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Distribution of generalized mex-related integer partitions

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Abstract. The minimal excludant or "*mex*" function for an integer partition π of a positive integer n, $mex(\pi)$, is the smallest positive integer that is not a part of π. Andrews and Newman introduced $\sigma mex(n)$ to be the sum of $mex(\pi)$ taken over all partitions π of n. Ballantine and Merca generalized this combinatorial interpretation to $\sigma_r mex(n)$, as the sum of least r-gaps in all partitions of n. In this article, we study the arithmetic density of $\sigma_2 mex(n)$ and $\sigma_3 mex(n)$ modulo 2^k for any positive integer k.

Keywords. Minimal excludant, Integer partition; Eta-quotients; Modular forms; Distribution.

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1. Introduction and statement of results

In 2015, Fraenkel and Peled [FrPe15] defined the minimal excludant or "*mex*" function on a set S of positive integers as the least positive integer not in S. Perhaps the notion of the *mex* function was introduced in the 1930s, and best known for the applications in combinatorics and game theory [Gr39, Sp35].

A partition of a non-negative integer n is a non-increasing sequence of positive integers whose sum is n. Let π be a partition of n and $\mathcal{P}(n)$ be the set of all partitions of n. Recently, Andrews and Newman [AnNe19] considered the minimal excludant function applied to integer partitions. The minimal excludant of π, denoted $mex(\pi)$, is the smallest positive integer which is not a part of π. Thus if π is $6+4+3+2+1$, a partition of 16, then $mex(\pi) = 5$. For each positive integer n, we have

$$\sigma mex(n) := \sum_{\pi \in \mathcal{P}(n)} mex(\pi).$$

For example, $\sigma mex(4) = 9$ with the relevant *mex* partitions being: $mex(4) = 1$, $mex(3+1) = 2$, $mex(2+2) = 1$, $mex(2+1+1) = 3$, and $mex(1+1+1+1) = 2$. The generating function for $\sigma mex(n)$ is given by

$$\sum_{n=0}^{\infty} \sigma mex(n) q^n = (-q;q)_\infty^2,$$

where the q-shifted factorial $(a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1}), \quad |q|<1$. Let $\mathcal{D}_2(n)$ be the set of partitions of n into distinct parts using two colors and let $D_2(n) = |\mathcal{D}_2(n)|$. In [AnNe19], the authors give two proofs of the following theorem.

Theorem 1.1. *Given an integer $n > 0$, we have*

$$\sigma mex(n) = D_2(n).$$

They also studied the parity of σmex function and proved that $\sigma mex(n)$ is almost always even and is odd exactly when n is of the form $j(3j \pm 1)$, where j is a non-negative integer.

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In some literature, the minimal excludant of a partition π is referred to as the least gap or smallest gap of π. The r-gap of a partition π is the least positive integer that does not appear at least r times as a part of π. Let $s_r(\pi)$ be the smallest part of the partition π appearing less than r times and $S_r(n) = \sum_{\pi\in\mathcal{P}(n)} s_r(\pi)$.
For example, all the 3-gaps in the partitions of 5 are: $s_3(5) = 1$, $s_3(4+1) = 1$, $s_3(3+2) = 1$, $s_3(3+1+1) = 1$, $s_3(2+2+1) = 1$, $s_3(2+1+1+1) = 2$, $s_3(1+1+1+1+1) = 2$. Therefore $S_3(5) = \sum_{\pi\in\mathcal{P}(5)} s_3(\pi) = 9$.
Ballantine and Merca [Ba, Ba20] generalized Theorem 1.1 to the sum $S_r(n)$ of r-gaps in all partitions of n. To keep notation uniform, in this article we use $\sigma_r mex(n)$ for $S_r(n)$. The generating function for $\sigma_r mex(n)$ is given by

$$\sum_{n=0}^{\infty} \sigma_r mex(n) q^n = \frac{(q^{2r};q^{2r})_\infty}{(q;q)_\infty (q^r;q^{2r})_\infty}. \tag{1.1}$$

In [Ba20], Ballantine and Merca proved the following identity.

Theorem 1.2. *For $n \geq 0$ and $r \geq 1$ we have*

$$\sum_{k=0}^{\infty} p(n - rT_k) = \sigma_r mex(n),$$

where $p(n)$ counts the number of partition of n and T_k is the k-th triangular number.

Recently, Ray and Barman [RaBa20, Theorem 1.6], studied the divisibility of Uncu's partition function $\mathcal{EO}_u(n)$. After some elementary calculations we observe that the generating function of $\mathcal{EO}_u(2n)$ and $\sigma mex(n)$ (or $\sigma_1 mex(n)$) are the same. So using the result mentioned above, $\sigma mex(n)$ is almost always divisible by 2^k for any positive integer k.

A well-known conjecture of Parkin and Shanks [PS67], for integer partitions $p(n)$, states that the even and odd values of $p(n)$ are equally distributed, that is,

$$\lim_{X\to\infty} \frac{\#\{0 \leq n \leq X : p(n) \equiv a \pmod 2\}}{X} = \frac{1}{2},$$

where $a \in \{0, 1\}$. Little is known regarding this conjecture. In the following theorem we prove that $\sigma_2 mex(n)$ and $\sigma_3 mex(n)$ are almost always even. More generally we prove the following result.

Theorem 1.3. (Main Theorem) *Let k be a positive integer and $r \in \{2, 3\}$. Then*

$$\lim_{X\to+\infty} \frac{\#\left\{0 \leq n < X : \sigma_r mex(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

In other words for almost every non-negative integer n lying in an arithmetic progression, the integer $\sigma_r mex(n)$ is a multiple of 2^k where $r \in \{2, 3\}$.

2. Preliminaries

In this section, we recall some definitions and facts relating to the arithmetic of classical modular forms. For more details, one can consult [On04, Ko93]. Let $\mathbb{H}$ denote the upper-half plane. The complex vector space of modular forms of weight ℓ (a positive integer) with respect to a congruence subgroup Γ will be denoted by $M_\ell(\Gamma)$.

Definition 2.1. *[On04, Definition 1.15] Let χ be a Dirichlet character modulo N (a positive integer). Then a modular form $f \in M_\ell(\Gamma_1(N))$ has Nebentypus character χ if*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_\ell(\Gamma_0(N), \chi)$. Here $\Gamma_0(N)$ will be as usual the Hecke congruence subgroup of level N.

Recall that Dedekind's eta-function is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q = e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and r_δ is an integer.

We now recall two theorems from [On04, p. 18], which help us check the modularity of eta-quotients that show up in our study.

Theorem 2.2. *[On04, Theorem 1.64] Suppose that $f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that*

$$\ell = \frac{1}{2}\sum_{\delta \mid N} r_\delta \in \mathbb{Z},$$
$$\sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Then

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. Here

$$\chi(d) := \left(\frac{(-1)^\ell \prod_{\delta \mid N} \delta^{r_\delta}}{d}\right).$$

Suppose that f is an eta-quotient satisfying the conditions of Theorem 2.2. If f is also holomorphic at all of the cusps of $\Gamma_0(N)$, then $f \in M_\ell(\Gamma_0(N), \chi)$. To check the holomorphicity at cusps of $f(z)$ it suffices to check that the orders at the cusps are non-negative. The necessary criterion for determining orders of an eta-quotient at cusps is the following.

Theorem 2.3. *[On04, Theorem 1.65] Let c, d, and N be positive integers with $d \mid N$ and $\gcd(c,d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions of Theorem 2.2 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24}\sum_{\delta \mid N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta}.$$

3. Proof of main results

In this section, we prove Theorem 1.3. We prove the following lemmas.

Lemma 3.1. *Let k be a positive integer and $r \in \{2,3\}$. Then*

$$\frac{\eta(24rz)^{2^k-1}}{\eta(24z)\eta(48rz)^{2^{k-1}-2}} \equiv \sum_{n=0}^{\infty} \sigma_r mex(n) q^{24n+3r-1} \pmod{2^k}.$$

Proof. From (1.1), the generating function of $\sigma_r mex(n)$ is given by

$$\sum_{n=0}^{\infty} \sigma_r mex(n) q^n = \frac{(q^{2r};q^{2r})_\infty^2}{(q;q)_\infty (q^r;q^r)_\infty}. \tag{3.2}$$

Consider

$$\mathcal{A}(z) = \prod_{n=1}^{\infty} \frac{(1-q^{24rn})^2}{(1-q^{48rn})} = \frac{\eta(24rz)^2}{\eta(48rz)}.$$

By the binomial theorem, for any positive integers r and k we have

$$(q^r;q^r)_\infty^{2^k} \equiv (q^{2r};q^{2r})_\infty^{2^{k-1}} \pmod{2^k}.$$

Therefore,

$$\mathcal{A}^{2^{k-1}}(z) = \frac{\eta(24rz)^{2^k}}{\eta(48rz)^{2^{k-1}}} \equiv 1 \pmod{2^k}.$$

Define $\mathcal{B}_{r,k}(z)$ by

$$\mathcal{B}_{r,k}(z) = \frac{\eta(48rz)^2}{\eta(24z)\eta(24rz)} \mathcal{A}^{2^{k-1}}(z).$$

Now, modulo 2^k, we have

$$\begin{aligned} \mathcal{B}_{r,k}(z) &= \frac{\eta(48rz)^2}{\eta(24z)\eta(24rz)} \frac{\eta(24rz)^{2^k}}{\eta(48rz)^{2^{k-1}}} \\ &\equiv \frac{\eta(48rz)^2}{\eta(24z)\eta(24rz)} \\ &= q^{3r-1} \frac{(q^{48r};q^{48r})_\infty^2}{(q^{24};q^{24})_\infty (q^{24r};q^{24r})_\infty}. \end{aligned} \tag{3.3}$$

Since

$$\mathcal{B}_{r,k}(z) = \frac{\eta(24rz)^{2^k-1}}{\eta(24z)\eta(48rz)^{2^{k-1}-2}},$$

combining (3.2) and (3.3), we obtain the required result. □

Lemma 3.2. *Let $k > 1$ be a positive integer and $r \in \{2,3\}$. Then*

$$\mathcal{B}_{r,k}(z) = \frac{\eta(24rz)^{2^k-1}}{\eta(24z)\eta(48rz)^{2^{k-1}-2}} \in M_{2^k-2}\left(\Gamma_0(L), \chi(\bullet)\right),$$

where

$$L = \begin{cases} 1152 & \text{if } r = 2, \\ 576 & \text{if } r = 3. \end{cases}$$

Proof. First, we use Theorem 2.2 and find the following:

1. The weight of the *eta*-quotient $\mathcal{B}_{r,k}(z)$ is 2^{k-2}.

2. Suppose the level of the *eta*-quotient $\mathcal{B}_{r,k}(z)$ is $48ru$, where u is the smallest positive integer satisfying the following identity.
$$\frac{48ru}{24r}(2^k-1)-\frac{48ru}{24}-\frac{48ru}{48r}(2^{k-1}-2)\equiv 0 \pmod{24}$$
Equivalently, we have
$$u\left(3\cdot 2^{k-1}-2r\right)\equiv 0 \pmod{24}.$$
Since $k>1$, we have $u=12$ if $r=2$ and $u=4$ if $r=3$. Hence level of the *eta*-quotient $\mathcal{B}_{r,k}(z)$ is
$$\begin{cases} 1152 & \text{if } r=2, \\ 576 & \text{if } r=3. \end{cases}$$

3. The Nebentypus character is
$$\chi(\bullet)=\left(\frac{(-1)^{2^{k-2}}(24r)^{2^k-1}(24)^{-1}(48r)^{-2^{k-1}+2}}{\bullet}\right).$$

By Theorem 2.3, the cusps of $\Gamma_0(L)$ are given by $\frac{c}{d}$ where $d \mid L$ and $\gcd(c,d)=1$. Now note that *eta*-quotient $\mathcal{B}_{r,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if
$$(2^k-1)\,\frac{\gcd(d,24r)^2}{24r}-\frac{\gcd(d,24)^2}{24}-(2^{k-1}-2)\frac{\gcd(d,48r)^2}{48r}\geq 0.$$
Equivalently, if and only if
$$2(2^k-1)\,\frac{\gcd(d,24r)^2}{\gcd(d,48r)^2}-2r\,\frac{\gcd(d,24)^2}{\gcd(d,48r)^2}-(2^{k-1}-2)\geq 0. \tag{3.4}$$

Case (i). When $r=2$ then the left side of (3.4) can be written as
$$2(2^k-1)\,\frac{\gcd(d,48)^2}{\gcd(d,96)^2}-4\,\frac{\gcd(d,24)^2}{\gcd(d,96)^2}-(2^{k-1}-2)\geq 0. \tag{3.5}$$

To check the positivity of (3.5), we have to find all the possible divisors of 1152. We define three sets as follows
$$\mathcal{H}_1=\{2^\alpha 3^\beta : 0\leq\alpha\leq 3, 0\leq\beta\leq 2\},$$
$$\mathcal{H}_2=\{2^\alpha 3^\beta : \alpha=4, 0\leq\beta\leq 2\},$$
$$\mathcal{H}_3=\{2^\alpha 3^\beta : 5\leq\alpha\leq 7, 0\leq\beta\leq 2\}.$$

Note that $\mathcal{H}_1\cup\mathcal{H}_2\cup\mathcal{H}_3$ contains all positive divisors of 1152. In the following table we compute all necessary data to prove the positivity of (3.5).

Values of d such that $d\|1152$	$\frac{\gcd(d,48)^2}{\gcd(d,96)^2}$	$\frac{\gcd(d,24)^2}{\gcd(d,96)^2}$	Values of (3.5)
$d \in \mathcal{H}_1$	1	1	$2^{k-1}3-4$
$d \in \mathcal{H}_2$	1	1/4	$2^{k-1}3-1$
$d \in \mathcal{H}_3$	1/4	1/16	5/4

Since $k>1$, it is clear from the above table that the quantities in (3.5) are always greater than equal to 0 for any positive integer $d \mid 1152$.

Case (ii). When $r=3$ then the left side of (3.4) can be written as

$$2(2^k-1)\,\frac{\gcd(d,72)^2}{\gcd(d,144)^2}-6\,\frac{\gcd(d,24)^2}{\gcd(d,144)^2}-(2^{k-1}-2)\geq 0. \tag{3.6}$$

To check the positivity of (3.6), we have to find all the possible divisors of 576. We define four sets as follows

$$\mathcal{G}_1=\{2^\alpha 3^\beta : 0\leq\alpha\leq 3, 0\leq\beta\leq 1\}, \qquad \mathcal{G}_2=\{2^\alpha 3^\beta : 0\leq\alpha\leq 3, \beta=2\},$$
$$\mathcal{G}_3=\{2^\alpha 3^\beta : 4\leq\alpha\leq 6, 0\leq\beta\leq 1\}, \qquad \mathcal{G}_4=\{2^\alpha 3^\beta : 4\leq\alpha\leq 6, \beta=2\}.$$

Note that $\mathcal{G}_1\cup\mathcal{G}_2\cup\mathcal{G}_3\cup\mathcal{G}_4$ contains all positive divisors of 576. In the following table we compute all necessary data to prove the positivity of (3.6).

Values of d such that $d\|576$	$\frac{\gcd(d,72)^2}{\gcd(d,144)^2}$	$\frac{\gcd(d,24)^2}{\gcd(d,144)^2}$	Values of (3.6)
$d \in \mathcal{G}_1$	1	1	$2^{k-1}3-6$
$d \in \mathcal{G}_2$	1	1/9	$2^{k-1}3-2/3$
$d \in \mathcal{G}_3$	1/4	1/4	0
$d \in \mathcal{G}_4$	1/4	1/36	4/3

Since $k>1$, it is clear from the above table that the quantities in (3.6) are always greater than equal to 0 for any positive integer $d \mid 576$.

Therefore, by **Case (i)** and **Case (ii)**, the *eta*-quotient $\mathcal{B}_{r,k}(z)$, where $r\in\{2,3\}$ and $k>1$, are holomorphic at every cusp $\frac{c}{d}$ and hence it is a modular form on $\Gamma_0(L)$ with Nebentypus character $\chi(\bullet)$. This completes the proof of Lemma 3.2. □

We state the following result of Serre.

Theorem 3.3. *[On04, Theorem 2.65] Let k,m be positive integers. If $f(z)\in M_k(\Gamma_0(N),\chi(\bullet))$ has the Fourier expansion $f(z)=\sum_{n=0}^{\infty}c(n)q^n\in\mathbb{Z}[[q]]$, then there is a constant $\alpha>0$ such that*

$$\#\{n\leq X : c(n)\not\equiv 0 \pmod{m}\}=\mathcal{O}\left(\frac{X}{\log^\alpha X}\right).$$

Proof of Theorem 1.3. Suppose $k>1$ is a positive integer and $r\in\{2,3\}$. From Lemma 3.2, we have

$$\mathcal{B}_{r,k}(z)=\frac{\eta(24rz)^{2^k-1}}{\eta(24z)\eta(48rz)^{2^{k-1}-2}}\in M_{2^{k-2}}\left(\Gamma_0(L),\chi(\bullet)\right).$$

Also the Fourier coefficients of the *eta*-quotient $\mathcal{B}_{r,k}(z)$ are integers. So, by Theorem 3.3 and Lemma 3.1, we can find a constant $\alpha > 0$ such that

$$\#\left\{n \leq X : \sigma_r mex(n) \not\equiv 0 \pmod{2^k}\right\} = \mathcal{O}\left(\frac{X}{\log^\alpha X}\right).$$

Hence

$$\lim_{X \to +\infty} \frac{\#\left\{n \leq X : \sigma_r mex(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

This completes the proof of Theorem 1.3. □

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