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Some new congruences for l-regular partitions modulo 13, 17, and 23

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Dedicated to the memory of Alan Baker

Abstract. A partition of n is l-regular if none of its parts is divisible by l. Let $b_l(n)$ denote the number of l-regular partitions of n. In this paper, using theta function identities due to Ramanujan, we establish some new infinite families of congruences for $b_l(n)$ modulo l, where $l = 17, 23$, and for $b_{65}(n)$ modulo 13.

Keywords. congruences, l-regular partitions, theta function identities.

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1. Introduction

We first recall a few terminologies and notations. Let n be a positive integer. A partition of n is a non-increasing sequence of positive integers whose sum is n, and the members of the sequence are called parts. For a positive integer $l \geq 2$, a partition of n is said to be l-regular if none of its parts is divisible by l. Let $b_l(n)$ denote the number of l-regular partitions of n. By convention, we assume that $b_l(0) = 1$. The generating function for $b_l(n)$ is given by

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{f_l}{f_1}, \tag{1.1}$$

where $|q| < 1$ and for any positive integer k, f_k is defined by

$$f_k := \prod_{i=1}^{\infty} \left(1 - q^{ki}\right).$$

It can be easily verified that for any prime number l,

$$f_l \equiv f_1^l \pmod{l}. \tag{1.2}$$

In recent years, various congruences satisfied by l-regular partitions for different values of l has been established. For instance, Lovejoy and Penniston [LoPe01] established criteria for 3-divisibility of $b_3(n)$. Dandurand and Penniston [DaPe09] employed the theory of complex multiplication to give a precise description of those n such that $l|b_l(n)$ for $l \in \{5, 7, 11\}$. Cui and Gu [CuiGu13] derived infinite families of congruences modulo 2 for $b_l(n)$ where $l \in \{2, 4, 5, 8, 13, 16\}$. Webb [Web11] established an infinite family of congruences modulo 3 for $b_{13}(n)$. Furcy and Penniston [FuPe12] obtained families of congruences modulo 3 for other values of l which are congruent to 1 modulo 3. Xia and Yao [XiYa14a] found some infinite families of congruences for $b_9(n)$ modulo 2, and Cui and Gu [CuiGu15] derived congruences for $b_9(n)$ modulo 3. Xia [Xia15] established infinite families of congruences for $b_l(n)$ modulo l where $l \in \{13, 17, 19\}$, for example, for any $n \geq 0$, $k \geq 0$,

$$b_{13}\left(5^{12k}n + \frac{1}{2}\left(5^{12k} - 1\right)\right) \equiv b_{13}(n) \pmod{13}.$$

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For a complete literature review, see [Xia15]. For more related works, see [AdDa18, AhLo01, AHS10, CaWe14, CuiGu18, DLY14, Kei14, LiWa14, Pen02, Pen08, Ran17, Tan18, Wan17, Wan18a, Wan18b, Xia14, XiYa14b, Yao14, ZJY18].

In this paper, we establish infinite families of Ramanujan-type congruences for $b_l(n)$ modulo l, where $l \in \{17, 23\}$, and for $b_{65}(n)$ modulo 13. In fact, no such congruence has been established for 23-regular partitions till now, to the best of the authors' knowledge. The following are the main results.

Theorem 1.1. *For any $n \geq 0$, $k \geq 0$,*

$$b_{17}\left(5^{4k}n + \frac{2}{3}\left(5^{4k} - 1\right)\right) \equiv 2^k b_{17}(n) \pmod{17}, \tag{1.3}$$

and for $r \in \{0, 1, 2, 4\}$,

$$b_{17}\left(5^{4k+4}n + \frac{1}{3}\left(5^{4k+3}(3r+1) - 2\right)\right) \equiv 0 \pmod{17}. \tag{1.4}$$

Theorem 1.2. *For any $n \geq 0$, $k \geq 0$,*

$$b_{23}\left(5^{24k}n + \frac{11}{12}\left(5^{24k} - 1\right)\right) \equiv 14^k b_{23}(n) \pmod{23}, \tag{1.5}$$

and for $r \in \{0, 1, 2, 3\}$,

$$b_{23}\left(5^{24k+24}n + \frac{1}{12}\left(5^{24k+23}(12r+7) - 11\right)\right) \equiv 0 \pmod{23}. \tag{1.6}$$

Theorem 1.3. *For any $n \geq 0$, $k \geq 0$,*

$$b_{65}\left(5^{12k}n + \frac{8}{3}\left(5^{12k} - 1\right)\right) \equiv 12^k b_{65}(n) \pmod{13}, \tag{1.7}$$

and for $r \in \{2, 4\}$,

$$b_{65}\left(5^{12k+12}n + \frac{1}{3}\left(5^{12k+11}(3r+1) - 8\right)\right) \equiv 0 \pmod{13}. \tag{1.8}$$

2. Preliminaries

In this section we shall state a preliminary lemma.

Lemma 2.1. *For*

$$R(q) := \prod_{n=1}^{\infty} \frac{\left(1 - q^{5n-4}\right)\left(1 - q^{5n-1}\right)}{\left(1 - q^{5n-3}\right)\left(1 - q^{5n-2}\right)},$$

we have

A. ([Ram62], [Wat38])

$$f_1 = f_{25}\left(\frac{1}{R(q^5)} - q - q^2 R(q^5)\right). \tag{2.9}$$

B. ([Wat29a],[Wat29b])

$$\frac{1}{R(q)^5} - 11q - q^2R(q)^5 = \frac{f_1^6}{f_5^6}. \tag{2.10}$$

C.

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\right). \tag{2.11}$$

Note that the identity in (2.11) can be easily established by first substituting q by q^5 in (2.10) and then using (2.9).

3. Congruences for 17-Regular Partitions

Proof of Theorem 1.1. Taking $l = 17$ in (1.1), we get

$$\sum_{n=0}^{\infty} b_{17}(n)q^n = \frac{f_{17}}{f_1}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{17}(n)q^n \equiv f_1^{16} \pmod{17}. \tag{3.12}$$

Replacing f_1 using (2.9) and then extracting the terms involving q^{5n+1}, we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^{5n+1} \equiv f_{25}^{16}\left(\frac{q}{R(q^5)^{15}} + \frac{13q^6}{R(q^5)^{10}} + \frac{8q^{11}}{R(q^5)^5} - q^{16} - 8q^{21}R(q^5)^5 \right.$$
$$\left. + 13q^{26}R(q^5)^{10} - q^{31}R(q^5)^{15}\right) \pmod{17}.$$

If we divide by q and substitute q by $q^{1/5}$, then we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \equiv f_5^{16}\left(\left(\frac{1}{R(q)^5} - 11q - q^2R(q)^5\right)^3 + 12q\left(\frac{1}{R(q)^5} - 11q - q^2R(q)^5\right)^2 \right.$$
$$\left. + 14q^2\left(\frac{1}{R(q)^5} - 11q - q^2R(q)^5\right) + 7q^3\right) \pmod{17}.$$

By (2.10), we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \equiv f_5^{16}\left(\frac{f_1^{18}}{f_5^{18}} + 12q\frac{f_1^{12}}{f_5^{12}} + 14q^2\frac{f_1^6}{f_5^6} + 7q^3\right) \pmod{17}.$$

In view of (3.12), we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \equiv \frac{f_1^{18}}{f_5^2} + 12qf_1^{12}f_5^4 + 14q^2f_1^6f_5^{10} + 7\sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \tag{3.13}$$

Replacing f_1 using (2.9), we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \equiv \frac{f_{25}^{18}}{f_5^2}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{18} + 12qf_5^4f_{25}^{12}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{12}$$
$$+ 14q^2f_5^{10}f_{25}^6\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{6} + 7\sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}.$$

If we extract the terms involving q^{5n+3}, divide by q^3, and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n \equiv \frac{f_5^{18}}{f_1^2}\left(q^3\right) + 12f_1^4f_5^{12}\left(3\frac{f_1^{12}}{f_5^{12}} + 3q^2\right) + 14f_1^{10}f_5^6\left(11\frac{f_1^6}{f_5^6} + 9q\right)$$
$$+ 7\sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n \equiv 7qf_1^{10}f_5^6 + 2q^2f_1^4f_5^{12} + q^3\frac{f_5^{18}}{f_1^2} + 10\sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}.$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n \equiv 7qf_5^6f_{25}^{10}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{10} + 2q^2f_5^{12}f_{25}^4\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{4}$$
$$+ q^3f_5^6f_{25}^{10}\left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5)\right.$$
$$\left. + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\right)^2 + 10\sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}.$$

If we extract the terms involving q^{5n+1}, divide by q, and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n \equiv 7f_1^6f_5^{10}\left(\frac{f_1^{12}}{f_5^{12}} + 12q\frac{f_1^6}{f_5^6} + q^2\right) + 2f_1^{12}f_5^4\left(12q\right) + f_1^6f_5^{10}\left(10q\frac{f_1^6}{f_5^6} + 6q^2\right)$$
$$+ 10\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \pmod{17},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n \equiv 7\frac{f_1^{18}}{f_5^2} + 16qf_1^{12}f_5^4 + 13q^2f_1^6f_5^{10} + 10\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \pmod{17}.$$

Now by (3.13), we get

$$\sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n \equiv 2\sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \tag{3.14}$$

Comparing the coefficients of the terms of the form q^{5n+3} in (3.14), we can conclude that for any $n \geq 0$,

$$b_{17}(5^4n+416) \equiv 2b_{17}(n) \pmod{17},$$

or,

$$b_{17}\left(5^4n+\frac{2}{3}\left(5^4-1\right)\right)\equiv 2b_{17}(n) \pmod{17}. \tag{3.15}$$

Now (1.3) follows from (3.15), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form q^{5n+r} where $r\in\{0,1,2,4\}$ in (3.14), we can conclude that for any $n\geq 0$,

$$b_{17}\left(5^3(5n+r)+\frac{5^3-2}{3}\right)\equiv 0 \pmod{17},$$

or,

$$b_{17}\left(5^4n+\frac{1}{3}\left(5^3(3r+1)-2\right)\right)\equiv 0 \pmod{17}. \tag{3.16}$$

Now (1.4) follows from (1.3) and (3.16).

4. Congruences for 23-Regular Partitions

Proof of Theorem 1.2. Taking $l=23$ in (1.1), we get

$$\sum_{n=0}^{\infty} b_{23}(n)q^n=\frac{f_{23}}{f_1}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{23}(n)q^n\equiv f_1^{22} \pmod{23}. \tag{4.17}$$

Replacing f_1 using (2.9) and then extracting the terms involving q^{5n+2}, we get

$$\sum_{n=0}^{\infty} b_{23}(5n+2)q^{5n+2}\equiv f_{25}^{22}\left(\frac{2q^2}{R(q^5)^{20}}+\frac{21q^7}{R(q^5)^{15}}+\frac{3q^{12}}{R(q^5)^{10}}+\frac{8q^{17}}{R(q^5)^5}-q^{22}-8q^{27}R\left(q^5\right)^5\right.$$
$$\left.+3q^{32}R\left(q^5\right)^{10}-21q^{37}R\left(q^5\right)^{15}+2q^{42}R\left(q^5\right)^{20}\right) \pmod{23}.$$

If we divide by q^2 and substitute q by $q^{1/5}$, then we get

$$\sum_{n=0}^{\infty} b_{23}(5n+2)q^n\equiv f_5^{22}\left(2\left(\frac{1}{R(q)^5}-11q-q^2R(q)^5\right)^4+17q\left(\frac{1}{R(q)^5}-11q-q^2R(q)^5\right)^3\right.$$
$$\left.+17q^2\left(\frac{1}{R(q)^5}-11q-q^2R(q)^5\right)^2+14q^4\right) \pmod{23}.$$

By (2.10), we get

$$\sum_{n=0}^{\infty} b_{23}(5n+2)q^n\equiv f_5^{22}\left(2\frac{f_1^{24}}{f_5^{24}}+17q\frac{f_1^{18}}{f_5^{18}}+17q^2\frac{f_1^{12}}{f_5^{12}}+14q^4\right) \pmod{23}.$$

In view of (4.17), we get

$$\sum_{n=0}^{\infty} b_{23}(5n+2)q^n\equiv 2\frac{f_1^{24}}{f_5^2}+17qf_1^{18}f_5^4+17q^2f_1^{12}f_5^{10}+14\sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \tag{4.18}$$

Replacing f_1 using (2.9), we get

$$\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \equiv 2\frac{f_{25}^{24}}{f_5^2}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{24} + 17qf_5^4f_{25}^{18}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{18}$$
$$+ 17q^2f_5^{10}f_{25}^{12}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{12} + 14\sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}.$$

If we extract the terms involving q^{5n+4}, divide by q^4, and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n \equiv 2\frac{f_5^{24}}{f_1^2}\left(q^4\right) + 17f_1^4f_5^{18}\left(19\frac{f_1^{18}}{f_5^{18}} + 7q^3\right) + 17f_1^{10}f_5^{12}\left(8\frac{f_1^{12}}{f_5^{12}} + 3q^2\right)$$
$$+ 14\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n \equiv 5q^2f_1^{10}f_5^{12} + 4q^3f_1^4f_5^{18} + 2q^4\frac{f_5^{24}}{f_1^2} + 13\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \tag{4.19}$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n \equiv 5q^2f_5^{12}f_{25}^{10}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{10} + 4q^3f_5^{18}f_{25}^4\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^4$$
$$+ 2q^4f_5^{12}f_{25}^{10}\left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5)\right.$$
$$\left. + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\right)^2 + 13\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

If we extract the terms involving q^{5n+2}, divide by q^2, and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n \equiv 5f_1^{12}f_5^{10}\left(\frac{f_1^{12}}{f_5^{12}} + 20q\frac{f_1^6}{f_5^6} + 16q^2\right) + 4f_1^{18}f_5^4\left(18q\right) + 2f_1^{12}f_5^{10}\left(10q\frac{f_1^6}{f_5^6} + 10q^2\right)$$
$$+ 13\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n \equiv 5\frac{f_1^{24}}{f_5^2} + 8qf_1^{18}f_5^4 + 8q^2f_1^{12}f_5^{10} + 13\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}.$$

Now by (4.18), we get

$$\sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n \equiv 8\frac{f_1^{24}}{f_5^2} + 22qf_1^{18}f_5^4 + 22q^2f_1^{12}f_5^{10} + 21\sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}.$$

Replacing f_1 using (2.9), we get

$$\sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n \equiv 8\frac{f_{25}^{24}}{f_5^2}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{24} + 22qf_5^4f_{25}^{18}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{18}$$
$$+ 22q^2f_5^{10}f_{25}^{12}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^{12} + 21\sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}.$$

If we extract the terms involving q^{5n+4}, divide by q^4, and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{23}(5^4n+572)q^n \equiv 8\frac{f_5^{24}}{f_1^2}\left(q^4\right) + 22f_1^4f_5^{18}\left(19\frac{f_1^{18}}{f_5^{18}}+7q^3\right) + 22f_1^{10}f_5^{12}\left(8\frac{f_1^{12}}{f_5^{12}}+3q^2\right) + 21\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}\left(5^4n+572\right)q^n \equiv 20q^2f_1^{10}f_5^{12} + 16q^3f_1^4f_5^{18} + 8q^4\frac{f_5^{24}}{f_1^2} + 17\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Now by (4.19), we get

$$\sum_{n=0}^{\infty} b_{23}\left(5^4n+572\right)q^n \equiv 4\sum_{n=0}^{\infty} b_{23}\left(5^2n+22\right)q^n + 11\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23},$$

or,

$$\sum_{n=0}^{\infty} b_{23}\left(5^4n+\frac{11}{12}\left(5^4-1\right)\right)q^n \equiv 4\sum_{n=0}^{\infty} b_{23}\left(5^2n+22\right)q^n + 11\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Substituting n by 5^2n+22, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{23}\left(5^6n+\frac{11}{12}\left(5^6-1\right)\right)q^n
&\equiv 4\sum_{n=0}^{\infty} b_{23}\left(5^4n+\frac{11}{12}\left(5^4-1\right)\right)q^n + 11\sum_{n=0}^{\infty} b_{23}\left(5^2n+22\right)q^n \pmod{23},\\
&\equiv 4\left(4\sum_{n=0}^{\infty} b_{23}\left(5^2n+22\right)q^n + 11\sum_{n=0}^{\infty} b_{23}(n)q^n\right) + 11\sum_{n=0}^{\infty} b_{23}\left(5^2n+22\right)q^n \pmod{23},\\
&\equiv 4\sum_{n=0}^{\infty} b_{23}\left(5^2n+22\right)q^n + 21\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.
\end{aligned}$$

Repeating the same process as above, it follows that

$$\sum_{n=0}^{\infty} b_{23}\left(5^{22}n+\frac{11}{12}\left(5^{22}-1\right)\right)q^n \equiv 18\sum_{n=0}^{\infty} b_{23}\left(5^2n+22\right)q^n + 20\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Now by (4.19), we get

$$\sum_{n=0}^{\infty} b_{23}\left(5^{22}n+\frac{11}{12}\left(5^{22}-1\right)\right)q^n \equiv 21q^2f_1^{10}f_5^{12} + 3q^3f_1^4f_5^{18} + 13q^4\frac{f_5^{24}}{f_1^2} + \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{23}\left(5^{22}n+\frac{11}{12}\left(5^{22}-1\right)\right)q^n
&\equiv 21q^2f_5^{12}f_{25}^{10}\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right)^{10} + 3q^3f_5^{18}f_{25}^4\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right)^4\\
&\quad + 13q^4f_5^{12}f_{25}^{10}\left(\frac{1}{R(q^5)^4}+\frac{q}{R(q^5)^3}+\frac{2q^2}{R(q^5)^2}+\frac{3q^3}{R(q^5)}+5q^4-3q^5R(q^5)+2q^6R(q^5)^2\right.\\
&\quad \left. -q^7R(q^5)^3+q^8R(q^5)^4\right)^2 + \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.
\end{aligned}$$

If we extract the terms involving q^{5n+2}, divide by q^2, and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{23}\left(5^{23}n+\frac{7\cdot 5^{23}-11}{12}\right)q^n \equiv 21f_1^{12}f_5^{10}\left(\frac{f_1^{12}}{f_5^{12}}+20q\frac{f_1^6}{f_5^6}+16q^2\right)+3f_1^{18}f_5^4\left(18q\right) + 13f_1^{12}f_5^{10}\left(10q\frac{f_1^6}{f_5^6}+10q^2\right)+\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}\left(5^{23}n+\frac{7\cdot 5^{23}-11}{12}\right)q^n \equiv 21\frac{f_1^{24}}{f_5^2}+6qf_1^{18}f_5^4+6q^2f_1^{12}f_5^{10}+\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}.$$

Now by (4.18), we get

$$\sum_{n=0}^{\infty} b_{23}\left(5^{23}n+\frac{7\cdot 5^{23}-11}{12}\right)q^n \equiv 14\sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \tag{4.20}$$

Comparing the coefficients of the terms of the form q^{5n+4} in (4.20), we can conclude that for any $n \geq 0$,

$$b_{23}\left(5^{24}n+\frac{11}{12}\left(5^{24}-1\right)\right) \equiv 14b_{23}(n) \pmod{23}. \tag{4.21}$$

Now (1.5) follows from (4.21), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form q^{5n+r} where $r \in \{0,1,2,3\}$ in (4.20), we can conclude that for any $n \geq 0$,

$$b_{23}\left(5^{23}(5n+r)+\frac{7\cdot 5^{23}-11}{12}\right) \equiv 0 \pmod{23},$$

or,

$$b_{23}\left(5^{24}n+\frac{1}{12}\left(5^{23}(12r+7)-11\right)\right) \equiv 0 \pmod{23}. \tag{4.22}$$

Now (1.6) follows from (1.5) and (4.22).

5. Congruences for 65-Regular Partition

Proof of Theorem 1.3. Taking $l = 65$ in (1.1), we get

$$\sum_{n=0}^{\infty} b_{65}(n)q^n = \frac{f_{65}}{f_1}. \tag{5.23}$$

Replacing $1/f_1$ using (2.11), we get

$$\sum_{n=0}^{\infty} b_{65}(n)q^n = f_{65}\cdot\frac{f_{25}^5}{f_5^6}\left(\frac{1}{R(q^5)^4}+\frac{q}{R(q^5)^3}+\frac{2q^2}{R(q^5)^2}+\frac{3q^3}{R(q^5)}+5q^4 - 3q^5R(q^5)+2q^6R(q^5)^2-q^7R(q^5)^3+q^8R(q^5)^4\right).$$

If we extract the terms involving q^{5n+4}, divide by q^4, and substitute q by $q^{1/5}$, then by (2.10), we get

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n = 5f_{13}\cdot\frac{f_5^5}{f_1^6}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n \equiv 5f_1^7 f_5^5 \pmod{13}.$$

Replacing f_1 using (2.9), we get

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n \equiv 5f_5^5 f_{25}^7\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right)^7 \pmod{13}.$$

If we extract the terms involving q^{5n+2}, divide by q^2, and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65}(5^2n+14)q^n \equiv 5f_1^5 f_5^7\left(\frac{f_1^6}{f_5^6}+8q\right) \equiv 5f_1^{11}f_5 + qf_1^5 f_5^7 \pmod{13}.$$

Replacing f_1 using (2.9), we get

$$\sum_{n=0}^{\infty} b_{65}(5^2n+14)q^n \equiv 5f_5 f_{25}^{11}\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right)^{11} + qf_5^7 f_{25}^5\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right)^5 \pmod{13}.$$

If we extract the terms involving q^{5n+1}, divide by q, and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65}(5^3n+39)q^n \equiv 5f_1 f_5^{11}\left(2\frac{f_1^{12}}{f_5^{12}}+11q\frac{f_1^6}{f_5^6}+8q^2\right) + f_1^7 f_5^5\left(\frac{f_1^6}{f_5^6}\right) \pmod{13},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65}(5^3n+39)q^n \equiv 11\frac{f_1^{13}}{f_5} + 3qf_1^7 f_5^5 + q^2 f_1 f_5^{11} \pmod{13},$$

or,

$$\sum_{n=0}^{\infty} b_{65}\left(5^3n+\frac{5^3-8}{3}\right)q^n \equiv 11\frac{f_1^{13}}{f_5} + 3qf_1^7 f_5^5 + q^2 f_1 f_5^{11} \pmod{13}. \tag{5.24}$$

Replacing f_1 using (2.9), we get

$$\sum_{n=0}^{\infty} b_{65}\left(5^3n+\frac{5^3-8}{3}\right)q^n \equiv 11\frac{f_{25}^{13}}{f_5}\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right)^{13} + 3qf_5^5 f_{25}^7\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right)^7 + q^2 f_5^{11} f_{25}\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right) \pmod{13}.$$

If we extract the terms involving q^{5n+3}, divide by q^3, and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65}\left(5^4 n+\frac{2\cdot 5^4-8}{3}\right)q^n \equiv 11\frac{f_5^{13}}{f_1}\left(12q^2\right)+3f_1^5f_5^7\left(\frac{f_1^6}{f_5^6}+8q\right)+f_1^{11}f_5\left(12\right) \pmod{13},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65}\left(5^4 n+\frac{2\cdot 5^4-8}{3}\right)q^n \equiv 2f_1^{11}f_5+11qf_1^5f_5^7+2q^2\frac{f_5^{13}}{f_1} \pmod{13}.$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_{65}&\left(5^4 n+\frac{2\cdot 5^4-8}{3}\right)q^n \\ &\equiv 2f_5f_{25}^{11}\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right)^{11}+11qf_5^7f_{25}^5\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right)^5 \\ &\quad+2q^2f_5^7f_{25}^5\left(\frac{1}{R(q^5)^4}+\frac{q}{R(q^5)^3}+\frac{2q^2}{R(q^5)^2}+\frac{3q^3}{R(q^5)}+5q^4-3q^5R(q^5)+2q^6R(q^5)^2\right. \\ &\quad\left.-q^7R(q^5)^3+q^8R(q^5)^4\right) \pmod{13}.\end{aligned}$$

If we extract the terms involving q^{5n+1}, divide by q, and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_{65}\left(5^5 n+\frac{5^5-8}{3}\right)q^n &\equiv 2f_1f_5^{11}\left(2\frac{f_1^{12}}{f_5^{12}}+11q\frac{f_1^6}{f_5^6}+8q^2\right)+11f_1^7f_5^5\left(\frac{f_1^6}{f_5^6}\right) \\ &\quad+2f_1^7f_5^5\left(5q\right) \pmod{13},\end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65}\left(5^5 n+\frac{5^5-8}{3}\right)q^n \equiv 2\frac{f_1^{13}}{f_5}+6qf_1^7f_5^5+2q^2f_1f_5^{11} \pmod{13}. \tag{5.25}$$

Repeating the same process as above to derive (5.25) from (5.24), it follows that

$$\sum_{n=0}^{\infty} b_{65}\left(5^{11} n+\frac{5^{11}-8}{3}\right)q^n \equiv \frac{f_1^{13}}{f_5}\equiv\frac{f_{13}}{f_5} \pmod{13}. \tag{5.26}$$

In (2.9), if we substitute q by q^{13}, then we get

$$f_{13}=f_{325}\left(\frac{1}{R(q^{65})}-q^{13}-q^{26}R(q^{65})\right).$$

Replacing f_{13} in (5.26), we get

$$\sum_{n=0}^{\infty} b_{65}\left(5^{11} n+\frac{5^{11}-8}{3}\right)q^n \equiv \frac{f_{325}}{f_5}\left(\frac{1}{R(q^{65})}-q^{13}-q^{26}R(q^{65})\right) \pmod{13}. \tag{5.27}$$

If we extract the terms involving q^{5n+3}, divide by q^3, and substitute q by $q^{1/5}$, then we get

$$\sum_{n=0}^{\infty} b_{65}\left(5^{12}n + \frac{2 \cdot 5^{12} - 8}{3}\right) q^n \equiv 12q^2 \frac{f_{65}}{f_1} \pmod{13},$$

which, by (5.23), yields

$$\sum_{n=0}^{\infty} b_{65}\left(5^{12}n + \frac{2 \cdot 5^{12} - 8}{3}\right) q^n \equiv 12 \sum_{n=0}^{\infty} b_{65}(n) q^{n+2} \pmod{13}. \tag{5.28}$$

Comparing the coefficients of the terms of the form q^{n+2} in (5.28), we can conclude that for any $n \geq 0$,

$$b_{65}\left(5^{12}n + \frac{8}{3}\left(5^{12} - 1\right)\right) \equiv 12 b_{65}(n) \pmod{13}. \tag{5.29}$$

Now (1.7) follows from (5.29), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form q^{5n+r} where $r \in \{2, 4\}$ in (5.27), we can conclude that for any $n \geq 0$,

$$b_{65}\left(5^{11}(5n + r) + \frac{5^{11} - 8}{3}\right) \equiv 0 \pmod{13},$$

or,

$$b_{65}\left(5^{12}n + \frac{1}{3}\left(5^{11}(3r + 1) - 8\right)\right) \equiv 0 \pmod{13}. \tag{5.30}$$

Now (1.8) follows from (1.7) and (5.30).

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