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A transference inequality for rational approximation to points in geometric progression

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Dedicated to the memory of Alan Baker

Abstract. We establish a transference inequality conjectured by Badziahin and Bugeaud relating exponents of rational approximation of points in geometric progression.

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Let $\xi \in \mathbb{R}$. For each integer $n \geq 1$, one defines $\omega_n(\xi)$ as the supremum of all $\omega \in \mathbb{R}$ for which there exist infinitely many non-zero polynomials $P(x) \in \mathbb{Z}[x]$ of degree at most n with

$$|P(\xi)| \leq \|P\|^{-\omega},$$

where $\|P\|$ stands for the largest absolute value of the coefficients of P. These quantities form the basis of Mahler's classification of numbers [Mah32]. Dually, one defines $\lambda_n(\xi)$ as the supremum of all $\lambda \in \mathbb{R}$ for which there exist infinitely many non-zero points $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ such that

$$\max_{1\leq m\leq n} |x_0\xi^m - x_m| \leq \|\mathbf{x}\|^{-\lambda},$$

where $\|\mathbf{x}\|$ stands for the maximum norm of $\mathbf{x}$.

Suppose now that ξ is transcendental over $\mathbb{Q}$, i.e. that $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$. In [BaBu19], D. Badziahin and Y. Bugeaud prove that for any integers k, n with $2 \leq k \leq n$, we have

$$\lambda_n(\xi) \geq \frac{\omega_k^{\mathrm{lead}}(\xi) - n + k}{(k-1)\omega_k^{\mathrm{lead}}(\xi) + n}\,, \tag{1}$$

where $\omega_k^{\mathrm{lead}}(\xi)$ is defined as $\omega_k(\xi)$ but by restricting to polynomials $P(x) \in \mathbb{Z}[x]$ of degree k whose coefficient $c_k(P)$ of x^k has largest absolute value $|c_k(P)| = \|P\|$. They conjecture that this inequality remains true if $\omega_k^{\mathrm{lead}}(\xi)$ is replaced by $\omega_k(\xi)$ and they prove this is indeed the case if $k = 2$ or $k = n-1$. Their proof for $k = 2$ is based on the fact that $\lambda_n(\xi) = \lambda_n(1/\xi)$. The purpose of this note is to prove this conjecture based on their inequality (1), the invariance of λ_n and ω_k by general fractional linear transformations with rational coefficients, and the following observation.

Theorem. *Let $k \geq 1$ be an integer and let $r_0, \dots, r_k$ be distinct integers. There is an integer $M \geq 1$ such that, for each $\xi \in \mathbb{R}$, there exists at least one index $i \in \{0, 1, \dots, k\}$, with $r_i \neq \xi$, for which the point $\xi_i = 1/(M(\xi - r_i))$ satisfies $\omega_k^{\mathrm{lead}}(\xi_i) = \omega_k(\xi_i) = \omega_k(\xi)$.*

Proof. We first note that there exist positive constants C_1 and C_2 such that

$$\|P\| \leq C_1 \max\{|P(r_i)|\,;\, 0 \leq i \leq k\} \quad \text{and} \quad \max\{\|P(x + r_i)\|\,;\, 0 \leq i \leq k\} \leq C_2\|P\|$$

for any polynomial $P \in \mathbb{R}[x]$ of degree at most k. Let $\xi \in \mathbb{R}$. Choose an integer M with $M \geq C_1C_2$ and a sequence of polynomials $(P_j)_{j\geq 1}$ in $\mathbb{Z}[x]$ of degree at most k with strictly increasing norms such

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that

$$\lim_{j\to\infty} -\frac{\log|P_j(\xi)|}{\log\|P_j\|} = \omega_k(\xi). \tag{2}$$

Then, choose $i \in \{0,1,\dots,k\}$ such that $\|P_j\| \le C_1|P_j(r_i)|$ for an infinite set S of positive integers j, and set

$$Q_j(x) = (Mx)^k P_j\Big(\frac{1}{Mx} + r_i\Big) \in \mathbb{Z}[x]$$

for each $j \in S$. For those values of j, the absolute value of the coefficient of x^k in $Q_j(x)$ is $|c_k(Q_j)| = M^k|P_j(r_i)|$ while its other coefficients have absolute value at most

$$M^{k-1}\|P_j(x+r_i)\| \le C_2M^{k-1}\|P_j\| \le C_1C_2M^{k-1}|P_j(r_i)| \le M^k|P_j(r_i)|,$$

thus $|c_k(Q_j)| = \|Q_j\|$. We also have $r_i \neq \xi$ and

$$|Q_j(\xi_i)| = (M|\xi_i|)^k|P_j(\xi)|$$

where $\xi_i = 1/(M(\xi - r_i))$. As the ratio $\|Q_j\|/\|P_j\|$ is bounded from above and from below by positive constants, we deduce from (2) that $-\log|Q_j(\xi_i)|/\log\|Q_j\|$ converges to $\omega_k(\xi)$ as j goes to infinity in S. Altogether, this means that $\omega_k^{\text{lead}}(\xi_i) \ge \omega_k(\xi)$. However, it is well known (and easy to prove) that $\omega_k(\xi) = \omega_k(\xi_i)$ because ξ_i is the image of ξ by a linear fractional transformation with rational coefficients. The conclusion follows because $\omega_k(\xi_i) \ge \omega_k^{\text{lead}}(\xi_i)$ by the very definition of ω_k^{lead}.

Applying the formula (1) with ξ replaced by ξ_i and using the fact that $\lambda_n(\xi_i) = \lambda_n(\xi)$, we reach the desired inequality.

Corollary 1. $\lambda_n(\xi) \ge \dfrac{\omega_k(\xi) - n + k}{(k-1)\omega_k(\xi) + n}$ *for any* $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$ *and any integers* $2 \le k \le n$.

We thank Yann Bugeaud and Victor Beresnevich for pointing out that our theorem formalizes principles that are implicit in Lemmas 3 and 4 of [Bak76] as well as on pages 25–26 of [Spr69] (for the choice of $r_i = i$). The next corollary is an application to metrical theory that was suggested by Yann Bugeaud.

Corollary 2. *Let* $k \ge 1$ *be an integer and let* $w \in \mathbb{R}$*. The sets* $S = \{\xi \in \mathbb{R}\,;\, \omega_k(\xi) \ge w\}$ *and* $S^{\text{lead}} = \{\xi \in \mathbb{R}\,;\, \omega_k^{\text{lead}}(\xi) \ge w\}$ *have the same Hausdorff dimension. This remains true if we replace the large inequalities* $\ge$ *by strict inequalities* $>$ *in the definitions of* S *and* S^{lead}.

Proof. We have $S^{\text{lead}} \subset S$ and, for an appropriate choice of integers $r_0,\dots,r_k$ and M, the theorem gives $S \subseteq \bigcup_{i=0}^k \tau_i^{-1}(S^{\text{lead}})$ where $\tau_i(\xi) = 1/(M(\xi - r_i))$ for each $\xi \in \mathbb{R}\setminus\{r_i\}$ $(0 \le i \le k)$. The conclusion follows by the invariance of the Hausdorff dimension under fractional linear transformations.

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