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Linear forms in logarithms and exponential Diophantine equations

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Dedicated to the memory of Alan Baker

Abstract. This paper aims to show two things. Firstly the importance of Alan Baker's work on linear forms in logarithms for the development of the theory of exponential Diophantine equations. Secondly how this theory is the culmination of a series of greater and smaller discoveries.

Keywords. Alan Baker, linear forms in logarithms, exponential Diophantine equations.

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1. Hilbert's seventh problem

After Hermite had proved that e is transcendental in 1873, Lindemann used his method to prove that e^c is transcendental when $c \neq 0$ is algebraic. This result implies the transcendence of π in view of the relation $e^{\pi i} = -1$. In a famous lecture in 1900 Hilbert stated 23 open problems for the new century (cf. [Br76]). The seventh problem is to prove the transcendence of a^c when a and c are algebraic numbers with $a \neq 0, 1$ and $c \notin \mathbb{Q}$. In 1920 Hilbert told that he expected that this problem would be solved later than the Riemann Hypothesis and Fermat's Last Theorem [Re70].

In 1929 Gelfond proved the transcendence of $e^\pi = (-1)^{-i}$ and Kuzmin the transcendence of $2^{\sqrt{2}}$ a year later. The full proof of Hilbert's seventh problem that a^c is transcendental if a and c are algebraic numbers with $a \neq 0, 1$ and $c \notin \mathbb{Q}$ was obtained by both Gelfond and Schneider in 1934, [Gel60, Sch57]. Because of the relation $a^{\log b/\log a} = b$ it cannot be that $a (\neq 0, 1), b$ and $\log b/\log a (\notin \mathbb{Q})$ are all algebraic. It follows that for algebraic numbers a, b with $a, b \neq 0, 1$ the fraction $\log b/\log a$ is either rational or transcendental. Actually Gelfond gave transcendence measures for a^c and $\log b/\log a$ ([Gel60], p. 134). As a special case he derived the following approximation measure ([Gel60], p. 174):
The inequalities

$$0 < |b_1 \log a_1 + b_2 \log a_2| < e^{-(\log B)^{2+\varepsilon}}, \;\; B \geq |b_1| + |b_2|, \;\; \varepsilon > 0$$

do not have a solution in rational integers b_1, b_2 *for* $B > B_0$*, where* $B_0 = B_0(a_1, a_2, \log a_2/\log a_1, \varepsilon)$ *can be effectively computed.*
Gelfond already indicated the relation with exponential Diophantine equations by remarking that his results imply that the solutions x, y, z of the equation

$$\alpha^x + \beta^y = \gamma^z,$$

with algebraic α, β, γ and the condition that at least one of them is not an algebraic unit and γ is not a power of 2, can effectively be determined, at least in principle ([Gel60], p. 127).

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2. Zeros of exponential polynomials

During my mathematics study at the University of Amsterdam Henk Jager mentioned some open problems of Erdős and Turán, after he had spent half a year in Budapest. One problem concerned exponential polynomials $f(z) = \sum_{k=1}^{\ell} P_k(z)e^{\omega_k z}$ where $P_k(z) \in \mathbb{C}[z]$ of degree $\rho_k - 1$ and $\omega_k \in \mathbb{C}$ for $k = 1, \ldots, \ell$ and z is a complex variable. In 1960 Turán had proved that in case the polynomials P_k are constants, the number of zeros of f in a square of side length L is not larger than

$$6L\Delta + n\log\left(2 + \frac{n}{\delta L}\right) + \log(2n),$$

where $\Delta = \max_k |\omega_k|, \delta = \min_{k\neq j} |\omega_k - \omega_j|, n = \sum_k \rho_k$, [Tu60]. Four years later Dancs and Turán derived a bound for exponential polynomials with polynomials P_k of any degree. Coates and Van der Poorten obtained similar results. By combining Jensen's inequality and Turán's First Main Theorem I was able to prove the upper bound

$$3L\Delta + 3(n-1)$$

independent of δ, [Ti71a]. This result is the best possible apart from a multiplicative factor.

I was invited to participate at an Oberwolfach meeting on number theory in 1968 where I met many number theorists for the first time, among them Alan Baker (and Davenport, Ljunggren and Mordell for the first and last time). During this conference Coates showed me how I should be able to use my zeros estimate to remove the dependence on δ in an algebraic dependence result of Gelfond. I am grateful to him that he gave me the chance to work it out. It resulted in theorems of the following type:

Suppose both the numbers $\alpha_1, \alpha_2, \alpha_3$ and the numbers η_1, η_2, η_3 are linearly independent over the rationals. Then a field containing the rational numbers and the twelve numbers $\alpha_k, e^{\alpha_k \eta_j} (k, j = 0, 1, 2)$ has at least transcendence degree 2, see [Ti71b] and [Wa00] pp. 588-606.

I spent the academic years 1968/69 in Budapest with Turán and 1970/71 at the Institute for Advanced Study in Princeton. The latter was a special number theory year. There were many famous number theorists who participated, among them Alan Baker, Bombieri, Chowla, Montgomery, Ramachandra, Schmidt, Selberg, Serre and Stark. These visits were very valuable for me. In particular, I am grateful to Ramachandra who made me familiar with Baker's theory on linear forms in logarithms.

3. Linear forms in logarithms

Already Gelfond obtained an estimate for linear forms in arbitrary many logarithms of algebraic numbers, [Gel60], p.34:
If the algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are multiplicatively independent, then the inequality

$$\Lambda := |b_1 \log\alpha_1 + \cdots + b_n \log\alpha_n| < e^{-\varepsilon B}, \ \varepsilon > 0, \ B = \max_{1\leq i\leq n} |b_i|, \tag{3.1}$$

where $b_1, \ldots, b_n$ are rational integers and the number $\log\alpha$ may be an arbitrary, but fixed, value of the logarithm, has only a finite number of solutions in rational integers $b_1, b_2, \ldots, b_n$.
He further proved a p-adic version of this statement. Gelfond's proof of (3.1) was ineffective in the sense that the proof does not enable one to compute all the solutions for given α's and ε. He gave an effective proof for $n = 2$ with a better right-hand side $e^{-(\log B)^{2+\varepsilon}}$, [Gel60], p.174.

In 1966 Baker gave an effective proof of (3.1) with an improved right-hand side $e^{-(\log B)^{2n+1+\varepsilon}}$, [Ba66-68] I. In 1967 Baker further improved the upper bound to $e^{-(\log B)^{n+\varepsilon}}$, [Ba66-68] III. A year later Feldman obtained a lower bound which is the best possible with respect to B:

$$e^{-C(\log A)^{\kappa}(\log B)}$$

where $A(\geq 4)$ is the maximum of the heights of the numbers α_j $(j = 1, \ldots, n)$, where κ depends only on n, and C depends only on n and the degree d of the field generated by $\alpha_1, \ldots, \alpha_n$ over the rationals [Fel68]. Here the height of an algebraic number is the maximal absolute value of the coefficients of the minimal defining polynomial. In 1968 Baker proved that if $0 < \Lambda < e^{-\delta B}$ for some δ with $0 < \delta \leq 1$ and if $d \geq 4$, then $B < C(\log A)^\kappa$ for any $\kappa > n+1$, where C depends only on n, d, κ and the maximum A' of the heights of $\alpha_1, \ldots, \alpha_{n-1}$, [Ba68]. This refinement is particularly useful in the many applications where the height of α_n is much larger than the heights of the other α_j's. After several refinements by Baker and others, Baker obtained in 1972 that if $\Lambda \neq 0$, then

$$\Lambda > C^{-\log A \log B},$$

where $C > 0$ depends only on n, d and A', [Ba72]. This result is best possible with respect to B if A is fixed and with respect to A if B is fixed.

After several refinements by Baker and others, Baker derived a completely explicit lower bound in 1977, [Ba77]:

If $\Lambda \neq 0$, *then* $\Lambda > B^{-C\Omega \log \Omega'}$ *where* $C = (16nd)^{200n}$ *and* $\Omega = \log A_1 \cdots \log A_n$, $\Omega' = \Omega / \log A_n$.

Thanks to a technical improvement, Philippon and Waldschmidt succeeded in removing the factor $\log \Omega'$ in the exponent (with another value of C), [PhWa88]. Waldschmidt in 1991 and Baker and Wűstholz [BaWu93] in 1993 improved the constant further, resulting in:

If $\Lambda \neq 0$, *then* $\Lambda > B^{-C\Omega}$ *where* $C = (16nd)^{2n+4}$.

Numerous improvements and variants have been given, e.g. in case $n = 1$ [Fel67], $n = 2$ [MiWa89], $\beta_n = -1$ [Sh76], the α's are near 1 [Sh74], using Weil height for the α's [BaWu93], using another value of B, [PhWa88] Theorem 2.2.

Already in 1968, Baker extended the theory to linear forms of the form $\beta_0 + \beta_1 \log \alpha_1 + \ldots \beta_n \log \alpha_n$ where both $\alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots, \beta_n$ are algebraic numbers, so providing a full generalization of the Gelfond-Schneider theorem. Generally speaking, all the above results for the 'rational' case also hold in this 'algebraic' case but with slightly worse bounds. Since the 'algebraic' results are not relevant for the exponential Diophantine equations I want to discuss, I refer for such results to [Ba77] for developments until 1977 and to [FeNe98, Wa00] for later developments. Many more results and references can be found there, also for the rational case.

Relevant for the theory of exponential Diophantine equations are the results on linear forms in the rational p-adic case. Coates, van der Poorten and later Yu derived p-adic analogues of the above estimates both in the rational and in the algebraic case. For this, I refer to [vdP77] for results obatined until 1977 and to [Yu02] for later developments. For other extensions of the theory on linear forms in logarithms and applications, e.g. in transcendence theory, I refer to [Ba88, FeNe98, Wa00, Wű02].

4. Baker's method in a nutshell

Basically Baker's method consists of four steps. (For the first three steps I follow Baker [Ba75], pp.14-17. The fourth step is a shortcut as used by Cijsouw [Cij74].) We consider the case that

$$|b_1 \log \alpha_1 + \cdots + b_{n-1} \log \alpha_{n-1} - \log \alpha_n|$$

is very small.

Step 1. Construction of an exponential polynomial in n variables with many vanishing partial derivatives in many integer points.

Choose integers $p(\lambda_0, \ldots, \lambda_{n-1})$, *not all zero, with absolute values at most* e^{h^3}, *such that the function*

$$\Phi(z_0, \ldots, z_{n-1}) = \sum_{\lambda_0=0}^{L} \cdots \sum_{\lambda_{n-1}=0}^{L} p(\lambda_0, \ldots, \lambda_{n-1}) z_0^{\lambda_0} \alpha_1^{\gamma_1 z_1} \cdots \alpha_{n-1}^{\gamma_{n-1} z_{n-1}},$$

where $\gamma_r = \lambda_r + \lambda_n b_r$ $(1 \le r \le n)$ *and* $L = [h^{2-1/(4n))}]$, *satisfies*

$$\Phi_{m_0,\ldots,m_{n-1}}(\ell,\ldots,\ell) = 0 \tag{4.2}$$

for all integers ℓ *with* $1 \le \ell \le h$ *and all nonnegative integers* $m_0,\ldots,m_{n-1}$ *with* $m_0+\cdots+m_{n-1} \le h^2$.
This can be done by Siegel's lemma using the fact that the number of variables is more than twice the number of linear equations.

Step 2. For $m_0,\ldots,m_{n-1}$ as above some properties of the function $f(z) := \Phi_{m_0,\ldots,m_{n-1}}(z_0,\ldots,z_{n-1})$.
For any number z *we have* $|f(z)| \le c_1^{h^3+L|z|}$. *Further, for any positive integer* ℓ, *either* $f(\ell) = 0$ *or* $|f(\ell)| > c_2^{-h^3-L|z|}$ *where* $c_1, c_2, \ldots$ *are suitably chosen positive numbers depending only on* $n, b_1,\ldots,b_{n-1},\alpha_1,\ldots,\alpha_n$.
The first bound is obtained by a straightforward estimation.
If $f(\ell) \ne 0$, then a Liouville-type argument can be used to show that the algebraic number $f(\ell)$ is not very close to 0.

Step 3. Show that $f(\ell) = 0$ for more values of $\ell, m_0,\ldots,m_{n-1}$.
Let J *be any integer satisfying* $0 \le J \le 8n^2$. *Then* (4.2) *holds for* $1 \le \ell \le h^{1+J/8n} =: R_J$ *and* $m_0+\cdots+m_{n-1} \le h^2/2^J =: S_J$.
By induction on J. Suppose the statement is true for J. Then apply the maximum modulus principle to the analytic function

$$\frac{f(z)}{((z-1)\cdots(z-R_J))^{S_{J+1}}}.$$

We use Step 2 to derive an upper bound for $f(z)$ on $|z| = R_{J+1}$ and compute a lower bound on that circle for the absolute value of the denominator. Hence we obtain an upper bound for the fraction on the disc $|z| \le R_{J+1}$. It follows that $|f(\ell)| < c_2^{-h^3-L\ell}$ for $1 \le \ell \le R_{J+1}$. By Step 2 this implies $f(\ell) = 0$ for $1 \le \ell \le R_{J+1}, m_0+\cdots+m_{n-1} \le S_{J+1}$.

Step 4. Show that $\phi(z) := \Phi(z,\ldots,z)$ has so many zeros in the disk $|z| \le R_{8n^2} = h^{n+1}$ that all its coefficients have to be zero.
In Section 2, an upper bound for the number of zeros of an exponential polynomial has been given. If the bound is not reached, then the fact that the ℓ's are equidistant is used to get sharper upper bounds for the number of zeros. (We have a good lower bound for δ.) Step 4 is usually the most complicated part of the proof. Special papers have been written on zero and multiplicity estimates, see e.g. [MaWu85].

5. Applications to exponential Diophantine equations

Baker applied his linear forms estimates to obtain effective versions of results of Thue and Siegel. He derived upper bounds for the absolute values of the solutions x, y of the equations:
a) $f(x,y) = m$ where $f \in \mathbb{Z}[x,y]$ is an irreducible homgeneous polynomial of degree $n \ge 3$ and m is a nonzero integer, [Ba66-68] : I,
b) $y^2 = x^3 + m$ where m is a nonzero integer, [Ba66-68] : II,
c) $f(x) = y^2$ where $f(x) \in \mathbb{Z}[x]$ has at least three simple zeros, [Ba69],
d) $f(x) = y^m$ where $f(x) \in \mathbb{Z}[x]$ has at least two simple zeros and $m \ge 3$, [Ba69],
e) integer points on curves of genus 1 (together with Coates), [BaCo70].

p-Adic analoges of a) and b) were derived by Coates implying that m may be any integer composed of a finite set of fixed primes [Coa69-70]. Results c) and d) were extended to $f(x,z) = y^m$ where

$f(x,z) \in \mathbb{Z}[x,z]$ is a homogeneous polynomial satisfying corresponding conditions and z is composed of a finite set of fixed primes.Thereafter many improvements were obtained on the bounds by several people. The p-adic analogues are examples of exponential equations where when exponents are variables their bases are fixed.

An example of a) is the equation $ax^n - by^n = c$ for given nonzero integers a, b, c and $n \geq 3$. It was quite a surprise that Baker's method made it possible to bound the exponent n while the base is a variable. We restrict the rest of the paper to the study of such equations. The principle is quite simple. If $ax^n - by^n = c$ for given positive integers a, b, c and unknowns n, x, y, then

$$0 < \log \frac{a}{b} + n \log \frac{x}{y} < \frac{a}{b} \left(\frac{x}{y}\right)^n - 1 = \frac{c}{by^n}.$$

If n is large, then x, y have similar size. So we may assume $x < 2y$ for $n > n_0(a,b,c)$. Hence the height of x/y is at most $2y$. By Baker's estimate we have

$$\log \frac{a}{b} + n \log \frac{x}{y} > e^{-C_1 \log n \log y}, \quad C_1 = C_1(a,b).$$

On combining both inequalities and taking a logarithm, we find

$$C_2 + n \log y < C_1 \log n \log y, \quad C_2 = C_2(b,c).$$

The factor $\log y$ cancels, and we find an upper bound n_1 for n in terms of a, b, c. For every n with $3 \leq n \leq n_1$, we can apply the above result a). Thus for given a, b, c the solutions (n, x, y) can be computed, at least in principle [Ti75].

Using estimates for linear forms in logarithms, it could even be proved that a bound for the solutions of the Catalan equation $x^m - y^n = 1$ in integer variables m, n, x, y each greater than 1 can be derived, [Ti76]. Furthermore, Schinzel and Tijdeman proved the following extension of the above results b), c) and d):

Let $f(x) \in \mathbb{Q}[x]$ have at least two distinct roots. Then $f(x) = y^n$ with $|y| > 1$ implies that n is bounded by a computable number depending only on f, [ScTi76].

We say that equations of the form $f(x) = y^n$ are of type A).

Later on estimates for linear forms in logarithms were applied to a variety of exponential Diophantine equations, such as:

B) $1^k + 2^k + \cdots + x^k = y^n$ for given integer k in unknowns n, x, y, [GTV76],
C) $\prod_{i=1}^{k}(x + id) = y^n$ for given d, k in unknowns n, x, y, [Sh88],
D) $\frac{x^m - 1}{x - 1} = y^n$ in integers x, y, m, n subject to some restriction, [ShTi76],
E) $u_m = y^n$ where (u_m) is a binary recurrence sequence and m, n, y are unknowns, [Pe82].

Numerous generalizations and variations of these results have been published. See for example, [Sh06], [ShSt83], [ShTi86].

An important step forward was taken when B.M.M. de Weger and others used the LLL-algorithm to reduce the huge upper bounds from the linear forms estimates to, roughly, their logarithmic value, [dWe89]. It made it possible to compute all the solutions of some Diophantine equations after using estimates for linear forms. But as far as I know, all these equations are polynomial or purely exponential (in the sense that the bases of exponential unknowns are fixed).

6. Later developments

Probably the most famous exponential Diophantine equation is the Fermat equation $x^n + y^n = z^n$ in integers n, x, y, z all greater than 2. It remained uncertain whether there exists a solution until 1995 when Wiles in collaboration with Taylor proved that there are no such solutions, [Wi95].

The 'modular' method developed by Wiles turned out to be a powerful tool to solve exponential Diophantine equations completely, see e.g. [BeSk04].

In 2004, Mihăilescu used methods from pure algebraic number theory to prove that the only solution of the Catalan equation $x^m - y^n = 1$ in integers $m > 1, n > 1, x > 1, y > 1$ is given by $(m, n, x, y) = (2, 3, 3, 2)$. See also Schoof [Sch08] and Bilu et al. [BBM14].

Nowadays exponential Diophantine equations are often solved by combining various methods including Baker's method and the modular method. In this way Bennett proved that if a, b, n are integers with $ab \neq 0, n \geq 3$, then the equation $|ax^n - by^n| = 1$ has at most one solution in positive integers x, y, [Be01]. It implies that for $a > 0, n > 2$ the only solution of the equation $(a+1)x^n - ay^n = 1$ is $x = y = 1$, (a type A) result). Bugeaud et al. proved that 8 and 144 are the only perfect powers in the Fibonacci sequence (a type E) result), [BMS06]. Győry, Hajdu and Pintér proved that the product of k consecutive terms in a coprime arithmetic progression with $3 \leq k \leq 35$ cannot be a perfect power, (a type C) result), [GHP09]. Bennett, Győry and Pintér solved equation B) for $1 \leq k \leq 11$, [BGP04]. Many similar results could be mentioned.

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