ON WARING'S PROBLEM: $g(4) \leqslant 21$

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(Dedicated to the Memory of Dr. S. S. PILLAI)

§ 1. Developing the ideas of Chen-jing-run [2] we prove the following

Theorem :

Every natural number is expressible as a sum of atmost twenty one fourth powers.

In the usual notation, our result reads : $19 < g(4) \leqslant 21$. This is an improvement of the result of Thomas [9] who proved that $g(4) \leqslant 22$. We recall that the general problem for the k^{th} powers in the place of fourth powers is nearly complete, which is due to Dickson [5] and Pillai (independent of each other); (see "S. S. Pillai" by K. Chandrasekharan : Jour. of Indian Math. Soc. 15 (1951) (1–10) for the list of complete works of Dr. S. S. Pillai). On the other hand, in the case of $G(4)$, it has been proved by Davenport [4] that $G(4) = 16$. Incidentally, we remark that Auluck [1] proved that every integer $> 10^{10^{89 \cdot 39}}$ is a sum of nineteen fourth powers. It was improved by Thomas [8] who proved that every integer $\geqslant 10^{1408 \cdot 3}$ is representable as a sum of 19 fourth powers (Theorem 12.1 ; pp. 152 of [8]). Our method improves the bound; but since it does not prove any thing substantial, we are not including the proof of this fact.

§ 2. **Notations :**

Let $e(x)$ stand for $e^{2\pi i x}$ and

$eq(x)$ for $e^{\frac{2\pi i x}{q}}$

Define $S_{a,q} = \sum_{x=1}^{q}{}' e^{\frac{2\pi i a x^4}{q}}$ where the accent here (and elsewhere) shows that the summation is restricted to only those x for which $(x, q) = 1$.

Let $N (\geq 10^{560})$ be a given integer to be represented as a sum of 21 biquadratics. Define two integers N_0 and N_1 by

$$N_0 \leq N \leq 2N_0 \text{ and } N_0 - P^{3\frac{3}{4}} \leq N_1 < N_0$$

Define $P = [N^{\frac{1}{4}}]$ and $T(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^4)$

For any real number α, $0 < \alpha \leq 1$, there exist two integers h, q with $1 \leq h < q \leq 8P^3$, $(h, q) = 1$ such that

$$\left| \alpha - \frac{h}{q} \right| \leq \frac{1}{8qP^3}.$$

The major arc $\Omega = \left\{ \alpha \,/\, \left| \alpha - \frac{h}{q} \right| < \frac{1}{8qP^3} \text{ for some } q < P^{\frac{1}{2}} \right\}$

and the minor arc $\mathfrak{m} = \left\{ \alpha \,/\, \left| \alpha - \frac{h}{q} \right| \leq \frac{1}{8qP^3} \text{ for some } q, \; P^{\frac{1}{2}} < q \leq 8P^3 \right\}$

The singular series $S(n\,;\,m)$ is defined by

$$S(n) = S(n, m) = \sum_{q=1}^{\infty} \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^m e^{\frac{-2\pi i a n}{q}}.$$

The truncated singular series $S_1(n, m)$ is defined by

$$S_1(n) = S_1(n, m) = \sum_{q \leqslant P^{\frac{1}{4}}} \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q}\right)^m e^{\frac{-2\pi ian}{q}}.$$

θ is a constant (depending on various parameters), such that $|\theta| \leqslant 1$.

$$\psi = \psi(a) = \int_0^P e^{2\pi iax^4}\, dx$$

$$W(N') = \int_0^1 T_a^m e^{-2\pi iaN'}\, da$$

$$W_0(N) = \int_\Omega T_a^m e^{-2\pi iaN'}\, da$$

$$R(N_0) = \int_{-\infty}^{\infty} \psi^m e^{-2\pi iaN_0}\, da$$

§ 3. An upper bound for $S_{a,q}$

Let us recall that $S_{a,q} = \sum_{\substack{x=1 \\ (x,q)=1}}^{q} e_q(ax^4)$.

We then have,

Lemma 1 : *$S_{a,q}$ is a multiplicative function of q*

Proof: A proof can be found in Davenport [3] (lemma 6 in page 31).

Because of lemma 1, it is sufficient to have the bound for S_{a,p^α}. In this direction, we have

Lemma 2 : (a) *For any prime* $p \neq 2$, $|S_{a,p}| \leqslant (\delta - 1) p^{\frac{1}{2}}$ where $\delta = (4, p-1)$

(b) *For any prime* $p \neq 2$, $|S_{a,p^\nu}| = p^{\nu-1}$ If $2 \leqslant \nu \leqslant 4$

(c) *For any prime* $p \geqslant 2$,

$$|S_{a,p^{\nu}}| = p^3 \, |S_{a,p^{\nu-4}}|$$

if $\nu > 4$.

Proof: A proof can be found in Davenport [3] (lemma 12 in page 42, lemma 13 in page 43, and lemma 14 in page 44).

One can now get a bound for $|S_{a,q}|$ from lemmas 1 and 2. But, for small primes (namely $p \leqslant 79$), we find the bound for $S_{a,p}$ by actual computation (and the bound is better than the one given by lemma 2) and deduce

Lemma 3: *We have* $|S_{a,q}| \leqslant (4{\cdot}3)\, q^{\frac{3}{4}}$ *if* $(a,q) = 1$.

Proof: A proof of the lemma can be found in Thomas [8] (Theorem 2.1, Page 38).

§ 4: A bound for T (α) in the minor arc:

Let us recall that $T(\alpha)$ is defined by

$$T(\alpha) = \sum_{1 \leqslant x \leqslant P} e(\alpha x^4)$$

Minor arc is defined by

$$\mathfrak{m} = \left\{ \alpha \;\middle|\; \left| \alpha - \frac{h}{p} \right| \leqslant \frac{1}{8q\,P^3} \text{ for some } q,\ P^{\frac{1}{2}} < q \leqslant 8P^3 \right\}$$

Lemma 4: *Let* $d(m)$ *be the number of divisors of* m.

Then $$\sum_{1 \leqslant i \leqslant n} (d(i))^j \leqslant A_j\, n\, (j + \log n)^{2^j - 1}$$

where A_j *depends only on* j *In particular one can take*

$$A_1 = 1; \quad A_2 = \tfrac{1}{3}; \quad A_3 = \tfrac{1}{24}; \quad A_4 = \frac{1}{24 \times 192}.$$

Proof: The values of A_j for every integer j is given by Mardjanichvili [7]. The values of A_1, A_2, A_3 and A_4 are discussed in Chen [2] (lemma 8)

Lemma 5 : *Let $h_1(n)$ denote the number of solutions of the equation* $n = l_1 l_2$, $1 \leqslant l_1 \leqslant P$; $1 \leq l_2 \leq P$ *in integers.*

Then, $\sum_{n \leq P^2} (h_1(n))^2 \leq 2P^2 (\log P + 3)$

Proof : We have

$$\sum_{n \leqslant P^2} (h_1(n))^2 = \sum_{n \leqslant P^2} \Big(\sum_{\substack{d \mid n \\ d \leqslant P;\ \frac{n}{d} \leqslant P}} 1 \Big)^2$$

$$= \sum_{n \leq P^2} \sum_{\substack{d_1 \mid n \\ d_1 \leq P;\ \frac{n}{d_1} \leq P}} \sum_{\substack{d_2 \mid n \\ d_2 \leq P;\ \frac{n}{d_2} \leq P}} 1.$$

$$\leq 2 \sum_{d_1 \leq P} \sum_{d_2 < d_1} \sum_{\substack{n \leq P d_2 \\ n \equiv 0 (\mathrm{mod}\ [d_1, d_2])}} 1$$

$$\leq 2 \sum_{d_1 \leq P} \sum_{d_2 < d_1} \frac{P d_2}{[d_1, d_2]}$$

$$\leq 2 \sum_{d_1 \leq P} \sum_{l \mid d_1} \sum_{d_2 \leq \frac{d_1}{l}} \frac{P d_2 l}{d_1 d_2}$$

$$\leqslant 2P^2 (\log P + 3).$$

Lemma 6 : *Let $h_2(n)$ denote the number of solutions of the equation* $n = l_1 l_2$, $1 \leq l_1 \leq P$; $1 \leq l_2 \leq P^2/4$ *in integers.*

Then

$$\sum_{n \leq P^3/4} (h_2(n))^4 \leq \frac{P^3}{12} (\log P + 3)^{11}$$

Proof : We have

$$\sum_{n \leq P^3/4} (h_2(n))^4 = \sum_{n \leq P^3/4} \Big(\sum_{\substack{l \mid n \\ l \leq P;\ \frac{n}{l} \leq P^2/4}} 1 \Big)^4$$

$$\leq 24 \sum_{l_1 \leq P} \sum_{l_2 \leq l_1} \sum_{l_3 \leq l_2} \sum_{l_4 \leq l_3} \frac{P^2 \, l_4}{4\,[l_1, l_2, l_3, l_4]}$$

$$\leq 6P^2 \sum_{1 \leqslant l_1 \leqslant P} \sum_{d_1 | l_1} \sum_{l_2 \leqslant \frac{l_1}{d_1}} \sum_{d_2 | l_1 l_2} \sum_{l_3 \leqslant \frac{d_1 l_2}{d_2}}$$

$$\sum_{d_3 | l_1 l_2 l_3} \sum_{d_4 \leq \frac{d_2 l_3}{d_3}} \frac{d_3 \, l_4}{l_1 \, l_2 \, l_3 \, l_4}$$

$$\leq 6P^2 \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} \sum_{l_2 \leq \frac{l_1}{d}} \sum_{d_2 | l_1 l_2}$$

$$\sum_{l_3 \leqslant \frac{d_1 l_2}{d_2}} \frac{d(l_1)\, d(l_2)\, d(l_3)\, d_2}{l_1 \, l_2}$$

$$\leq 6P^2 \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} \sum_{l_2 \leq \frac{l_1}{d}}$$

$$\sum_{d_2 | l_1 l_2} \frac{d(l_1)\, d(l_2) \cdot (\log P + 1)}{l_1}$$

$$\leq 6P^2 \, (\log P + 1) \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1}$$

$$\sum_{l_2 \leq \frac{l_1}{d}} \frac{d^2(l_1)\, d^2(l_2)\, d_1}{l_1}$$

$$\leq 2P^2 \, (\log P + 1) \sum_{1 \leq l_1 \leq P} \sum_{d_1 | l_1} d^2(l_1) \cdot (\log P + 2)^3$$

$$\leq 2P^2 \, (\log P + 2)^4 \sum_{1 \leq l_1 < P} d^3(l_1)$$

$$\leq \frac{P^3}{12} (\log P + 3)^{11}$$

Lemma 7 : *Let* $\alpha = \frac{a}{q} + \frac{\theta}{8P^8 q}$. *Then, if* $P > 10^{50}$,

we have

$$\sum_{0 \leq n \leq P^3/4} \min \left(P, \frac{1}{2 \, \| 24 \alpha n \|} \right)^{4/3}$$

$$\leq 50 P^{\frac{10}{3}} \textit{ provided } 8P^8 > q > P ;$$

Proof : Let $q' = \frac{q}{(24, q)}$. The sum over n can be broken into not more than $\left(\frac{P^3}{4q'} + 1 \right)$ parts, in each of which n runs over atmost q' consecutive integers. Let us consider the typical sum say

$$\sum_{B < n \leqslant B + q'} \min \left(P, \frac{1}{2 \, \| 24 \alpha n \|} \right)^{\frac{4}{3}}.$$

Then it is easily seen that, there are atmost $\left(\frac{q'}{P} + 4 \right)$ values for n for which $\frac{1}{2 \, \| 24 \alpha n \|} \geqslant P$ and these values of n contribute atmost $\left(\frac{q'}{P} + 4 \right) P^{4/3} = q' P^{1/3} + 4 P^{4/3}$ to the sum. The remaining values of n contribute

$$\sum_{m \geqslant \frac{q'}{P}} \tfrac{1}{2} \left(\frac{q'}{m} \right)^{4/3} < 4 q' P^{1/3}.$$

Hence

$$\sum_{B < n \leqslant B + q'} \min \left(P, \frac{1}{2 \, \| 24 \alpha n \|} \right)^{4/3} \leqslant 5 q' P^{1/3} + 4 P^{4/3}$$

Hence the total sum is atmost

$$(5 q' P^{1/3} + 4 P^{4/3}) \left(\frac{P^3}{4q'} + 1 \right) < 50 P^{\frac{10}{3}}$$

Lemma 8 : *If $h_2(n)$ denotes the number of solutions of the equation $n = l_1 l_2$, $1 \leqslant l_1 \leqslant P$; $1 \leqslant l_2 \leqslant P^2/4$, in integers, then*

$$\sum_{n \leqslant P^3/4} h_2(n) \min\left(P, \frac{1}{2\,\|24\,\alpha n\|}\right)$$

$$\leqslant 11\, P^{\frac{18}{4}} (\log P + 3)^{\frac{11}{4}}.$$

where $\alpha = \dfrac{a}{q} + \dfrac{\theta}{8P^3 q}$; $P > 10^{50}$ *and* $P \leqslant q \leqslant 8P^3$

Proof : If $P \leqslant q \leqslant 8P^3$, then

$$\sum_{n \leqslant P^3/4} h_2(n) \min\left(P, \frac{1}{2\,\|24\,\alpha n\|}\right)$$

$$\leqslant \Bigg(\sum_{n \leqslant \frac{P^3}{4}} (h_2(n))^4\Bigg)^{\frac{1}{4}} \Bigg(\sum_{n \leqslant \frac{P^3}{4}} \left(\min\left(P, \frac{1}{2\,\|24\,\alpha n\|}\right)\right)^{\frac{4}{3}}\Bigg)^{\frac{3}{4}}$$

and the result follows from lemmas 6 and 7.

Lemma 9 : *Let* $\alpha = \dfrac{a}{q} + z$; $(a,q)=1$; $P > 10^{100}$, $P \leqslant q \leqslant 8\,P^3$

$|z| \leqslant \dfrac{1}{8q\,P^3}$. *Then*

$$T(\alpha) = \left|\sum_{x=1}^{P} e^{2\pi i \alpha x^4}\right| < (2.86)\, P^{\frac{29}{32}} (\log P + 3)^{\frac{15}{32}}$$

Proof : Let $l_1 = x - y$; $f(x, y) = 4x^3y + 6x^2y + 4xy^3$

$h(x\,;\,y\,;\,z) = 12\,xyz\,(x+y+z)$.

$h_1(n)$ and $h_2(n)$ are as defined in lemmas 5 and 6.

Then

$$|T(\alpha)|^2 \leqslant P + 2\left|\sum_{x=1}^{P} \sum_{\substack{y=1 \\ y \neq x}}^{P} e^{2\pi i \alpha (x^4 - y^4)}\right|$$

$$\leqslant P+2 \sum_{l=1}^{P} \left| \sum_{x=1}^{P-l} e^{2\pi i a f(x,l)} \right|$$

$$= P+2S_1, \quad \text{say}$$

$$| S_1 |^2 \leqslant \left(\sum_{l=1}^{P} 1 \right) \left(\sum_{l=1}^{P} \left| \sum_{x=1}^{P-l} e^{2\pi i a f(x,l} \right|^2 \right)$$

$$\leqslant P^3+2P \sum_{l_1=1}^{P} \sum_{l_2=1}^{P-l_1} \left| \sum_{x=1}^{P-l_1-l_2} e^{2\pi i a h(x;\, l_1, l_2)} \right|$$

$$= P^3 + 2PS \quad \text{say}$$

Now

$$| S | \leqslant \sum_{1 \leqslant n \leqslant \frac{P^2}{4}} h_1(n) \max_{l_1 l_2 = n} \left| \sum_{x=1}^{P-l_1-l_2} e^{2\pi i a . h(x; l_1, l_2)} \right|$$

Hence $| S |^2 \leqslant \left(\sum_{1 \leq n \leq P^2/4} h_1(n) \right)^2$

$$\left(\sum_{1 \leq n \leq P^2/4} \max_{l_1 l_2 = n} \left| \sum_{x=1}^{P-l_1-l_2} e^{2\pi i a h(x;\, l_1,\, l_2)} \right|^2 \right)$$

Now using lemma 5,

$$| S |^2 \leq 2P^3 (\log P + 3) \sum_{1 \leq n \leq P^2/4} \max_{l_1 l_2 = n}$$

$$\left(P + 2 \sum_{l_3 = 1}^{P} \min \left(P, \frac{1}{2 \| 24\, a\, l_1 l_2 l_3 \|} \right) \right)$$

$$\leq 2P^3 (\log P + 3) \sum_{1 < i < P^2/4}$$

$$\left(P + 2 \sum_{l=1}^{P} \min \left(P, \frac{1}{2 \| 24\, a\, li \|} \right) \right)$$

$$\leq 2P^{3}(\log P+3)\times$$

$$\left(\frac{P^{3}}{4}+2\sum_{0<n\leq P^{3}/4}h_2(n)\min\left(P,\frac{1}{2\,\|24\,\alpha\,n\|}\right)\right)$$

and using lemma 8,

$$|S|^{8}\leq 2P^{3}(\log P+3)\left(P^{3}/4+22P^{\frac{13}{4}}(\log P+3)^{\frac{11}{4}}\right)$$

and this gives

$$|T(\alpha)|\leq(2.86)\,P^{\frac{29}{32}}.\ (\log P+3)^{\frac{15}{32}}$$

Lemma 10 : *If* $\alpha=\dfrac{a}{q}+\beta$, *where* $q<P$, $|\beta|\leq\dfrac{1}{8q\,P^{3}}$,

and $P\geqslant 6$, *then*

$$|T(\alpha)|\leq q^{-\frac{1}{4}}(1+\log q)\min\left(16\,P,\ \tfrac{3}{4}\,|\beta|^{-1}\,P^{-3}\right)$$

Proof: This is lemma 9.5 (page 139) of Thomas [8]. A similar result is proved in lemma 9 of Davenport [4]. Even though the result in [8] is proved under the condition

$|\beta|\leq\dfrac{1}{64q\,P^{3}}$ it holds, in fact, for $|\beta|\leq\dfrac{1}{8q\,P^{3}}$.

Lemma 11 : *If* $\alpha=\dfrac{a}{q}+\beta$, $P^{\frac{1}{2}}\leq q\leq P$,

$|\beta|\leq\dfrac{1}{8q\,P^{3}}$, *and* $P\geqslant 10^{100}$, *then*

$$|T(\alpha)|\leq(2.86)\,P^{\frac{29}{32}}(\log P+3)^{\frac{15}{32}}$$

Proof: This follows from lemma 10.

Lemma 12 : *In the minor arc, the following estimate holds :*

$$| T(\alpha) | \leq 2.86 \, P^{\frac{29}{32}} \, (\log P + 3)^{\frac{15}{32}}$$

Proof : The result follows from lemmas 9 and 11.

§ 5. A lower bound for $R(N_0)$

Let us recall that

$$\psi = \psi(\alpha) = \int_0^P e^{2\pi i \alpha x^4} \, dx$$

$$R(N_0) = \int_{-\infty}^{\infty} \psi^m \, e^{-2\pi i \alpha N_0} \, d\alpha$$

Define $B = B(\alpha) = \begin{cases} P & \text{if } |\alpha| \leqslant P^{-4} \\ \sqrt{2}\,|\alpha|^{-\frac{1}{4}} & \text{if } |\alpha| > P^{-4} \end{cases}$

Then it is easily seen that $|\psi| \leq B$

$$W(N') = \int_0^1 T_\alpha^m \, e^{-2\pi i \alpha N'} \, d\alpha$$

Lemma 13 : *(van der Corput) : Suppose* $f(x)$ *is a real function which is twice differentiable for* $A \leq x \leq B$ *Suppose further that, in this interval* $0 \leq f'(x) \leq \frac{1}{2}$ and $f''(x) \geqslant 0$.

Then $\displaystyle\sum_{A \leq n \leq B} e(f(n)) = \int_A^B e(f(x)) \, dx + 4\theta$

Proof : This is lemma 16 (page 65) of Davenport [3]. For the O — constants see lemma 13 (page 34) of Vinogradov [10].

Lemma 14: *If* $N_0 - P^{3\frac{3}{4}} \leqslant N' \leqslant N_0$, *then*

$$W(N') = R(N_0) + \int_{\frac{1}{8P^3}}^{1-\frac{1}{8P^3}} T_a^m e^{-2\pi i a N'} \, da$$

$$+ 2\theta \cdot 10^8 \cdot P^{m-5+3/4},$$

where $m = 9$ or 10 and $P > 10^{100}$

Proof: In $-\frac{1}{8P^3} \leqslant a \leqslant \frac{1}{8P^3}$ we have, by lemma 13,

$$T(a) = \sum_{1 \leqslant x \leqslant P} e^{2\pi i a x^4} = \int_0^P e^{2\pi i a x^4} \, dx + 9\theta$$

$$= \psi + 9\theta$$

hence $|(T(a))^m - \psi^m| \leqslant (9m)(B+5)^{m-1}$

Consequently we have

$$|(T(a))^m e^{-2\pi i a N'} - \psi^m e^{-2\pi i a N_0}|$$

$$= |T_a^m e^{-2\pi i a N'} - \psi^m e^{-2\pi i a N'}$$

$$+ \psi^m e^{-2\pi i a N'} - \psi^m e^{-2\pi i a N_0}|$$

$$\leqslant 9m(B+5)^{m-1} + B^m (2\pi a) P^{3\frac{3}{4}}$$

Hence

$$\int_{-\frac{1}{8P^3}}^{\frac{1}{8P^3}} |T_a^m e^{-2\pi i a N'} - \psi^m e^{-2\pi i a N_0}| \, da$$

$$\leqslant 2 \int_0^1 \left((9m)\,(B+3)^{m-1} + B^m\,(2\pi a)\,.\,P^{3\frac{3}{4}} \right) da$$

$$\leqslant 2 \int_0^{P^{-4}} \left(9m\,(P+5)^{m-1} + (2\pi a)\,.\,P^{m+3\frac{3}{4}} \right) da$$

$$+ 2 \int_{P^{-4}}^{1} \Big(9m\,(\sqrt{2}\,a^{-\frac{1}{4}} + 5)^{m-1}$$

$$+ 2\pi\,P^{3\frac{3}{4}}\,(\sqrt{2}\,a^{-\frac{1}{4}})^m\,.\,a \Big)\,da$$

$$\leqslant 10^8\,P^{m-5+\frac{3}{4}}$$

Now, we have

$$W(N') = \int_{-\frac{1}{8P^8}}^{\frac{1}{8P^8}} (T(a))^m .\; e^{-2\pi i a N'}\,da$$

$$+ \int_{-\frac{1}{8P^8}}^{1-\frac{1}{8P^3}} (T(a))^m\,e^{-2\pi i a N'}\,da$$

Now, replace the integrand

$(T(a))^m\,e^{-2\pi i a N'}$ of the first integral on the right by $\psi^m\;e^{-2\pi i a N_0}$ and we have just proved that the error is atmost $10^3\;P^{m-5+3/4}$. Hence we have

$$W(N') = \int_{-\frac{1}{8P^3}}^{\frac{1}{8P^3}} \psi^m\,e^{-2\pi i a N_0}\,da +$$

$$+ \int_{\frac{1}{8P^3}}^{1 - \frac{1}{8P^3}} (T(a))^m \, e^{-2\pi i a N'} \, da$$

$$+ 10^3.\theta. \, P^{m - 5 + \frac{3}{4}}$$

Now we replace the first integral on the right by $R(N_0)$.

The error involved is at most,

$$2 \int_{\frac{1}{8P^3}}^{\infty} |\psi|^m \, da$$

$$\leqslant 2 \int_{\frac{1}{8P^3}}^{\infty} (\sqrt{2}\, a^{-\frac{1}{4}})^m \, da$$

$$\leqslant 10^3 \, P^{\frac{3m}{4} - 3}$$

$$\leqslant 10^3 \, P^{m - 5 + \frac{3}{4}}$$

and this proves the lemma.

We now take $M = \left[\frac{P^{3\frac{3}{4}}}{2} \right]$ and apply lemma 14 with $N' = N - N'' - N'''$, $0 \leqslant N''$, $N''' \leqslant M$ and add the M^2 equations. This gives

Lemma 15 : *We have, with* $M = \left[\frac{P^{3\frac{3}{4}}}{2} \right]$,

$$\sum_{N''} \sum_{N'''} W(N - N'' - N''') = M^2 \, R(N_0) +$$

$$+ \int_{\frac{1}{8P^3}}^{1-\frac{1}{8P^3}} (T(\alpha))^m \, e^{-2\pi i\alpha N} \left(\sum_{N''} e^{-2\pi i\alpha N''} \right)^2 d\alpha$$

$$+ 2\theta \cdot 10^3 \cdot M^2 \quad P^{m-5+\frac{3}{4}}$$

Lemma 16 : *The integral on the right of lemma 15 is atmost* $8\, P^{m+3}$.

Proof : Since $| T(\alpha) | \leq P$, $| e^{-2\pi i\alpha N} | \leq 1$,

$$\left| \sum_{N''} e^{-2\pi i\alpha N''} \right| \leq \frac{1}{\| \alpha \|},$$

the result is clear.

Lemma 17 : *If* $K_r(N)$ *denotes the number of integer solutions of the equation* $x_1^n + x_2^n + \ldots\ldots + x_r^n \leq N$, *then*

$$K_r(N) = T_r \cdot N^{\frac{r}{n}} - \theta \cdot r \cdot N^{\frac{r-1}{n}},$$

$$\text{where} \quad T_r = \frac{(\Gamma(\frac{5}{4}))^r}{\Gamma\left(1 + \frac{r}{4}\right)} \quad \text{and} \quad 0 \leq \theta \leq 1$$

Proof : This is lemma 3 in (page 22) in Vinogradov [10].

Lemma 18 : *If* $9 \leq m \leq 11$,

$$\sum_{N''=1}^{M} \sum_{N'''=1}^{M} W(N - N'' - N''')$$

$$\geq 0.102 \cdot M^2 \cdot N^{\frac{m}{4} - 1}$$

Proof : We have

$$\sum_{N''=1}^{M} W(N - N'' - N''')$$

$$= \sum_{N''=1}^{M} (K_m(N-N''-N''') - K_m(N-N''-N'''-1))$$

$$= K_m (N - N'') - K_m (N - N'' - M)$$

$$= T_m \left((N - N'')^{\frac{m}{4}} - (N - N'' - M)^{\frac{m}{4}} \right)$$

$$- 2\theta\, m .\, N^{\frac{m-1}{4}}$$

$$\geqslant T_m \frac{m}{8} M N^{\frac{m}{4} - 1} - 2\theta . m . N^{\frac{m-1}{4}}$$

$$\geqslant T_m \frac{m}{10} M\, N^{\frac{m}{4} - 1}$$

$$\geqslant 0.102 . M .\, N^{\frac{m}{4} - 1}$$

and hence the result.

Lemma 19 : *The following inequality holds*

$$| R (N_0) | \geqslant \tfrac{1}{16} N^{\frac{m}{4} - 1}$$

Proof : The inequality follows from lemmas 15 and 18.

§ 6 : A lower bound for the singular series

Let us recall that the singular series is defined by

$$S(n) = S(n ; m) = \sum_{q=1}^{a} \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^m . \, e^{\frac{-2\pi i a n}{q}}$$

and the truncated singular series

$$S_1 (n) \pm S_1 (n ; m) = \sum_{q \leqslant P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^m . \, e^{\frac{-2\pi i a n}{q}}$$

Here (and elsewhere), the accent shows that the summation is restricted to only those a's, for which $(a, q) = 1$.

Define $A_m(n; q) = \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^m e^{\frac{-2\pi i a n}{q}}$

Let us define

$$\chi_p(n, m) = \sum_{i=0}^{\infty} A_m(n; p^i)$$

Lemma 20 : *We have* $S(n; 11) > 0.5304\ \chi_2(n, 11)$

Proof: This is Theorem 4.1 of Thomas [8] (page 98)

Lemma 21 : *Suppose* $1 < n \leqslant m \leqslant 15$. *Then*

$$\chi_2(n; m) = 16\ {}^m c_n\ 2^{-m},$$

where $${}^m c_n = \frac{m!}{n!\,(m-n)!}$$

Proof: This is Theorem 4.2 of Thomas [8] (page 98)

Lemma 22 : *If* $n \equiv 2, 3, 4, 5, 6, 7$ or $8 \pmod{16}$, *then*

$$\chi_2(n, 11) \geqslant 0.42$$

Proof: If $n_1 \equiv n_2 \pmod{16}$, then

$\chi_2(n_1, m) = \chi_2(n_2, m)$. Hence it follows from lemma 21, that $\chi_2(n, 11) \geqslant 16 \cdot {}^{11}c_2 \cdot 2^{-11} \geqslant 0.42$

Lemma 23 : *The singular series* $|S(n)| > 0.222$ *if* $m = 11$ *provided* $n \equiv 2, 3, 4, 5, 6, 7$ or $8 \pmod{16}$

Proof. The result follows from lemmas 20 and 22

Lemma 24 : *The truncated singular series*

$$| S_1 (n) | \geqslant 0.22 \quad \textit{if } m = 11$$

provided $n \equiv 2, 3, 4, 5, 6, 7$ or $8 \pmod{16}$

Proof : We have

$$| S (n) - S_1 (n) |$$

$$= \left| \sum_{q > P^{\frac{1}{8}}} \left(\sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^{11} e^{-2\pi i \frac{an}{q}} \right) \right|$$

$$\leqslant \sum_{q \geqslant P^{\frac{1}{8}}} \left(\sum_{a=1}^{q}{}' \left| \frac{S_{a,q}}{q} \right|^{11} \right)$$

$$\leqslant \sum_{q > P^{\frac{1}{8}}} 5^{11} \cdot q^{-\frac{7}{4}}$$

$$\leqslant 5^{11} \times 2 \times P^{\frac{8}{8}}$$

$$\leqslant 0.01, \text{ since } P > 10^{39}$$

and hence the result.

§ 6. The estimate on the major arc

Let us recall that the major arc is defined by

$$\Omega = \{ \alpha \mid \left| \alpha - \frac{a}{q} \right| \leqslant \frac{1}{8qP^{8}} ; \ 1 \leqslant a \leqslant q; (a,q) = 1; q \leqslant P^{\frac{1}{8}}$$

and the integral

$$W_0 (N_1) = \int_{\Omega} (T(\alpha))^m \cdot e^{-2\pi i N_1 \alpha} \, d\alpha$$

Lemma 25 : *The following approximation holds :*

If $\alpha = \frac{a}{q} + z$, *and* $| N_1 - N_0 | \leqslant P^{3\frac{1}{4}}$, *then*

$$(T(\alpha))^m . e^{-2\pi i N_1 \alpha} = \psi^m \left(\frac{S_{a,q}}{q}\right)^m e^{\frac{-2\pi i a N_1}{q} - 2\pi i z N_0}$$

$$+ \theta \left(5^{m+1} q^{\frac{5 \ m}{4}} B^{m-1} + 5^{m} {}^{5} . q^{\frac{5-m}{4}} B^m (2\pi z) P^{3\frac{1}{4}} \right)$$

Proof: We have

$$T(\alpha) = \sum_{y=0}^{q-1} \sum_{-yq^{-1} \leqslant t < (P-y)q^{-1}} e^{2\pi i \left(\frac{ay^4}{q} + z(qt+y)^4\right)}$$

$$= \sum_{y=0}^{q-1} e^{\frac{2\pi i a y^4}{q}} D_y(z) , \text{ say}$$

since $\frac{d}{dt}(z(qt+y)^4) \leqslant \frac{1}{2}$, we have, by lemma 13,

$$D_s(z) = \int_{\cdot yq^{-1}}^{(P-y)q^{-1}} e^{2\pi i z (qt+y)^4} dt + 4\theta$$

$$= \frac{1}{q} \int_0^P e^{2\pi i z x^4} dx + 4\theta$$

Hence $T(\alpha) = \psi \frac{S_{a,q}}{q} + 4\theta q$

Since $|z| \leqslant \frac{1}{8q P^3}$, $q < p^{\frac{1}{2}}$ we have $zq^{-\frac{1}{4}} \geqslant 8q$

Hence $\left| (T(a))^m - \left(\psi \frac{S_{a,q}}{q} \right)^m \right|$

$$\leqslant 4m.\, q\, (4q^{-\frac{1}{4}} B)^{m-1}$$

$$< 5^{m+1} . \, q^{\frac{5-m}{4}} . \, B^{m-1}$$

Hence $\left| (T(a))^m e^{-2\pi i N_1 a} - \left(\psi \frac{S_{a,q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} \right|$

$$\leqslant \left| T_a^m e^{-2\pi i N_1 a} - \left(\psi \frac{S_{a,q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_1} \right|$$

$$+ \left| \left(\psi \frac{S_{a,q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_1} - \left(\psi \frac{S_{a,q}}{q} \right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} \right|$$

$$\leqslant 5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + \left| \frac{S_{a,q}}{q} \right|^{m-5} \times B^m (2\pi z) P^{3\frac{1}{4}}$$

$$\leqslant 5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} \times B^m (2\pi z) P^{3\frac{1}{4}}$$

Lemma 26 : *We have*

$$W_0(N_1) = \sum_{q \leqslant p^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{-\frac{1}{8q P^3}}^{\frac{1}{8q P^3}} \left(\psi \frac{S_{a,q}}{q}\right)^m \times e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} dz$$

$$+ 5^m q^{\frac{5-m}{4}} P^{m-5+\frac{1}{4}} . \theta$$

Proof: From lemma 25, we have

$$W_0(N) = \int_{\Omega} (T(\alpha))^m e^{-2\pi i N_1 \alpha} d\alpha$$

$$= \sum_{q \leqslant p^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{-\frac{1}{8qP^3}}^{\frac{1}{8qP^3}} \left(\psi \frac{S_{a,q}}{q}\right)^m e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0} dz$$

$$+\theta \sum_{q \leqslant p^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{-\frac{1}{8qP^3}}^{\frac{1}{8qP^3}} \left(5^{m+1} q^{\frac{5-m}{4}} B^{m-1}\right.$$

$$\left. + 5^{m-5} q^{\frac{5-m}{4}} B^m . (2\pi z) P^{3\frac{1}{4}}\right) dz$$

Now, using the value of B, the integral in the error term is easily seen to be at most

$$2 \int_0^1 (5^{m+1} q^{\frac{5-m}{4}} B^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} B^m . 2\pi z . P^{3\frac{1}{4}}) \; dz$$

$$\leqslant 2\int_0^{P^{-4}} \left(5^{m+1} q^{\frac{5-m}{4}} P^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} P^m 2\pi z . P^{3\frac{1}{4}}\right) dz$$

$$+ 2\int_{P^{-4}}^{1} \left(5^{m-1} q^{\frac{5-m}{4}} \left(\sqrt{2}\,|z|^{-\frac{1}{4}}\right)^{m-1} + 5^{m-5} q^{\frac{5-m}{4}} \left(\sqrt{2}\,|z|^{-\frac{1}{4}}\right)^{m} . 2\pi z . P^{3\frac{1}{4}}\right) dz$$

$$\leqslant 5^m \cdot q^{\frac{5\ m}{4}} P^{m-5+\frac{1}{4}}.$$

Lemma 27 : *We have*

$$W_0(N_1) = \sum_{q \leqslant P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{-\infty}^{\infty} \left(\psi \frac{S_{a,q}}{q}\right)^m \times e^{-2\pi i \frac{a}{q} N_1 - 2\pi i z N_0}\, dz$$

$$+ \theta \cdot 2\, 5^m . P^{m-5+\frac{1}{4}} . A$$

where

$$A = \begin{cases} P^{\frac{1}{2}} & \text{if } m = 9 \\ \log P & \text{if } m = 10 \\ 1 & \text{if } m = 11 \end{cases}$$

The result follows almost immediately from lemma 26 ; we have only to prove that the error in extending the range of integration to $[-\infty, \infty]$ is small.

Actually the error is atmost

$$2 \sum_{q \leqslant P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{\frac{1}{8q P^{3}}}^{\infty} \left|\left(\psi \frac{S_{a,q}}{q}\right)\right|^m dz$$

$$\leq 2 \sum_{q \leq P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \int_{\frac{1}{8qP^{3}}}^{\infty} (\sqrt{2}\,|z|^{-\frac{1}{4}})^{m} \cdot (5q^{-\frac{1}{4}})^{m}\, dz$$

$$\leq 5^{m} \cdot P^{m-5+\frac{1}{4}} A.$$

Lemma 28 : *If* $|N_1 - N_0| \leq P^{3\frac{1}{4}}$, *then*

$$W_0(N_1) = S_1(N_1)\, R(N_0) + 2\theta\, 5^{m}\, P^{m-5+\frac{1}{4}} \text{ where}$$

$$S_1(N_1) = \sum_{q \leq P^{\frac{1}{2}}} \sum_{a=1}^{q}{}' \left(\frac{S_{a,q}}{q} \right)^{m} e^{-2\pi i \frac{a}{q} N_1}$$

$$\text{and } R(N_0) = \int_{-\infty}^{\infty} \psi^{m}\, e^{-2\pi i z N_0}\, dz$$

Proof : This follows from lemma 27.

Lemma 29 : If $|N_1 - N_0| \leq P^{3\frac{1}{4}}$, *and* $N_1 \equiv 2, 3, 4, 5, 6, 7,$ or 8 (mod 16) *then* $W_0(N_1) > 0.02\, N^{\frac{m-1}{4}}$ for $m = 11$.

Proof : This follows from lemmas 19, 24 and 28

§ 8 : A lower bound for the number of integers less than a given integer which are representable as a sum of five biquadratics.

Lemma 30 : *Let P be a positive integer and assume* $P > 100$. *Let* δ *and C be fixed positive reals. Let* μ *be a fixed number in the interval* (0, 1) *and suppose*

$\mathbf{U} = \{ u_1, u_2 \ldots\ldots u_U \}$ *is a set of (distinct) integers in the interval* $[0, P^{3+\mu}]$ *where*

$$U \geq C \cdot P^{3(1-\mu)-\delta}.$$

Then the number of solutions M *of the equation*

$$x^4 + u_i = y^4 + u_j$$

where u_h *and* u_j *vary over* $\mathbf{U}$ *and*

$$P \leqslant x,\ y \leqslant 2P$$

and x, y *have the same fixed parity modulo* 2 *does not exceed*

$$C^{-\frac{3}{4}} P^2\ U^2\ P^{3\mu-4+\frac{3\delta}{4}} \left(C^{-\frac{1}{4}} P^{\frac{\delta}{4}} + \tfrac{1}{2} . P^{\frac{(3+\mu)\,\varepsilon}{4}} \{ \tfrac{1}{4} K_2(\varepsilon) . (192)^{-\varepsilon} \}^{\frac{1}{4}}\right)$$

In particular if we take

$$\varepsilon = \frac{\delta}{3+\mu}, \quad \text{then}$$

$$M \leqslant C^{-\frac{3}{4}} P^2\ U^2\ P^{3\mu-4+\delta} \{ C^{-\frac{1}{4}} + \tfrac{1}{2}(\tfrac{1}{4}K_2(\varepsilon).(192)^{-\varepsilon})^{\frac{1}{4}} \}$$

Here $K_2(\varepsilon)$ *is defined by*

$$d_4(m) \leqslant K_2(\varepsilon)\, m^{\varepsilon} \quad \textit{for all} \quad m \geqslant 2$$

and we have

$$\{ K_2(0.20).(\tfrac{1}{192})^{0.20} \}^{\frac{1}{4}} \leqslant (6.2124170)\ 10^6$$

Proof: This is lemma (7.1) of Thomas [8] (page 118). The bound for M when $\varepsilon = \frac{\delta}{3+\mu}$ is given in (7.7) in Thomas [8] (page 119). The bound for $K_2(0.20)$ is given in page 127 of Thomas [8].

Lemma 31 : *Let* $1 \leqslant l < 16$; *Let* f *be a fixed integer in* $(0, 1, \ldots\ldots l)$. *Let* f_1 *be a fixed integer in the set* $(0, 1, \ldots\ldots l+1)$. *Let it be given that, for all integers* $X \geqslant X_0$, *the number of integers less than* X, *which are congruent to* $f \pmod{16}$ *and which are representable as a sum of* l *biquadratics is atleast* $C_l . P^l$. *Then the number of integers less than*

Y, which are congruent to f_1 (mod 16) and which are representable as a sum of $(l+1)$ biquadratics is atleast

$C_{l+1} P^{l+1}$ *where*

$$C_{l+1} = \tfrac{1}{64} C_l^{\frac{3}{4}} \left(C_l^{-\frac{1}{4}} + (2 \cdot 3) \times 10^5 \right)^{-1}$$

and $$\nu_{l+1} = \tfrac{1}{4} + \frac{3\,\nu_l}{3 + \nu_l + \varepsilon}$$

Provided $$Y \geqslant 10^8 \,.\, X^{\frac{4}{3+\mu}}$$

with $$\mu = \frac{3\,(1 - \nu_l - \varepsilon)}{3 + \nu_l + \varepsilon}$$

Here ε *is any positive number.*

Proof : Define $\delta = (3 + \mu)\,\varepsilon$

Note that $3\,(1 - \mu) - \delta = (3 + \mu)\,\nu_l$

and $-4\,\nu_{l+1} = -4 + 3\mu + \delta$

We observe that $\left[Y^{\frac{3+\mu}{4}} \right] \geqslant X$

Choose f_2 from the set $(0, 1)$ such that

$f_2 + f = f_1$.

Let $\mathbf{U} = \{ x ;\ 0 \leqslant x \leqslant [Y^{\frac{3+\mu}{4}}] ;$

$$x = x_0^4 + x_1^4 + x_2^4 + x^4$$

where $x_i \equiv f_1 \pmod 2 \}$

Let $U = \text{Card}\ \mathbf{U}$.

By the hypothesis,

$$U > C_l \, Y^{\frac{(3+\mu)\nu_l}{4}} = C_l \, Y^{\frac{3(1-\mu)-\delta}{4}}.$$

Let $r(m)$ denote the number of solutions of $M = u_h + y^4$, where u_h runs over the set $\mathbf{U}$ and

$$z^{\frac{1}{4}} \leqslant y \leqslant 2z^{\frac{1}{4}}; \qquad y \equiv f_2 \pmod 2)$$

Then $\sum_m r(m) \geqslant \frac{1}{2} P\,U$; with $P = [\, z^{\frac{1}{4}} \,]$

$$m \leqslant 16z + y^{\frac{3+\mu}{4}}$$

Also $\sum_m (r(m))^2$ does not exceed the number of solutions of $x^4 + u_h = y^4 + u_j$ subject to the conditions of lemma 30.

Hence $\sum (r(m))^2$

$$\leqslant C_l^{-\frac{3}{4}} \,.\, P^2 \, U^2 \, P^{3\mu - 4 + \delta} \; \{ C^{-\frac{1}{4}} + (2.3) \times 10^5 \}$$

$$\sum_m 1 \geqslant \frac{(\sum r(m))^2}{\sum (r(m))^2}$$

$$r(m) \neq 0$$

$$\geqslant \frac{1}{4} \, C_l^{\frac{3}{4}} \,.\, (C_l^{-\frac{1}{4}} + (2.3) \times 10^5)^{-1} \,.\, P^{-(3\mu - 4 + \delta)}$$

$$\geqslant \frac{1}{4} \, C_l^{\frac{3}{4}} \, (C_l^{-\frac{1}{4}} + (2.3) \times 10^5)^{-1} . \quad P^{4\nu_{l+1}}$$

$$\geqslant \frac{1}{4} \, C_l^{\frac{3}{4}} \, (C_l^{-\frac{1}{4}} + (2.3) \times 10^5)^{-1} . \quad ([\, Z^{\frac{1}{4}} \,]^{4\nu_{l+1}})$$

Choosing Z such that $16Z + Y^{\frac{3+\mu}{4}} = Y$, we have

$$\sum_{\substack{m \leqslant Y \\ r(m) \neq 0}} 1 \geqslant \tfrac{1}{64} C_l^{\frac{3}{4}} \left(C_l^{-\frac{1}{4}} + (2.3) \times 10^{6} \right)^{-1} Y^{\nu_{l+1}}$$

and hence the result.

Lemma 32 : *Let f be a fixed integer from the set $\{0, 1, 2, 3, 4\}$. Suppose $n \geqslant 2.10^{44}$. Then the number of non negative $m \equiv f \,(\mathrm{mod}\ 16)$ such that $m \leqslant n$ and such that m is the sum of four fourth powers is greater than*

$$(1.1950)\, 10^{-18} \,.\, n^{0.745633624425}$$

Proof: This is lemma 7.4 of Thomas [8] (page 127)

Lemma 33 : *Let f_1 be a fixed integer from the set $\{0, 1, 2, 3, 4, 5\}$. Suppose $n \geqslant 10^{60}$. Then the number of non negative $m \equiv f$ (mod 16) such that $m \leqslant n$ and such that m is the sum of five fourth powers is greater than*

$$(0.8) \times 10^{-21} \,.\, n^{0.8168}.$$

Proof: The result follows from a direct application of lemmas 31 and 32. In the notation of lemma 31, we have

$$C_l = (1.195) \times 10^{-18}$$

Hence

$$C_{l+1} = \tfrac{1}{64} \,.\, (1.195 \times 10^{-18})^{\frac{3}{4}} \left((1.195 \times 10^{-18})^{-\frac{1}{4}} + (2.3) \times 10^{6} \right)^{-1}$$

$$\geqslant 2.17 \times 10^{-21}.$$

$$\nu_l = 0.745633$$

$$\text{Hence } \nu_{l+1} = 0.25 + \frac{3 \times 0.745633}{3.945633} \geqslant 0.8168$$

and hence the result.

§ 7 : An asymptotic formula

Let N be an integer $> 10^{560}$ and $P = [N^{\frac{1}{4}}]$;

Let N_0 and N_1 be integers satisfying

$$\frac{N}{2} \leqslant N_0 \leqslant N; \quad N_0 - P^{4\frac{1}{5}} \leqslant N_1 \leqslant N_0$$

Let f be an integer in the set (0, 1, 2, 3, 4, 5) such that

$$N - 2f \equiv 2, 3, 4, 5, 6, 7 \text{ or } 8 \pmod{16}.$$

Let μ_1, μ_2 go independently over the same sequence of numbers, which is less than $\frac{N}{4}$ and can be represented as a sum of five biquadratics.

Let U denote the number of numbers μ.

Lemma 34 : *Every integer $N > 10^{560}$ can be represented as a sum of twenty one biquadratics*

Proof : Let $I(N) =$

$$\int_0^1 (T(\alpha))^{11} \sum_{\mu_1} \sum_{\mu_2} e^{2\pi i(\mu_1 + \mu_2)\alpha} \, e^{-2\pi i N\alpha} \, d\alpha$$

It is sufficient to prove that $I(N) > 0$

Now $I(N) = \int_{\Omega} + \int_{m}$

where Ω is the major arc and m is the minor arc.

By lemma 30

$$\int_{\Omega} (T(\alpha))^{11} \sum_{\mu_1} \sum_{\mu_2} e^{2\pi i(\mu_1 + \mu_2)\alpha} \, e^{-2\pi i N\alpha} \, d\alpha$$

$$= \sum_{\mu_1} \sum_{\mu_2} W_0(N - \mu_1 - \mu_2)$$

$$> U^2 \,.\, (0.02) \,.\, N^{\frac{7}{4}}$$

$$> U^2 \,.\, (0.02) \,.\, P^7$$

By lemma 13, we have

$$\int_{\mathfrak{m}} (T(a))^{11} \,.\, \sum_{\mu_1} \sum_{\mu_2} e^{2\pi i(\mu_1+\mu_2)a} \; e^{-2\pi i N a} \; da$$

$$\leqslant \max_{a \varepsilon \mathfrak{m}} |\, T(a) \,|^{11} \int_{\mathfrak{m}} \left| \sum_{\mu} e^{2\pi i \mu a} \right|^2 da$$

$$< \max_{a \varepsilon \mathfrak{m}} |\, T(a) \,|^{11} \,.\, U$$

$$\leqslant \left(3 \, P^{\frac{29}{32}} (\log P+3)^{\frac{15}{32}}\right)^{11} \,.\, U$$

We have only to check that

$$U^2 \, (0.02) \, P^7 > 3^{11} \,.\, P^{10-\frac{1}{32}} \, (\log P+3)^{5+\frac{5}{32}} \; U$$

Now using the value

$$U \geqslant \tfrac{1}{4} \; P^{3.2672} \times (0.8) \times 10^{-21}$$

we have only to check that

$$P^{0.2984} \geqslant 10^{28.65} \, (\log \, P+3)^{5 \cdot \frac{5}{32}}$$

Taking $P \geqslant 10^{140}$, we see that the inequality is satisfied and this proves the result.

§ 8 : **The ascent**

Lemma 35 : *Let l be an integer* $\geqslant 0$;

$$\textit{Let} \quad \nu = \frac{1 - \dfrac{l}{L_0}}{n} \; ;$$

$$L_0 > l; \; (\nu \, L_0)^{\frac{n}{n-1}} \geqslant L_0.$$

Compute L_t by

$$\log L_t = \left(\frac{n}{n-1}\right)^t (\log L_0 + n \log \nu) - n \log \nu$$

If all integers between l and L_0 inclusive are sums of k integral n^{th} powers, then all integers between l and L_t, inclusive are sums of $(k + t)$ integral n^{th} powers ≥ 0.

Proof: This is Theorem 12 (page 711) of Dickson [5].

Lemma 36 : *Every natural number in the range* $[13793, 10^{143}]$ *is a sum of sixteen biquadratics.*

Proof: This is Theorem 3·4 of Thomas [9].

Lemma 37 : *Every natural number less than* 10^{560} *is a sum of twenty one biquadratics.*

Proof: In the notation of lemma 36, we take $l = 13793$; $L_0 = 10^{143}$; $t = 5$; $n = 4$ and this proves the result.

§ 9 : Completion of the proof

The proof of the main theorem follows from lemmas 34 and 37.

§ 10 : Acknowledgement

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