

Some New Congruences for l -Regular Partitions Modulo 13, 17, and 23

S Abinash, T Kathiravan, K Srilakshmi

► **To cite this version:**

S Abinash, T Kathiravan, K Srilakshmi. Some New Congruences for l -Regular Partitions Modulo 13, 17, and 23. Hardy-Ramanujan Journal, Hardy-Ramanujan Society, 2020, Volume 42 - Special Commemorative volume in honour of Alan Baker. hal-02301897v2

HAL Id: hal-02301897

<https://hal.archives-ouvertes.fr/hal-02301897v2>

Submitted on 30 Apr 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Some new congruences for l -regular partitions modulo 13, 17, and 23

S. Abinash, T. Kathiravan and K. Srilakshmi

Dedicated to the memory of Alan Baker

Abstract. A partition of n is l -regular if none of its parts is divisible by l . Let $b_l(n)$ denote the number of l -regular partitions of n . In this paper, using theta function identities due to Ramanujan, we establish some new infinite families of congruences for $b_l(n)$ modulo l , where $l = 17, 23$, and for $b_{65}(n)$ modulo 13.

Keywords. congruences, l -regular partitions, theta function identities.

2010 Mathematics Subject Classification. 11P83, 05A17

1. Introduction

We first recall a few terminologies and notations. Let n be a positive integer. A partition of n is a non-increasing sequence of positive integers whose sum is n , and the members of the sequence are called parts. For a positive integer $l \geq 2$, a partition of n is said to be l -regular if none of its parts is divisible by l . Let $b_l(n)$ denote the number of l -regular partitions of n . By convention, we assume that $b_l(0) = 1$. The generating function for $b_l(n)$ is given by

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{f_l}{f_1}, \quad (1.1)$$

where $|q| < 1$ and for any positive integer k , f_k is defined by

$$f_k := \prod_{i=1}^{\infty} (1 - q^{ki}).$$

It can be easily verified that for any prime number l ,

$$f_l \equiv f_1^l \pmod{l}. \quad (1.2)$$

In recent years, various congruences satisfied by l -regular partitions for different values of l has been established. For instance, Lovejoy and Penniston [LoPe01] established criteria for 3-divisibility of $b_3(n)$. Dandurand and Penniston [DaPe09] employed the theory of complex multiplication to give a precise description of those n such that $l|b_l(n)$ for $l \in \{5, 7, 11\}$. Cui and Gu [CuiGu13] derived infinite families of congruences modulo 2 for $b_l(n)$ where $l \in \{2, 4, 5, 8, 13, 16\}$. Webb [Web11] established an infinite family of congruences modulo 3 for $b_{13}(n)$. Furcy and Penniston [FuPe12] obtained families of congruences modulo 3 for other values of l which are congruent to 1 modulo 3. Xia and Yao [XiYa14a] found some infinite families of congruences for $b_9(n)$ modulo 2, and Cui and Gu [CuiGu15] derived congruences for $b_9(n)$ modulo 3. Xia [Xia15] established infinite families of congruences for $b_l(n)$ modulo l where $l \in \{13, 17, 19\}$, for example, for any $n \geq 0$, $k \geq 0$,

$$b_{13} \left(5^{12k}n + \frac{1}{2} (5^{12k} - 1) \right) \equiv b_{13}(n) \pmod{13}.$$

For a complete literature review, see [Xia15]. For more related works, see [AdDa18, AhLo01, AHS10, CaWe14, CuiGu18, DLY14, Kei14, LiWa14, Pen02, Pen08, Ran17, Tan18, Wan17, Wan18a, Wan18b, Xia14, XiYa14b, Yao14, ZJY18].

In this paper, we establish infinite families of Ramanujan-type congruences for $b_l(n)$ modulo l , where $l \in \{17, 23\}$, and for $b_{65}(n)$ modulo 13. In fact, no such congruence has been established for 23-regular partitions till now, to the best of the authors' knowledge. The following are the main results.

Theorem 1.1. *For any $n \geq 0$, $k \geq 0$,*

$$b_{17} \left(5^{4k}n + \frac{2}{3}(5^{4k} - 1) \right) \equiv 2^k b_{17}(n) \pmod{17}, \quad (1.3)$$

and for $r \in \{0, 1, 2, 4\}$,

$$b_{17} \left(5^{4k+4}n + \frac{1}{3}(5^{4k+3}(3r+1) - 2) \right) \equiv 0 \pmod{17}. \quad (1.4)$$

Theorem 1.2. *For any $n \geq 0$, $k \geq 0$,*

$$b_{23} \left(5^{24k}n + \frac{11}{12}(5^{24k} - 1) \right) \equiv 14^k b_{23}(n) \pmod{23}, \quad (1.5)$$

and for $r \in \{0, 1, 2, 3\}$,

$$b_{23} \left(5^{24k+24}n + \frac{1}{12}(5^{24k+23}(12r+7) - 11) \right) \equiv 0 \pmod{23}. \quad (1.6)$$

Theorem 1.3. *For any $n \geq 0$, $k \geq 0$,*

$$b_{65} \left(5^{12k}n + \frac{8}{3}(5^{12k} - 1) \right) \equiv 12^k b_{65}(n) \pmod{13}, \quad (1.7)$$

and for $r \in \{2, 4\}$,

$$b_{65} \left(5^{12k+12}n + \frac{1}{3}(5^{12k+11}(3r+1) - 8) \right) \equiv 0 \pmod{13}. \quad (1.8)$$

2. Preliminaries

In this section we shall state a preliminary lemma.

Lemma 2.1. *For*

$$R(q) := \prod_{n=1}^{\infty} \frac{(1 - q^{5n-4})(1 - q^{5n-1})}{(1 - q^{5n-3})(1 - q^{5n-2})},$$

we have

A. ([Ram62], [Wat38])

$$f_1 = f_{25} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right). \quad (2.9)$$

B. ([Wat29a],[Wat29b])

$$\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 = \frac{f_1^6}{f_5^6}. \tag{2.10}$$

C.

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6} \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right). \tag{2.11}$$

Note that the identity in (2.11) can be easily established by first substituting q by q^5 in (2.10) and then using (2.9).

3. Congruences for 17-Regular Partitions

Proof of Theorem 1.1. Taking $l = 17$ in (1.1), we get

$$\sum_{n=0}^{\infty} b_{17}(n)q^n = \frac{f_{17}}{f_1}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{17}(n)q^n \equiv f_1^{16} \pmod{17}. \tag{3.12}$$

Replacing f_1 using (2.9) and then extracting the terms involving q^{5n+1} , we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5n+1)q^{5n+1} &\equiv f_{25}^{16} \left(\frac{q}{R(q^5)^{15}} + \frac{13q^6}{R(q^5)^{10}} + \frac{8q^{11}}{R(q^5)^5} - q^{16} - 8q^{21} R(q^5)^5 \right. \\ &\quad \left. + 13q^{26} R(q^5)^{10} - q^{31} R(q^5)^{15} \right) \pmod{17}. \end{aligned}$$

If we divide by q and substitute q by $q^{1/5}$, then we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5n+1)q^n &\equiv f_5^{16} \left(\left(\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^3 + 12q \left(\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^2 \right. \\ &\quad \left. + 14q^2 \left(\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right) + 7q^3 \right) \pmod{17}. \end{aligned}$$

By (2.10), we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \equiv f_5^{16} \left(\frac{f_1^{18}}{f_5^{18}} + 12q \frac{f_1^{12}}{f_5^{12}} + 14q^2 \frac{f_1^6}{f_5^6} + 7q^3 \right) \pmod{17}.$$

In view of (3.12), we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \equiv \frac{f_1^{18}}{f_5^{18}} + 12q f_1^{12} f_5^4 + 14q^2 f_1^6 f_5^{10} + 7 \sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \tag{3.13}$$

Replacing f_1 using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5n+1)q^n &\equiv \frac{f_{25}^{18}}{f_5^2} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{18} + 12q f_5^4 f_{25}^{12} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{12} \\ &\quad + 14q^2 f_5^{10} f_{25}^6 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^6 + 7 \sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \end{aligned}$$

If we extract the terms involving q^{5n+3} , divide by q^3 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n &\equiv \frac{f_5^{18}}{f_1^2} \left(q^3 \right) + 12f_1^4 f_5^{12} \left(3\frac{f_1^{12}}{f_5^{12}} + 3q^2 \right) + 14f_1^{10} f_5^6 \left(11\frac{f_1^6}{f_5^6} + 9q \right) \\ &\quad + 7 \sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n \equiv 7q f_1^{10} f_5^6 + 2q^2 f_1^4 f_5^{12} + q^3 \frac{f_5^{18}}{f_1^2} + 10 \sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}.$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n &\equiv 7q f_5^6 f_{25}^{10} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{10} + 2q^2 f_5^{12} f_{25}^4 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^4 \\ &\quad + q^3 f_5^6 f_{25}^{10} \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) \right. \\ &\quad \left. + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right)^2 + 10 \sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}. \end{aligned}$$

If we extract the terms involving q^{5n+1} , divide by q , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n &\equiv 7f_1^6 f_5^{10} \left(\frac{f_1^{12}}{f_5^{12}} + 12q \frac{f_1^6}{f_5^6} + q^2 \right) + 2f_1^{12} f_5^4 \left(12q \right) + f_1^6 f_5^{10} \left(10q \frac{f_1^6}{f_5^6} + 6q^2 \right) \\ &\quad + 10 \sum_{n=0}^{\infty} b_{17}(5n+1)q^n \pmod{17}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n \equiv 7\frac{f_1^{18}}{f_5^2} + 16q f_1^{12} f_5^4 + 13q^2 f_1^6 f_5^{10} + 10 \sum_{n=0}^{\infty} b_{17}(5n+1)q^n \pmod{17}.$$

Now by (3.13), we get

$$\sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n \equiv 2 \sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \quad (3.14)$$

Comparing the coefficients of the terms of the form q^{5n+3} in (3.14), we can conclude that for any $n \geq 0$,

$$b_{17}(5^4n+416) \equiv 2b_{17}(n) \pmod{17},$$

or,

$$b_{17} \left(5^4 n + \frac{2}{3} (5^4 - 1) \right) \equiv 2b_{17}(n) \pmod{17}. \tag{3.15}$$

Now (1.3) follows from (3.15), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form q^{5n+r} where $r \in \{0, 1, 2, 4\}$ in (3.14), we can conclude that for any $n \geq 0$,

$$b_{17} \left(5^3(5n+r) + \frac{5^3-2}{3} \right) \equiv 0 \pmod{17},$$

or,

$$b_{17} \left(5^4 n + \frac{1}{3} (5^3(3r+1) - 2) \right) \equiv 0 \pmod{17}. \tag{3.16}$$

Now (1.4) follows from (1.3) and (3.16).

4. Congruences for 23-Regular Partitions

Proof of Theorem 1.2. Taking $l = 23$ in (1.1), we get

$$\sum_{n=0}^{\infty} b_{23}(n)q^n = \frac{f_{23}}{f_1}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{23}(n)q^n \equiv f_1^{22} \pmod{23}. \tag{4.17}$$

Replacing f_1 using (2.9) and then extracting the terms involving q^{5n+2} , we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5n+2)q^{5n+2} &\equiv f_5^{22} \left(\frac{2q^2}{R(q^5)^{20}} + \frac{21q^7}{R(q^5)^{15}} + \frac{3q^{12}}{R(q^5)^{10}} + \frac{8q^{17}}{R(q^5)^5} - q^{22} - 8q^{27}R(q^5)^5 \right. \\ &\quad \left. + 3q^{32}R(q^5)^{10} - 21q^{37}R(q^5)^{15} + 2q^{42}R(q^5)^{20} \right) \pmod{23}. \end{aligned}$$

If we divide by q^2 and substitute q by $q^{1/5}$, then we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5n+2)q^n &\equiv f_5^{22} \left(2 \left(\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^4 + 17q \left(\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^3 \right. \\ &\quad \left. + 17q^2 \left(\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^2 + 14q^4 \right) \pmod{23}. \end{aligned}$$

By (2.10), we get

$$\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \equiv f_5^{22} \left(2 \frac{f_1^{24}}{f_5^{24}} + 17q \frac{f_1^{18}}{f_5^{18}} + 17q^2 \frac{f_1^{12}}{f_5^{12}} + 14q^4 \right) \pmod{23}.$$

In view of (4.17), we get

$$\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \equiv 2 \frac{f_1^{24}}{f_5^{24}} + 17q f_1^{18} f_5^4 + 17q^2 f_1^{12} f_5^{10} + 14 \sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \tag{4.18}$$

Replacing f_1 using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5n+2)q^n &\equiv 2\frac{f_{25}^{24}}{f_5^2} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{24} + 17qf_5^4f_{25}^{18} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{18} \\ &\quad + 17q^2f_5^{10}f_{25}^{12} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{12} + 14 \sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \end{aligned}$$

If we extract the terms involving q^{5n+4} , divide by q^4 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n &\equiv 2\frac{f_5^{24}}{f_1^2} \left(q^4 \right) + 17f_1^4f_5^{18} \left(19\frac{f_1^{18}}{f_5^{18}} + 7q^3 \right) + 17f_1^{10}f_5^{12} \left(8\frac{f_1^{12}}{f_5^{12}} + 3q^2 \right) \\ &\quad + 14 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n \equiv 5q^2f_1^{10}f_5^{12} + 4q^3f_1^4f_5^{18} + 2q^4\frac{f_5^{24}}{f_1^2} + 13 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \quad (4.19)$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n &\equiv 5q^2f_5^{12}f_{25}^{10} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{10} + 4q^3f_5^{18}f_{25}^4 \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^4 \\ &\quad + 2q^4f_5^{12}f_{25}^{10} \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) \right. \\ &\quad \left. + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4 \right)^2 + 13 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \end{aligned}$$

If we extract the terms involving q^{5n+2} , divide by q^2 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n &\equiv 5f_1^{12}f_5^{10} \left(\frac{f_1^{12}}{f_5^{12}} + 20q\frac{f_1^6}{f_5^6} + 16q^2 \right) + 4f_1^{18}f_5^4 \left(18q \right) + 2f_1^{12}f_5^{10} \left(10q\frac{f_1^6}{f_5^6} + 10q^2 \right) \\ &\quad + 13 \sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n \equiv 5\frac{f_1^{24}}{f_5^2} + 8qf_1^{18}f_5^4 + 8q^2f_1^{12}f_5^{10} + 13 \sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}.$$

Now by (4.18), we get

$$\sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n \equiv 8\frac{f_1^{24}}{f_5^2} + 22qf_1^{18}f_5^4 + 22q^2f_1^{12}f_5^{10} + 21 \sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}.$$

Replacing f_1 using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n &\equiv 8\frac{f_{25}^{24}}{f_5^2} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{24} + 22qf_5^4f_{25}^{18} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{18} \\ &\quad + 22q^2f_5^{10}f_{25}^{12} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{12} + 21 \sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \end{aligned}$$

If we extract the terms involving q^{5n+4} , divide by q^4 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{23}(5^4n + 572)q^n \equiv 8 \frac{f_5^{24}}{f_1^2} \left(q^4 \right) + 22 f_1^4 f_5^{18} \left(19 \frac{f_1^{18}}{f_5^{18}} + 7q^3 \right) + 22 f_1^{10} f_5^{12} \left(8 \frac{f_1^{12}}{f_5^{12}} + 3q^2 \right) + 21 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^4n + 572)q^n \equiv 20q^2 f_1^{10} f_5^{12} + 16q^3 f_1^4 f_5^{18} + 8q^4 \frac{f_5^{24}}{f_1^2} + 17 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Now by (4.19), we get

$$\sum_{n=0}^{\infty} b_{23}(5^4n + 572)q^n \equiv 4 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n + 11 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23},$$

or,

$$\sum_{n=0}^{\infty} b_{23} \left(5^4n + \frac{11}{12} (5^4 - 1) \right) q^n \equiv 4 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n + 11 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Substituting n by $5^2n + 22$, we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23} \left(5^6n + \frac{11}{12} (5^6 - 1) \right) q^n &\equiv 4 \sum_{n=0}^{\infty} b_{23} \left(5^4n + \frac{11}{12} (5^4 - 1) \right) q^n + 11 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n \pmod{23}, \\ &\equiv 4 \left(4 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n + 11 \sum_{n=0}^{\infty} b_{23}(n)q^n \right) + 11 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n \pmod{23}, \\ &\equiv 4 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n + 21 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \end{aligned}$$

Repeating the same process as above, it follows that

$$\sum_{n=0}^{\infty} b_{23} \left(5^{22}n + \frac{11}{12} (5^{22} - 1) \right) q^n \equiv 18 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n + 20 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Now by (4.19), we get

$$\sum_{n=0}^{\infty} b_{23} \left(5^{22}n + \frac{11}{12} (5^{22} - 1) \right) q^n \equiv 21q^2 f_1^{10} f_5^{12} + 3q^3 f_1^4 f_5^{18} + 13q^4 \frac{f_5^{24}}{f_1^2} + \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23} \left(5^{22}n + \frac{11}{12} (5^{22} - 1) \right) q^n &\equiv 21q^2 f_5^{12} f_{25}^{10} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{10} + 3q^3 f_5^{18} f_{25}^4 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^4 \\ &\quad + 13q^4 f_5^{12} f_{25}^{10} \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R(q^5)^2 \right. \\ &\quad \left. - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right)^2 + \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \end{aligned}$$

If we extract the terms involving q^{5n+2} , divide by q^2 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23} \left(5^{23}n + \frac{7 \cdot 5^{23} - 11}{12} \right) q^n &\equiv 21f_1^{12}f_5^{10} \left(\frac{f_1^{12}}{f_5^{12}} + 20q \frac{f_1^6}{f_5^6} + 16q^2 \right) + 3f_1^{18}f_5^4 \left(18q \right) \\ &\quad + 13f_1^{12}f_5^{10} \left(10q \frac{f_1^6}{f_5^6} + 10q^2 \right) + \sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23} \left(5^{23}n + \frac{7 \cdot 5^{23} - 11}{12} \right) q^n \equiv 21 \frac{f_1^{24}}{f_5^2} + 6q f_1^{18} f_5^4 + 6q^2 f_1^{12} f_5^{10} + \sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}.$$

Now by (4.18), we get

$$\sum_{n=0}^{\infty} b_{23} \left(5^{23}n + \frac{7 \cdot 5^{23} - 11}{12} \right) q^n \equiv 14 \sum_{n=0}^{\infty} b_{23}(n) q^{5n+4} \pmod{23}. \quad (4.20)$$

Comparing the coefficients of the terms of the form q^{5n+4} in (4.20), we can conclude that for any $n \geq 0$,

$$b_{23} \left(5^{24}n + \frac{11}{12} (5^{24} - 1) \right) \equiv 14b_{23}(n) \pmod{23}. \quad (4.21)$$

Now (1.5) follows from (4.21), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form q^{5n+r} where $r \in \{0, 1, 2, 3\}$ in (4.20), we can conclude that for any $n \geq 0$,

$$b_{23} \left(5^{23}(5n+r) + \frac{7 \cdot 5^{23} - 11}{12} \right) \equiv 0 \pmod{23},$$

or,

$$b_{23} \left(5^{24}n + \frac{1}{12} (5^{23}(12r+7) - 11) \right) \equiv 0 \pmod{23}. \quad (4.22)$$

Now (1.6) follows from (1.5) and (4.22).

5. Congruences for 65-Regular Partition

Proof of Theorem 1.3. Taking $l = 65$ in (1.1), we get

$$\sum_{n=0}^{\infty} b_{65}(n) q^n = \frac{f_{65}}{f_1}. \quad (5.23)$$

Replacing $1/f_1$ using (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65}(n) q^n &= f_{65} \cdot \frac{f_{25}^5}{f_5^6} \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 \right. \\ &\quad \left. - 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right). \end{aligned}$$

If we extract the terms involving q^{5n+4} , divide by q^4 , and substitute q by $q^{1/5}$, then by (2.10), we get

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n = 5f_{13} \cdot \frac{f_5^5}{f_1^6}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n \equiv 5f_1^7 f_5^5 \pmod{13}.$$

Replacing f_1 using (2.9), we get

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n \equiv 5f_5^5 f_{25}^7 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^7 \pmod{13}.$$

If we extract the terms involving q^{5n+2} , divide by q^2 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65}(5^2n+14)q^n \equiv 5f_1^5 f_5^7 \left(\frac{f_1^6}{f_5^6} + 8q \right) \equiv 5f_1^{11} f_5 + qf_1^5 f_5^7 \pmod{13}.$$

Replacing f_1 using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65}(5^2n+14)q^n &\equiv 5f_5 f_{25}^{11} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{11} \\ &\quad + qf_5^7 f_{25}^5 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^5 \pmod{13}. \end{aligned}$$

If we extract the terms involving q^{5n+1} , divide by q , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65}(5^3n+39)q^n \equiv 5f_1 f_5^{11} \left(2\frac{f_1^{12}}{f_5^{12}} + 11q\frac{f_1^6}{f_5^6} + 8q^2 \right) + f_1^7 f_5^5 \left(\frac{f_1^6}{f_5^6} \right) \pmod{13},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65}(5^3n+39)q^n \equiv 11\frac{f_1^{13}}{f_5} + 3qf_1^7 f_5^5 + q^2 f_1 f_5^{11} \pmod{13},$$

or,

$$\sum_{n=0}^{\infty} b_{65} \left(5^3n + \frac{5^3 - 8}{3} \right) q^n \equiv 11\frac{f_1^{13}}{f_5} + 3qf_1^7 f_5^5 + q^2 f_1 f_5^{11} \pmod{13}. \quad (5.24)$$

Replacing f_1 using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65} \left(5^3n + \frac{5^3 - 8}{3} \right) q^n &\equiv 11\frac{f_{25}^{13}}{f_5} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{13} + 3qf_5^5 f_{25}^7 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^7 \\ &\quad + q^2 f_5^{11} f_{25} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right) \pmod{13}. \end{aligned}$$

If we extract the terms involving q^{5n+3} , divide by q^3 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65} \left(5^4 n + \frac{2 \cdot 5^4 - 8}{3} \right) q^n \equiv 11 \frac{f_5^{13}}{f_1} \left(12q^2 \right) + 3f_1^5 f_5^7 \left(\frac{f_1^6}{f_5^6} + 8q \right) + f_1^{11} f_5 \left(12 \right) \pmod{13},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65} \left(5^4 n + \frac{2 \cdot 5^4 - 8}{3} \right) q^n \equiv 2f_1^{11} f_5 + 11q f_1^5 f_5^7 + 2q^2 \frac{f_5^{13}}{f_1} \pmod{13}.$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65} \left(5^4 n + \frac{2 \cdot 5^4 - 8}{3} \right) q^n & \equiv 2f_5 f_{25}^{11} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{11} + 11q f_5^7 f_{25}^5 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^5 \\ & + 2q^2 f_5^7 f_{25}^5 \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R(q^5)^2 \right. \\ & \left. - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right) \pmod{13}. \end{aligned}$$

If we extract the terms involving q^{5n+1} , divide by q , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65} \left(5^5 n + \frac{5^5 - 8}{3} \right) q^n & \equiv 2f_1 f_5^{11} \left(2 \frac{f_1^{12}}{f_5^{12}} + 11q \frac{f_1^6}{f_5^6} + 8q^2 \right) + 11f_1^7 f_5^5 \left(\frac{f_1^6}{f_5^6} \right) \\ & + 2f_1^7 f_5^5 \left(5q \right) \pmod{13}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65} \left(5^5 n + \frac{5^5 - 8}{3} \right) q^n \equiv 2 \frac{f_1^{13}}{f_5} + 6q f_1^7 f_5^5 + 2q^2 f_1 f_5^{11} \pmod{13}. \quad (5.25)$$

Repeating the same process as above to derive (5.25) from (5.24), it follows that

$$\sum_{n=0}^{\infty} b_{65} \left(5^{11} n + \frac{5^{11} - 8}{3} \right) q^n \equiv \frac{f_1^{13}}{f_5} \equiv \frac{f_{13}}{f_5} \pmod{13}. \quad (5.26)$$

In (2.9), if we substitute q by q^{13} , then we get

$$f_{13} = f_{325} \left(\frac{1}{R(q^{65})} - q^{13} - q^{26} R(q^{65}) \right).$$

Replacing f_{13} in (5.26), we get

$$\sum_{n=0}^{\infty} b_{65} \left(5^{11} n + \frac{5^{11} - 8}{3} \right) q^n \equiv \frac{f_{325}}{f_5} \left(\frac{1}{R(q^{65})} - q^{13} - q^{26} R(q^{65}) \right) \pmod{13}. \quad (5.27)$$

If we extract the terms involving q^{5n+3} , divide by q^3 , and substitute q by $q^{1/5}$, then we get

$$\sum_{n=0}^{\infty} b_{65} \left(5^{12}n + \frac{2 \cdot 5^{12} - 8}{3} \right) q^n \equiv 12q^2 \frac{f_{65}}{f_1} \pmod{13},$$

which, by (5.23), yields

$$\sum_{n=0}^{\infty} b_{65} \left(5^{12}n + \frac{2 \cdot 5^{12} - 8}{3} \right) q^n \equiv 12 \sum_{n=0}^{\infty} b_{65}(n) q^{n+2} \pmod{13}. \quad (5.28)$$

Comparing the coefficients of the terms of the form q^{n+2} in (5.28), we can conclude that for any $n \geq 0$,

$$b_{65} \left(5^{12}n + \frac{8}{3}(5^{12} - 1) \right) \equiv 12b_{65}(n) \pmod{13}. \quad (5.29)$$

Now (1.7) follows from (5.29), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form q^{5n+r} where $r \in \{2, 4\}$ in (5.27), we can conclude that for any $n \geq 0$,

$$b_{65} \left(5^{11}(5n+r) + \frac{5^{11} - 8}{3} \right) \equiv 0 \pmod{13},$$

or,

$$b_{65} \left(5^{12}n + \frac{1}{3}(5^{11}(3r+1) - 8) \right) \equiv 0 \pmod{13}. \quad (5.30)$$

Now (1.8) follows from (1.7) and (5.30).

References

- [AdDa18] C. Adiga and R. Dasappa, Congruences for 7 and 49-regular partitions modulo powers of 7, *Ramanujan J.* **46** (2018), 821–833. [MR-3826757](#)
- [AhLo01] S. Ahlgren and J. Lovejoy, The arithmetic of partitions into distinct parts, *Mathematika* **48** (2001), 203–211. [MR-1996371](#)
- [AHS10] G.E. Andrews, M.D. Hirschhorn, and J.A. Sellers, Arithmetic properties of partitions with even parts distinct, *Ramanujan J.* **23** (2010), 169–181. [MR-2739210](#)
- [CaWe14] R. Carlson and J.J. Webb, Infinite families of infinite families of congruences for k -regular partitions, *Ramanujan J.* **33** (2014), 14–23. [MR-3182537](#)
- [CuiGu13] S.-P. Cui and N.S.S. Gu, Arithmetic properties of l -regular partitions, *Adv. in Appl. Math.* **51** (2013), 507–523. [MR-3097009](#)
- [CuiGu15] S.-P. Cui and N. S. S. Gu, Congruences for 9-regular partitions modulo 3, *Ramanujan J.* **38** (2015), 503–512. [MR-3423010](#)
- [CuiGu18] S.-P. Cui and N.S.S. Gu, Congruences for 13-regular partitions, *Util. Math.* **107** (2018), 103–114. [MR-3793046](#)
- [DLY14] H. Dai, C. Liu, and H. Yan, On the distribution of odd values of 2^a -regular partition functions, *J. Number Theory* **143** (2014), 14–23. [MR-3227331](#)
- [DaPe09] B. Dandurand and D. Penniston, l -divisibility of l -regular partition functions, *Ramanujan J.* **19** (2009), 63–70. [MR-2501237](#)
- [FuPe12] D. Furcy and D. Penniston, Congruences for l -regular partition functions modulo 3, *Ramanujan J.* **27** (2012), 101–108. [MR-2886492](#)
- [Kei14] W.J. Keith, Congruences for 9-regular partitions modulo 3, *Ramanujan J.* **35** (2014), 157–164. [MR-3258606](#)

- [LiWa14] B. L. S. Lin and A. Y. Z. Wang, Generalisation of Keith's conjecture on 9-regular partitions and 3-cores, *Bull. Aust. Math. Soc.* **90** (2014), 204–212. [MR-3252001](#)
- [LoPe01] J. Lovejoy and D. Penniston, 3-regular partitions and a modular $K3$ surface, in *q-Series with Applications to Combinatorics, Number Theory, and Physics*, ed. B. C. Berndt and K. Ono, Contemp. Math., Vol. 291 (Amer. Math. Soc., Providence, RI, 2001), pp. 177–182.
- [Pen02] D. Penniston, The p^a -regular partition function modulo p^j , *J. Number Theory* **94** (2002), 320–325. [MR-1916276](#)
- [Pen08] D. Penniston, Arithmetic of l -regular partition functions, *Int. J. Number Theory* **4** (2008), 295–302. [MR-2404802](#)
- [Ram62] S. Ramanujan, in *Collected Papers*, ed. G. H. Hardy (New York, Chelsea, 1962).
- [Ran17] D. Ranganatha, Ramanujan-type congruences modulo powers of 5 and 7, *Indian J. Pure Appl. Math.* **48** (2017), 449–465. [MR-3694080](#)
- [Tan18] D. Tang, Congruences modulo powers of 5 for k -colored partitions, *J. Number Theory* **187** (2018), 198–214. [MR-3766907](#)
- [Wan17] L. Wang, Congruences for 5-regular partitions modulo powers of 5, *Ramanujan J.* **44** (2017), 343–358. [MR-3715418](#)
- [Wan18a] L. Wang, Arithmetic properties of 7-regular partitions, *Ramanujan J.* **47** (2018), 99–115. [MR-3857937](#)
- [Wan18b] L. Wang, Congruences modulo powers of 11 for some partition functions, *Proc. Amer. Math. Soc.* **146** (2018), 1515–1528. [MR-3754338](#)
- [Wat29a] G. N. Watson, Theorems Stated by Ramanujan (IX) : Two Continued Fractions, *J. London Math. Soc.* **4** (1929), 231–237. [MR-1575054](#)
- [Wat29b] G. N. Watson, Theorems Stated by Ramanujan (VII): Theorems on Continued Fractions, *J. London Math. Soc.* **4** (1929), 39–48. [MR-1574903](#)
- [Wat38] G. N. Watson, Ramanujans Vermutung über Zerfallungszahlen, *J. Reine Angew. Math.* **179** (1938), 97–128. [MR-1581588](#)
- [Web11] J. J. Webb, Arithmetic of the 13-regular partition function modulo 3, *Ramanujan J.* **25** (2011), 49–56. [MR-2787291](#)
- [Xia14] E. X. W. Xia, New infinite families of congruences modulo 8 for partitions with even parts distinct, *Electron. J. Combin.* **21** (2014), Paper 4.8, 10. [MR-3284057](#)
- [Xia15] E. X. W. Xia, Congruences for some l -regular partitions modulo l , *J. Number Theory* **152** (2015), 105–117. [MR-3319057](#)
- [XiYa14a] E. X. W. Xia and O. X. M. Yao, Parity results for 9-regular partitions, *Ramanujan J.* **34** (2014), 109–117. [MR-3210258](#)
- [XiYa14b] E. X. W. Xia and O. X. M. Yao, A proof of Keith's conjecture for 9-regular partitions modulo 3, *Int. J. Number Theory* **10** (2014), 669–674. [MR-3190002](#)
- [Yao14] O. X. M. Yao, New congruences modulo powers of 2 and 3 for 9-regular partitions, *J. Number Theory* **142** (2014), 89–101. [MR-3208395](#)
- [ZJY18] T. Y. Zhao, J. Jin, and O. X. M. Yao, Parity results for 11-, 13- and 17-regular partitions, *Colloq. Math.* **151** (2018), 97–109. [MR-3741528](#)

S. Abinash

Indian Institute of Science Education and Research Thiruvananthapuram,
Maruthamala P.O., Vithura, Thiruvananthapuram-695551, Kerala, India.
e-mail: sarmaabinash15@iisertvm.ac.in

T. Kathiravan

The Institute of Mathematical Sciences,
IV Cross Road, CIT Campus, Taramani, Chennai-600113, Tamil Nadu, India.
e-mail: kkathiravan98@gmail.com

K. Srilakshmi

Indian Institute of Science Education and Research Thiruvananthapuram,
Maruthamala P.O., Vithura, Thiruvananthapuram-695551, Kerala, India.
e-mail: srilakshmi@iisertvm.ac.in