads to (with a constant C depending on G only)

$$\sum_{m=1}^{M} \prod_{\substack{q \mid G(Qx+n_0+m) \\ q \leq 2(h+1)M}} \left(1 - \frac{1}{q^r}\right) \geq CM \geq 2M^{1/2},$$

when M is large enough. This ends the proof of Theorem 1.1.

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