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Bounds for d-distinct partitions

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In memory of the 100th anniversary of Ramanujan's eternal rest

Abstract. Euler's identity and the Rogers-Ramanujan identities are perhaps the most famous results in the theory of partitions. According to them, 1-distinct and 2-distinct partitions of n are equinumerous with partitions of n into parts congruent to ± 1 modulo 4 and partitions of n into parts congruent to ± 1 modulo 5, respectively. Furthermore, their generating functions are modular functions up to multiplication by rational powers of q. For $d \geq 3$, however, there is neither the same type of partition identity nor modularity for d-distinct partitions. Instead, there are partition inequalities and mock modularity related with d-distinct partitions. For example, the Alder-Andrews Theorem states that the number of d-distinct partitions of n is greater than or equal to the number of partitions of n into parts which are congruent to $\pm 1 \pmod{d+3}$. In this note, we present the recent developments of generalizations and analogs of the Alder-Andrews Theorem and establish asymptotic lower and upper bounds for the d-distinct partitions. Using the asymptotic relations and data obtained from computation, we propose a conjecture on a partition inequality that gives an upper bound for d-distinct partitions. Specifically, for $d \geq 4$, the number of d-distinct partitions of n is less than or equal to the number of partitions of n into parts congruent to $\pm 1 \pmod{m}$, where $m \leq \lfloor \frac{2d\pi^2}{3\log^2(d)+6\log d} \rfloor$.

Keywords. partitions; d-distinct partitions; partition identities; partition inequalities; asymptotic formulas

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1. Introduction

A partition of a positive integer n is a non-increasing finite sequence of positive integers whose sum is n. Let $p(n|\text{condition})$ be the partition function counting the number of partitions of n satisfying a certain *condition.* For positive integers a and d, the number of partitions into d-distinct parts with the smallest part at least a denoted by $q_d^{(a)}(n)$,

$$q_d^{(a)}(n) := p(n|\text{parts} \geq a \text{ and parts differ by at least } d), \tag{1.1}$$

has long been of interest in the theory of partitions. The generating function for $q_d^{(a)}(n)$ is given by

$$\sum_{n=0}^{\infty} q_d^{(a)}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{d\binom{n}{2}+na}}{(q;q)_n}, \tag{1.2}$$

where $(x;q)_m := (x;q)_\infty/(xq^m;q)_\infty$ with $(x;q)_\infty := \prod_{j=0}^{\infty}(1-xq^j)$ for any $m \in \mathbb{Z}$.

One of the oldest and the most famous results in partition theory is Euler's identity, which states that the number of 1-distinct partitions of n equals the number of partitions of n into odd parts. Equivalently,

$$\sum_{n=0}^{\infty} q_1^{(1)}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} = \frac{1}{(q;q^2)_\infty}. \tag{1.3}$$

This is easily proved by using another generating function representation of $q_1^{(1)}(n)$, which is $(-q;q)_\infty$.

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Two other famous partition identities due to Rogers and Ramanujan involve 2-distinct partitions. The celebrated Rogers-Ramanujan identities are

$$\sum_{n=0}^{\infty} q_2^{(1)}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.4}$$

$$\sum_{n=0}^{\infty} q_2^{(2)}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.5}$$

Combinatorially, they imply that the number of partitions of n into parts differing by at least 2 with parts $\geq c$ equals the number of partitions of n into parts that are congruent to $\pm c \pmod 5$. Here $c = 1$ for the first and $c = 2$ for the second Rogers-Ramanujan identity.

Another important result on 1-distinct and 2-distinct partitions is that their generating functions are modular functions up to multiplication by rational powers of q when $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$, the upper-half of the complex plane. These are easy consequences of Euler's identity and the Rogers-Ramanujan identities. The infinite product on the far right in (1.3) is $q^{-1/24}\eta(2\tau)/\eta(\tau)$, where $\eta(\tau) = q^{1/24}(q;q)_\infty$ is the Dedekind eta function, a modular form of weight $1/2$. The infinite products in (1.4) and (1.5) are modular functions as well when multiplied by $q^{-1/60}$ and $q^{11/60}$, respectively.

However, for any $d \geq 3$, this type of sum-product identity no longer exists for d-distinct partitions. It means if $d \geq 3$, the number of d-distinct partitions of n is not equal to the number of partitions of n with parts from fixed residue classes and its generating function is no more a modular function. In fact, D. H. Lehmer [Le46] proved that for any $d \geq 3$, there is no set N satisfying that $q_d^{(1)}(n) = p(n|\, parts\ in\ N)$ for all $n > 0$. This is also true for general $q_d^{(a)}(n)$ ($a \geq 1$ and $d \geq 3$). With regard to modularity, D. Zagier [Za07] also proved that for $d \geq 3$, the series

$$\sum_{n=0}^{\infty} \frac{q^{d\frac{n^2}{2}+\beta n+\gamma}}{(q;q)_n}$$

are never modular forms for any pair $(\beta, \gamma) \in \mathbb{Q}^2$. A. Folsom [Fo14] though proved that the generating function of $q_d^{(1)}(n)$ is a natural denominator of a new class of mixed mock modular form. She also obtained the analytic behavior of the generating function near the unit disc. For further discussion of modularity of the generating functions of d-distinct partitions, please refer to [Fo14].

Back to the combinatorial aspect, there are partition inequalities instead of partition identities related with d-distinct partitions for general d. For example, the Alder-Andrews Theorem states that the number of d-distinct partitions of n is greater than or equal to the number of partitions of n into parts which are congruent to $\pm 1 \pmod{d+3}$. We discuss the partition inequalities, including the Alder-Andrews Theorem and its analogs in the next section.

Then in Section 3, we prove the following asymptotic bounds for d-distinct partitions. First, for integers b, m, m_1, m_2 with $0 \leq m_1 < m_2 < m$ and $b \geq 1$, we define

$$Q_m^{(m_1,m_2)}(n) := p(n|\text{parts} \equiv m_1 \text{ or } m_2 \pmod m) \tag{1.6}$$

and

$$\Delta_{(d,m)}^{(a,b)}(n) := q_d^{(a)}(n) - Q_m^{(b,m-b)}(n). \tag{1.7}$$

Theorem 1.1. *Let a, d, m, m_1, m_2 be integers such that $0 \leq m_1 < m_2 < m$ and $a, d \geq 1$. For α_d the unique real root of $x^d + x - 1$ in the interval $(0, 1)$ and*

$$A_d := \frac{d}{2}\log^2 \alpha_d + \sum_{r=1}^{\infty} \frac{\alpha_d^{rd}}{r^2}, \tag{1.8}$$

we let $M_d := \lfloor \frac{\pi^2}{3A_d} \rfloor$. *Then*

$$\begin{cases} \lim_{n\to\infty} (q_d^{(a)}(n) - Q_m^{(m_1,m_2)}(n)) = \infty, & \text{if } m > M_d, \\ \lim_{n\to\infty} (q_d^{(a)}(n) - Q_m^{(m_1,m_2)}(n)) = -\infty, & \text{if } m \le M_d. \end{cases}$$

Remark 1.2. (i) *Let S be a set of all positive integers congruent to m_1 or m_2* $\pmod m$ *and S' be an arbitrary finite subset of S. Since*

$$p(n|\text{parts in } S - S') \sim Q_m^{(m_1,m_2)}(n),$$

Theorem 1.1 still holds when $Q_m^{(m_1,m_2)}(n)$ is replaced by $p(n|\text{parts in } S - S')$.

(ii) *If $d \ge 4$, then $A_d > \frac{\pi^2}{3(d+3)}$ [An71, p. 282]. Hence both Andrews asymptotic formula for $\Delta^{(1,1)}_{(d,d+3)}$ [An71, Theorem 2] in (3.18) and Theorem 3.3 below follow from the first asymptotics in Theorem 1.1.*

(iii) *If $M_d < m < d + 3$, then Theorem 1.1 implies that for sufficiently large n,*

$$q_d^{(1)}(n) \ge Q_m^{(1,m-1)}(n).$$

However this inequality does not hold in general. For example, $q_d^{(1)}(d+1) = 1$ while $Q_{d+2}^{(1,d+1)}(d+1) \ge 2$. Hence the choice of $d + 3$ is optimal in the Alder-Andrews Theorem.

(iv) *Observing their values, the inequality $q_d^{(1)}(n) \le Q_{M_d}^{(1,M_d-1)}(n)$ seems hold for all $n \ge 1$, if $d \ge 198$. It will be interesting to find the maximal value of $m < M_d$ so that this inequality holds for all $d, n \ge 1$.*

In the last section, we propose a conjecture on a partition inequality that gives an upper bound for d-distinct partitions for any $d \ge 4$.

Conjecture 1. *Let $d \ge 4$ and let* $m_d := \left\lfloor \dfrac{2d\pi^2}{3\log^2(d) + 6\log(d)} \right\rfloor$. *Then for all $n \ge 1$,*

$$q_d^{(1)}(n) \le Q_{m_d}^{(1,m_d-1)}(n).$$

2. Alder-Andrews Theorem and its analogs

I. Schur [Sch73] found that the number of partitions of n into parts differing by at least 3 is greater than the number of partitions of n into parts congruent to $\pm 1 \pmod 6$ and the excess occurs as partitions of n into parts differing by at least 3 with at least one pair of consecutive multiples of 3. In 1956, H. L. Alder [Al56] proposed a problem asking whether for all $d, n \ge 1$, $q_d^{(1)}(n)$ is greater than or equal to the number of partitions of n into parts congruent to $\pm 1 \pmod{d+3}$. In 1971, G. E. Andrews [An71] proved that it is true for $d = 2^r - 1$, $r \ge 4$. In 2004, A. J. Yee [Ye08] proved it for $d = 7$ and all $d \ge 32$. Finally, in 2011, C. Alfes, M. Jameson, and R. J. Lemke-Oliver [AJL11] completed the proof by showing that it holds for the rest cases. Hence we now have a theorem

Theorem 2.1. (Alder-Andrews Theorem) *For all $d, n \ge 1$,*

$$\Delta^{(1,1)}_{(d,d+3)}(n) = q_d^{(1)}(n) - Q_{d+3}^{(1,d+2)}(n) \ge 0.$$

This partition inequality is a generalization of Euler's identity (1.3), the first Rogers-Ramanujan identity (1.4), and Schur's theorem.

Recently, the first author and E. Y. Park [KaPa20] established an analogue of the Alder-Andrews Theorem that generalizes the 2nd Rogers-Ramanujan identity (1.5) for certain values of d and predicted that the analogue is valid for all $d \geq 1$. Namely, they proved that the number of d-distinct partitions of n with no part 1 is greater than or equal to the number of partitions of n into parts $\equiv \pm 2 \pmod{d+3}$ with no part $d+1$ when $d = 2^r - 2\,(r \geq 2, r \neq 3, 4)$. Letting

$$Q_m^{(b,-)}(n) := p(n|\text{parts} \equiv \pm b \pmod{m},\ \text{parts } \neq m-b) \tag{2.9}$$

and

$$\Delta_{(d,m)}^{(a,b,-)}(n) := q_d^{(a)}(n) - Q_m^{(b,-)}(n), \tag{2.10}$$

we may write the proposed partition inequality as

Conjecture 2. *[KaPa20, Conjecture 1.1] For all $d, n \geq 1$,*

$$\Delta_{(d,d+3)}^{(2,2,-)}(n) = q_d^{(2)}(n) - Q_{d+3}^{(2,-)}(n) \geq 0.$$

The $d = 2$ case is weaker than the second Rogers-Ramanujan identity (1.5), but in general, excluding the part $d+1$ from the latter partition in the difference is necessary, because

$$\Delta_{(d,d+3)}^{(2,2)}(n) < 0 \text{ for certain } d,\ n \geq 1.$$

More recently, Conjecture 2 has been proved by A. L. Duncan, S. Khunger, H. Swisher, and R. Tamura [DKST21] for all $d \geq 62$. So the conjecture is proved for all $d \geq 1$ except for $d = 1$, $3 \leq d \leq 29$, and $31 \leq d \leq 61$. Furthermore, Duncan et al. presented a more generalized conjecture for higher $a = b$ and proved it for infinite classes of n and d. For $1 \leq b \leq d+2$, define

$$Q_m^{(b,-,-)}(n) := p(n|\text{parts} \equiv \pm b \pmod{m},\ \text{parts } \neq b, m-b) \tag{2.11}$$

and

$$\Delta_{(d,m)}^{(a,b,-,-)}(n) := q_d^{(a)}(n) - Q_m^{(b,-,-)}(n). \tag{2.12}$$

Conjecture 3. *[DKST21, Conjecture 1.5] Let a and d be positive integers with $1 \leq a \leq d+2$. Then for all $n \geq 1$,*

$$\Delta_{(d,d+3)}^{(a,a,-,-)}(n) = q_d^{(a)}(n) - Q_{d+3}^{(a,-,-)}(n) \geq 0.$$

In [DKST21], three partial results of Conjecture 3 for $a \geq 3$ are proved and also asymptotic behavior of $\Delta_{(d,d+3)}^{(a,a,-,-)}(n)$ is investigated. The latter is presented in the next section.

3. Asymptotic bounds for d-distinct partitions

G. Meinardus established the asymptotic formulas for $q_d^{(a)}(n)$ in [Mei54]. We first recall his original theorem. Let k, ℓ and m be positive integers with $(m, k) = 1$. Now consider $F(u, v)$ a function of two complex variables u and v, which is holomorphic in a neighborhood of $u = 0$ for fixed v with $|v| < 1$ and satisfies the following functional equation

$$F(u, v) = F\left(uv^k, v\right) + uv^m F\left(uv^{kl}, v\right). \tag{3.13}$$

Assuming $F(0, v) = 1$ for all v, it is easy to see from (3.13) that $F(u, v)$ for every u is a holomorphic function of v in the unit circle, namely

$$F(u, v) = 1 + \sum_{\nu=1}^{\infty} \frac{v^{\frac{k\ell}{2}\nu(\nu-1)+m\nu} u^{\nu}}{(1-v^k)(1-v^{2k})\ldots(1-v^{\nu k})}. \tag{3.14}$$

One finds that the power series representation of $F(1, v)$ as follows:

Theorem 3.1. *[Mei54, Theorem 2] For $|v| < 1$,*

$$F(1, v) = F(v) = 1 + \sum_{n=1}^{\infty} r(n)v^n, \tag{3.15}$$

where

$$r(n) = C(k, \ell, m)n^{-3/4}e^{2\sqrt{\frac{A_\ell}{k}n}} \quad \text{as } n \to \infty.$$

Here, A_ℓ and α_ℓ are given in Theorem 1.1 and

$$C(k, \ell, m) = \frac{1}{2\sqrt{\pi}}\left(\frac{A_\ell}{k}\right)^{1/4}\left\{(\alpha_\ell)^{\ell+1-\frac{2m}{k}} \cdot \left(\ell\,(\alpha_\ell)^{\ell-1}+1\right)\right\}^{-\frac{1}{2}}.$$

Meinardus also found asymptotics for coefficients of certain infinite products. We state a part of his theorem after a slight modification.

Theorem 3.2. *[Mei53, Theorem 1] Let $\tau = y + 2\pi ix$, $y > 0$ and*

$$f(\tau) = \prod_{n=1}^{\infty}\left(1 - e^{-n\tau}\right)^{-a_n} \tag{3.16}$$

for non-negative real a_n. Then it follows that

(i) the Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad (s = \sigma + it)$$

converges for $\sigma > \alpha > 0$ and is extended to the line $\sigma = -c_0$ with $0 < c_0 < 1$. It has a simple pole at $s = \alpha$ with residue A and $D(s) = O(|t|^{c_1})$ for $|t| \to \infty$ for a positive constant c_1.

(ii) $f(\tau)$ converges and has a series expansion

$$f(\tau) = 1 + \sum_{n=1}^{\infty} r(n)e^{-n\tau},$$

where

$$r(n) = C \cdot n^{\aleph} \cdot \exp\left(n^{\frac{\alpha}{\alpha+1}}\left(1 + \frac{1}{\alpha}\left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{\frac{1}{\alpha+1}}\right)\right) \cdot \left(1 + O(n^{-\aleph_1})\right), n \to \infty.$$

Here, $\zeta(s)$ is the Riemann zeta function, $\aleph_1 = \frac{\alpha}{\alpha+1}\min\left(\frac{c_0}{\alpha} - \frac{\delta}{4}, \frac{1}{2} - \delta\right)$ for $\delta > 0$, $\aleph = \frac{2D(0)-2-\alpha}{2(1+\alpha)}$, and $C = e^{D'(0)} \cdot (2\pi(1+\alpha))^{-1/2} \cdot (A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1-2D(0)}{2(1+\alpha)}}$.

Substituting $v = q$, $m = a, k = 1$ and $\ell = d$ in equations in Theorem 3.1, we find that the asymptotics for $q_d^{(a)}(n)$ as

$$q_d^{(a)}(n) \sim \frac{A_d^{1/4}}{2\sqrt{\pi\alpha^{d+1-2a}\left(d\alpha^{d-1}+1\right)}}n^{-3/4}e^{2\sqrt{A_d n}}. \tag{3.17}$$

Using (3.17) and Theorem 3.2, Andrews [An71] established the asymptotic formula for all $d \geq 4$:

$$\lim_{n\to\infty} \Delta^{(1,1)}_{(d,d+3)}(n) = \infty. \tag{3.18}$$

The proofs of the Alder-Andrews Theorem by Alfes at al. [AJL11] also heavily depends on Meinardus Theorems.

Similarly, Duncan et al. found the following asymptotic result.

Theorem 3.3. *[DKST21, Theorem 1.9] Let $a \geq 1$ and $d \geq 4$ such that $a < \frac{d+3}{2}$ and $\gcd(a, d+3) = 1$. Then*

$$\lim_{n\to\infty} \Delta^{(a,a,-,-)}_{(d,d+3)}(n) = \infty.$$

As the proof in [DKST21] indicates, in order to prove Theorem 3.3. it suffices to use asymptotics of $q^{(a)}_d(n)$ and $Q^{(1,d+2)}_{d+3}(n)$ and show that

$$\lim_{n\to\infty} (q^{(a)}_d(n) - Q^{(1,d+2)}_{d+3}(n)) = \infty. \tag{3.19}$$

This is because, for $1 \leq a \leq d+2$ and $n \geq 1$,

$$\Delta^{(a,a,-,-)}_{(d,d+3)}(n) \geq \Delta^{(a,a,-)}_{(d,d+3)}(n) \geq \Delta^{(a,a)}_{(d,d+3)}(n)$$

and for $a < \frac{d+3}{2}$ and $\gcd(a, d+3) = 1$ and $n \geq 1$,

$$\Delta^{(a,a)}_{(d,d+3)}(n) = q^{(a)}_d(n) - Q^{(a,d+3-a)}_{d+3}(n) \geq q^{(a)}_d(n) - Q^{(1,d+2)}_{d+3}(n). \tag{3.20}$$

The inequality in (3.20) holds by a theorem of L. Xia ([Xi11], [DKST21, Theorem 6.1]).

Applying similar asymptotics used in the proof of Theorem 3.3, one can determine which partition function $Q^{(m_1,m_2)}_m(n)$ takes the role as an asymptotic upper bound or lower bound for $q^{(a)}_d(n)$ as stated in Theorem 1.1.

Proof of Theorem 1.1. We note that (3.17) implies that

$$\log q^{(a)}_d(n) \sim 2\sqrt{A_d n}. \tag{3.21}$$

Let $f(\tau) := \sum_{n=0}^{\infty} Q^{(m_1,m_2)}_m(n)e^{-n\tau}$. Then

$$f(\tau) = \prod_{k=0}^{\infty}\left(1 - e^{-(mk+m_1)\tau}\right)^{-1}\prod_{k=0}^{\infty}\left(1 - e^{-(mk+m_2)\tau}\right)^{-1}$$

and the corresponding Dirichlet series is written in terms of the Hurwitz zeta function as

$$D(s) = m^{-s}\left(\zeta\left(s, \frac{m_1}{m}\right) + \zeta\left(s, \frac{m_2}{m}\right)\right).$$

Note that $D(s)$ has a simple pole at $\alpha = 1$ with the residue $A = \frac{2}{m}$ and $D(0) = 1 - \left(\frac{m_1+m_2}{m}\right)$. Also,

$$D'(0) = \log\left(\Gamma\left(\frac{m_1}{m}\right)\Gamma\left(\frac{m_2}{m}\right)\right) - \log 2\pi + \left(\frac{m_1+m_2}{m} - 1\right)\log m.$$

Applying these values in Theorem 3.2, we find that

$$Q^{(m_1,m_2)}_m(n) \sim Cn^{\aleph}e^{2\sqrt{\frac{\pi^2 n}{3m}}},$$

where

$$C = \Gamma\left(\frac{m_1}{m}\right)\Gamma\left(\frac{m_2}{m}\right)2^{-2}\cdot 3^{-\frac{2m_1+2m_2-m}{4m}}\cdot \pi^{\frac{2m_1+2m_2-m}{2m}-\frac{3}{2}}\cdot m^{\frac{m_1+m_2-m}{m}-\frac{2m_1+2m_2-m}{4m}},$$

$$\aleph = -\frac{2m_1+2m_2+m}{4m}.$$

Hence

$$\log Q^{(m_1,m_2)}_m(n) \sim 2\sqrt{\frac{\pi^2 n}{3m}}. \tag{3.22}$$

By comparing (3.21) with (3.22), we have the results.

4. Upper bounds for d-distinct partitions

Now we fix $a = 1$ and consider bounds (not asymptotics) for $q_d^{(1)}(n)$ for a given $d \geq 1$.

1. If $m \geq d+3$, then it follows from the Alder-Andrews Theorem and fact that $q_d^{(1)}(n) \geq q_{d+k}^{(1)}(n)$ that for all $n \geq 1$,
$$q_d^{(1)}(n) \geq Q_m^{(1,m-1)}(n).$$

2. If $M_d < m < d+3$, then as mentioned in Remark 1.2, for sufficiently large n,
$$q_d^{(1)}(n) \geq Q_m^{(1,m-1)}(n).$$

3. If $m \leq M_d$, then Theorem 1.1 implies that for sufficiently large n,
$$q_d^{(1)}(n) \leq Q_m^{(1,m-1)}(n).$$

   Using SageMath (the Sage Mathematics Software System Version 8.4), we obtain the following conjectural data:

   (a) If it occurs that $q_d^{(1)}(n) > Q_{M_d}^{(1,M_d-1)}(n)$, then $d+2 \leq n \leq 3d+3$.

   (b) If $d \geq 198$, $q_d^{(1)}(n) \leq Q_{M_d}^{(1,M_d-1)}(n)$ for all $n \geq 1$.

We propose the following conjecture and Conjecture 1 from our inspection on data.

Conjecture 4. *Let $d \geq 198$. Then for all $n \geq 1$,*
$$q_d^{(1)}(n) \leq Q_{M_d}^{(1,M_d-1)}(n).$$

By choosing a smaller m than M_d, we observe that the inequality may hold for more values of d. If we set $\beta_d = d^{(-1/d)}$, then $\beta_d^d + \beta_d < 1$, which implies that $0 < \beta_d < \alpha_d < 1$. Using this with (1.8), we have
$$\begin{aligned} A_d &= \frac{d}{2}\log^2 \alpha_d + \sum_{r=1}^{\infty} \frac{(\alpha_d)^{rd}}{r^2} = \frac{d}{2}\log^2 \alpha_d + \sum_{r=1}^{\infty} \frac{(1-\alpha_d)^r}{r^2} \\ &< \frac{d}{2}\log^2 \beta_d + \sum_{r=1}^{\infty} \frac{(1-\beta_d)^r}{r^2} < \frac{d}{2}\log^2 \beta_d + \sum_{r=1}^{\infty} \frac{(1-\beta_d)^r}{r} \\ &= \frac{d}{2}\log^2 \beta_d - \log \beta_d = \frac{1}{2d}\log^2 d + \frac{1}{d}\log d. \end{aligned}$$
Hence
$$m_d := \lfloor \frac{2d\pi^2}{3\log^2(d) + 6\log(d)} \rfloor \leq M_d$$
and we use m_d in Conjecture 1.

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