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A heuristic guide to evaluating triple-sums

Eric T. Mortenson

Abstract. Using a heuristic that relates Appell–Lerch functions to divergent partial theta functions one can expand Hecke-type double-sums in terms of Appell–Lerch functions. We give examples where the heuristic can be used as a guide to evaluate analogous triple-sums in terms of Appell–Lerch functions or false theta functions.

Keywords. Hecke-type triple-sums, Appell–Lerch functions, mock theta functions, false theta functions

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1. Notation

Let q be a nonzero complex number with $|q|<1$ and define $\mathbb{C}^*:=\mathbb{C}-\{0\}$. Recall

$$(x)_n=(x;q)_n:=\prod_{i=0}^{n-1}(1-q^ix),\quad (x)_\infty=(x;q)_\infty:=\prod_{i=0}^{\infty}(1-q^ix),$$
$$j(x;q):=(x)_\infty(q/x)_\infty(q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^nq^{\binom{n}{2}}x^n,$$

and

$$j(x_1,x_2,\dots,x_n;q):=j(x_1;q)j(x_2,q)\cdots j(x_n;q),$$

where in the penultimate line the equivalence of product and sum follows from Jacobi's triple product identity. Let a and m be integers with m positive. Define

$$J_{a,m}:=j(q^a;q^m),\quad J_m:=J_{m,3m}=\prod_{i=1}^{\infty}(1-q^{mi}),\text{ and }\overline{J}_{a,m}:=j(-q^a;q^m).$$

We will use the following definition of an Appell–Lerch function [HiMo14]

$$m(x,q,z):=\frac{1}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^rq^{\binom{r}{2}}z^r}{1-q^{r-1}xz}. \tag{1.1}$$

2. Introduction

Appell–Lerch functions are the building blocks of Ramanujan's classical mock theta functions [HiMo14, Section 5]. For two of Ramanujan's fifth order mock theta functions [HiMo14, Section 5], [AnGa89, Hi88a]:

$$\chi_0(q):=\sum_{n\geq 0}\frac{q^n}{(q^{n+1})_n}=2-3m(q^7,q^{15},q^9)-3q^{-1}m(q^2,q^{15},q^4)+\frac{2J_5^2J_{2,5}}{J_{1,5}^2}, \tag{2.2}$$

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$$\chi_1(q) := \sum_{n\geq 0} \frac{q^n}{(q^{n+1})_{n+1}} = -3q^{-1}m(q^4,q^{15},q^3) - 3q^{-2}m(q,q^{15},q^2) - \frac{2J_5^2J_{1,5}}{J_{2,5}^2}. \tag{2.3}$$

In recent work [HiMo14], we used a heuristic (3.21) to express Hecke-type double-sums of the form [HiMo14, (1.4)]

$$\Big(\sum_{r,s\geq 0} - \sum_{r,s<0}\Big)(-1)^{r+s}x^ry^sq^{a\binom{r}{2}+brs+c\binom{s}{2}}, \tag{2.4}$$

where a, b, c are strictly positive integers, in terms of theta functions and the $m(x,q,z)$ function. As an example, a special case of more general results from [HiMo14, (1.7)] reads

$$\begin{aligned}&\Big(\sum_{r,s\geq 0} - \sum_{r,s<0}\Big)(-1)^{r+s}x^ry^sq^{\binom{r}{2}+2rs+\binom{s}{2}} \qquad (2.5)\\ &\quad = j(y;q)m\Big(\frac{q^2x}{y^2},q^3,-1\Big) + j(x;q)m\Big(\frac{q^2y}{x^2},q^3,-1\Big) - \frac{yJ_3^3j(-x/y;q)j(q^2xy;q^3)}{\overline{J}_{0,3}j(-qy^2/x,-qx^2/y;q^3)}.\end{aligned}$$

Objects that could be thought of as Hecke-type triple-sums have been the subject of recent work. Zwegers discovered that [Zw09, Theorem 1]

$$2 - \frac{1}{(q)_\infty^2}\Big(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\Big)(-1)^{k+l+m}q^{\binom{k}{2}+\binom{l}{2}+\binom{m}{2}+2kl+2km+2lm+k+l+m} = \chi_0(q), \tag{2.6}$$

$$\frac{1}{(q)_\infty^2}\Big(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\Big)(-1)^{k+l+m}q^{\binom{k}{2}+\binom{l}{2}+\binom{m}{2}+2kl+2km+2lm+2k+2l+2m} = \chi_1(q), \tag{2.7}$$

which suggests considering the following building block for triple-sums:

$$\mathfrak{g}_{a,b,c,d,e,f}(x,y,z,q) := \Big(\sum_{r,s,t\geq 0} + \sum_{r,s,t<0}\Big)(-1)^{r+s+t}x^ry^sz^tq^{a\binom{r}{2}+brs+c\binom{s}{2}+drt+est+f\binom{t}{2}}, \tag{2.8}$$

where a, b, c, d, e, and f are strictly positive integers. For related examples, see [BRZ15, Mo17]. Indefinite binary theta series for $\chi_0(q)$ and $\chi_1(q)$ can be found in [Ga19, Za07].

Kim and Lovejoy [KiLo17] recently found examples of triple-sums that evaluate to false theta functions. False theta functions are theta functions but with the wrong signs [AnWa07]. In particular, they found

$$\mathfrak{g}_{1,7,1,1,1,1}(q^2,q^3,q,q) + q^4\mathfrak{g}_{1,7,1,1,1,1}(q^6,q^7,q^2,q) = J_1^2\sum_{r=0}^{\infty}(-1)^rq^{3r(r+1)/2}, \tag{2.9}$$

$$\mathfrak{g}_{1,5,1,1,1,1}(q^2,q^2,q,q) + q^3\mathfrak{g}_{1,5,1,1,1,1}(q^5,q^5,q^2,q) = J_1J_{1,2}\sum_{r=0}^{\infty}(-1)^rq^{3r^2+2r}(1+q^{2r+1}). \tag{2.10}$$

In Section 3., we introduce a few basic facts and preliminary results. In Section 4., we remind the reader how the heuristic is used to understand the (mock) modularity of double-sums. In Section 5., we obtain the relation (5.46) that will be used to understand the modularity of triple-sums. We will also introduce a few basic facts about triple-sums.

In Section 6., we consider the mock theta functions $\chi_0(q)$ and $\chi_1(q)$. We use the relation (5.46) to demonstrate how the heuristic (3.21) guides us from their respective triple-sums in (2.6) and (2.7) to their Appell–Lerch sum expressions modulo a theta function, see (2.2) and (2.3). In Section 7., we demonstrate how the relation (5.46) guides us from Kim and Lovejoy's two triple-sums in

(2.9) and (2.10) to their respective false theta function forms. The work in Sections 6. and 7. is experimental and does not contain proofs, but perhaps more general formulas such as those found in [BRZ15, HiMo14, Mo17] are lurking in the background.

In Section 8., we use triple-sum relations from Section 5. in order to prove corollaries to Identities (2.6) and (2.7). We show

Corollary 2.1. *We have*

$$\mathfrak{g}_{1,2,1,2,2,1}(q^3,q^3,q^3,q)=0, \tag{2.11}$$
$$\mathfrak{g}_{1,2,1,2,2,1}(q,q,q^2,q)=J_1^2, \tag{2.12}$$
$$\mathfrak{g}_{1,2,1,2,2,1}(q,q^2,q^2,q)=J_1^2\chi_0(q), \tag{2.13}$$
$$\mathfrak{g}_{1,2,1,2,2,1}(q,q,q^3,q)=J_1^2(q\chi_0(q)+1-q), \tag{2.14}$$
$$\mathfrak{g}_{1,2,1,2,2,1}(q,q^2,q^3,q)=J_1^2(q\chi_1(q)+1). \tag{2.15}$$

One wonders how triple-sums whose parameters differ slightly from the left-hand sides of (2.6) and (2.7) might behave. In Section 9., we demonstrate how our heuristic methods lead us to suspect

$$\begin{aligned}\mathfrak{g}_{1,3,1,3,3,1}(q,q,q,q)&\\ &=3J_{1,2}\overline{J}_{3,8}m(-q^{27},q^{56},-1)+3q^{-2}J_{1,2}\overline{J}_{3,8}m(-q^{13},q^{56},-1)\\ &\quad-3q^{-7}J_{1,2}\overline{J}_{1,8}m(-q^{-1},q^{56},-1)-3q^{-16}J_{1,2}\overline{J}_{1,8}m(-q^{-15},q^{56},-1)+\theta(q),\end{aligned} \tag{2.16}$$

where $\theta(q)$ is a yet to be determined weight $3/2$ theta function.

3. Preliminaries

We will frequently use the following product rearrangements without mention

$$\overline{J}_{0,1}=2\overline{J}_{1,4}=\frac{2J_2^2}{J_1},\ \overline{J}_{1,2}=\frac{J_2^5}{J_1^2J_4^2},\ J_{1,2}=\frac{J_1^2}{J_2},\ \overline{J}_{1,3}=\frac{J_2J_3^2}{J_1J_6},$$
$$J_{1,4}=\frac{J_1J_4}{J_2},\ J_{1,6}=\frac{J_1J_6^2}{J_2J_3},\ \overline{J}_{1,6}=\frac{J_2^2J_3J_{12}}{J_1J_4J_6}.$$

We have the general identities:

$$j(q^nx;q)=(-1)^nq^{-\binom{n}{2}}x^{-n}j(x;q),\quad n\in\mathbb{Z}, \tag{3.17a}$$
$$j(x;q)=j(q/x;q)=-xj(x^{-1};q), \tag{3.17b}$$
$$j(x;q)=J_1j(x,qx,\dots,q^{n-1}x;q^n)/J_n^n\ \text{ if } n\geq 1, \tag{3.17c}$$
$$j(x^n;q^n)=J_nj(x,\zeta_nx,\dots,\zeta_n^{n-1}x;q^n)/J_1^n\ \text{ if } n\geq 1, \tag{3.17d}$$

where ζ_n is an n-th primitive root of unity. We also have from [Hi88a, Theorem 1.1]

$$j(-x,q)j(y,q)+j(x,q)j(-y,q)=2j(xy,q^2)j(qx^{-1}y,q^2). \tag{3.18}$$

The Appell–Lerch function $m(x,q,z)$ satisfies several functional equations and identities, which we collect in the form of a proposition.

Proposition 3.1. *For generic* $x,z\in\mathbb{C}^*$

$$m(x,q,z)=m(x,q,qz), \tag{3.19a}$$
$$m(x,q,z)=x^{-1}m(x^{-1},q,z^{-1}), \tag{3.19b}$$
$$m(qx,q,z)=1-xm(x,q,z), \tag{3.19c}$$
$$m(x,q,z_1)-m(x,q,z_0)=\frac{z_0J_1^3j(z_1/z_0;q)j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}. \tag{3.19d}$$

The $m(x,q,z)$ function has a useful (slightly rewritten) functional equation

$$m(x,q,z)=1-q^{-1}xm(q^{-1}x,q,z). \tag{3.20}$$

In recent work [HiMo14, Section 3] we introduced a heuristic point of view which guides our study of the Appell–Lerch function $m(x,q,z)$ and Hecke-type double-sums. If we iterate (3.20), we obtain

$$m(x,q,z)\sim\sum_{r\geq 0}(-1)^r q^{-\binom{r+1}{2}}x^r, \tag{3.21}$$

where '$\sim$' means mod theta. We cannot use an equal sign here, because the series on the right diverges for $|q|<1$. However, it is often useful to think of $m(x,q,z)$ as a partial theta series with q replaced by q^{-1}. Property (3.19d) says that changing z only adjusts the theta function, hence we have the expression $m(x,q,*)$ when working mod theta.

We define

$$\mathfrak{f}_{a,b,c}(x,y,q):=\Big(\sum_{r,s\geq 0}-\sum_{r,s<0}\Big)(-1)^{r+s}x^ry^sq^{a\binom{r}{2}+brs+c\binom{s}{2}}, \tag{3.22}$$

and note that we can also write

$$\mathfrak{f}_{a,b,c}(x,y,q)=\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)}\mathrm{sg}(r)(-1)^{r+s}x^ry^sq^{a\binom{r}{2}+brs+c\binom{s}{2}}, \tag{3.23}$$

$$=\sum_{r,s}\mathrm{sg}(r,s)(-1)^{r+s}x^ry^sq^{a\binom{r}{2}+brs+c\binom{s}{2}}, \tag{3.24}$$

where

$$\mathrm{sg}(r):=\begin{cases}1 & r\geq 0,\\ -1 & r<0,\end{cases} \tag{3.25}$$

and

$$\mathrm{sg}(r,s):=(\mathrm{sg}(r)+\mathrm{sg}(s))/2. \tag{3.26}$$

Our Hecke-type double-sum notation has several useful properties. We will later determine analogous properties for triple-sums. For $b<a$, we follow the usual convention:

$$\sum_{r=a}^{b}c_r:=-\sum_{r=b+1}^{a-1}c_r. \tag{3.27}$$

Proposition 3.2. (Proposition 6.2 [HiMo14]) *For $x,y\in\mathbb{C}^*$*

$$\mathfrak{f}_{a,b,c}(x,y,q)=-\frac{q^{a+b+c}}{xy}\mathfrak{f}_{a,b,c}(q^{2a+b}/x,q^{2c+b}/y,q). \tag{3.28}$$

Proposition 3.3. (Proposition 6.3 [HiMo14]) *For $x,y\in\mathbb{C}^*$ and $\ell,k\in\mathbb{Z}$*

$$\begin{aligned}\mathfrak{f}_{a,b,c}(x,y,q)&=(-x)^{\ell}(-y)^kq^{a\binom{\ell}{2}+b\ell k+c\binom{k}{2}}\mathfrak{f}_{a,b,c}(q^{a\ell+bk}x,q^{b\ell+ck}y,q)\\&\quad+\sum_{m=0}^{\ell-1}(-x)^mq^{a\binom{m}{2}}j(q^{mb}y;q^c)+\sum_{m=0}^{k-1}(-y)^mq^{c\binom{m}{2}}j(q^{mb}x;q^a).\end{aligned} \tag{3.29}$$

We have many expressions that evaluate Hecke-type double-sums in terms of Appell–Lerch functions [HiMo14]. We recall one that we will use here.

Theorem 3.4. *[HiMo14, Theorem 1.4] Let a, b, and c be positive integers with $ac < b^2$ and b divisible by a, c. Then*

$$\mathfrak{f}_{a,b,c}(x,y,q) = \mathfrak{h}_{a,b,c}(x,y,q,-1,-1) - \frac{1}{\overline{J}_{0,b^2/a-c}\overline{J}_{0,b^2/c-a}} \cdot \theta_{a,b,c}(x,y,q),$$

where

$$\begin{aligned}\mathfrak{h}_{a,b,c}(x,y,q,z_1,z_0) :&= j(x;q^a)m\Big(-q^{a\binom{b/a+1}{2}-c}(-y)(-x)^{-b/a}, q^{b^2/a-c}, z_1\Big)\\ &\quad + j(y;q^c)m\Big(-q^{c\binom{b/c+1}{2}-a}(-x)(-y)^{-b/c}, q^{b^2/c-a}, z_0\Big),\end{aligned}$$

and

$$\begin{aligned}\theta_{a,b,c}(x,y,q) :=& \sum_{d=0}^{b/c-1}\sum_{e=0}^{b/a-1}\sum_{f=0}^{b/a-1} q^{(b^2/a-c)\binom{d+1}{2}+(b^2/c-a)\binom{e+f+1}{2}+a\binom{f}{2}} j\big(q^{(b^2/a-c)(d+1)+bf}y; q^{b^2/a}\big)\\ &\cdot (-x)^f j\big(q^{b(b^2/(ac)-1)(e+f+1)-(b^2/a-c)(d+1)+b^3(b-a)/(2a^2c)}(-x)^{b/a}y^{-1}; q^{(b^2/a)(b^2/(ac)-1)}\big)\\ &\cdot \frac{J^3_{b(b^2/(ac)-1)} j\big(q^{(b^2/c-a)(e+1)+(b^2/a-c)(d+1)-c\binom{b/c}{2}-a\binom{b/a}{2}}(-x)^{1-b/a}(-y)^{1-b/c}; q^{b(b^2/(ac)-1)}\big)}{j\big(q^{(b^2/c-a)(e+1)-c\binom{b/c}{2}}(-x)(-y)^{-b/c}, q^{(b^2/a-c)(d+1)-a\binom{b/a}{2}}(-x)^{-b/a}(-y); q^{b(b^2/(ac)-1)}\big)}.\end{aligned}$$

Theorem 3.4 has the following two specializations.

Corollary 3.5. *We have*

$$\begin{aligned}\mathfrak{f}_{4,4,1}(x,y,q) &= \mathfrak{h}_{4,4,1}(x,y,q,-1,-1)\\ &\quad - \sum_{d=0}^{3} \frac{q^{3\binom{d+1}{2}} j(q^{3+3d}y;q^4) j(-q^{9-3d}x/y;q^{12}) J_{12}^3 j(-q^{9+3d}/y^3;q^{12})}{\overline{J}_{0,3}\overline{J}_{0,12} j(-q^6x/y^4;q^{12}) j(q^{3+3d}y/x;q^{12})},\end{aligned} \tag{3.30}$$

where

$$\mathfrak{h}_{4,4,1}(x,y,q,-1,-1) = j(x;q^4)m\big(-q^3y/x,q^3,-1\big) + j(y;q)m\big(q^6x/y^4,q^{12},-1\big). \tag{3.31}$$

Corollary 3.6. *We have*

$$\begin{aligned}\mathfrak{f}_{3,3,1}(x,y,q) &= \mathfrak{h}_{3,3,1}(x,y,q,-1,-1) &(3.32)\\ &\quad - \sum_{d=0}^{2} \frac{q^{d(d+1)} j\big(q^{2+2d}y;q^3\big) j\big(-q^{4-2d}x/y;q^6\big) J_6^3 j\big(q^{5+2d}/y^2;q^6\big)}{4\overline{J}_{2,8}\overline{J}_{6,24} j\big(q^3x/y^3;q^6\big) j\big(q^{2+2d}y/x;q^6\big)},\end{aligned}$$

where

$$\mathfrak{h}_{3,3,1}(x,y,q,-1,-1) = j(x;q^3)m(-q^2x^{-1}y,q^2,-1) + j(y;q)m(-q^3xy^{-3},q^6,-1). \tag{3.33}$$

Proposition 3.7. *For $m, n \in \mathbb{Z}$, $m \neq n$, we have*

$$\mathfrak{f}_{1,1,1}(q^m,q^n,q) = 0. \tag{3.34}$$

Proof of Proposition 3.7. Using the functional equation (3.29), we note that the two theta functions have $j(q^m;q) = j(q^n;q) = 0$, giving us

$$\mathfrak{f}_{1,1,1}(q^m,q^n,q) = (-1)^{\ell+k} q^{m\ell} q^{nk} q^{\binom{\ell}{2}+\ell k+\binom{k}{2}} \mathfrak{f}_{1,1,1}(q^{\ell+k+m},q^{\ell+k+n},q).$$

If we choose $\ell = -k$ then

$$\mathfrak{f}_{1,1,1}(q^m, q^n, q) = q^{m\ell - n\ell}\mathfrak{f}_{1,1,1}(q^m, q^n, q).$$

If we set $\ell = 1$ then

$$\mathfrak{f}_{1,1,1}(q^m, q^n, q) = q^{m-n}\mathfrak{f}_{1,1,1}(q^m, q^n, q).$$

Because $m \neq n$ and $|q| < 1$, we conclude

$$\mathfrak{f}_{1,1,1}(q^m, q^n, q) = 0. \qquad \square$$

Proposition 3.8. *For $t \in \mathbb{Z}$, we have*

$$\begin{aligned}&\mathfrak{f}_{1,7,1}(q^{2+t}, q^{3+t}, q) + q^{4+t}\mathfrak{f}_{1,7,1}(q^{6+t}, q^{7+t}, q) \\ &\quad = \mathfrak{f}_{4,4,1}(-q^{4+t}, q^{2+t}, q) - q^{-t}\mathfrak{f}_{4,4,1}(-q^{4-t}, q^{2-t}, q).\end{aligned} \tag{3.35}$$

$$\begin{aligned}&\mathfrak{f}_{1,5,1}(q^{2+t}, q^{2+t}, q) + q^{3+t}\mathfrak{f}_{1,5,1}(q^{5+t}, q^{5+t}, q) \\ &\quad = \mathfrak{f}_{3,3,1}(-q^{3+t}, q^{2+t}, q) - q^{-t}\mathfrak{f}_{3,3,1}(-q^{3-t}, q^{2-t}, q).\end{aligned} \tag{3.36}$$

Proof of Proposition 3.8. The proofs for the identities are similar, so we only do the first. A change of variables yields

$$\begin{aligned}
&\mathfrak{f}_{1,7,1}(q^{2+t}, q^{3+t}, q) + q^{4+t}\mathfrak{f}_{1,7,1}(q^{6+t}, q^{7+t}, q) \\
&= \sum_{\substack{u,v \\ u \equiv v \mod 2}} \operatorname{sg}(u,v)(-1)^{\frac{u-v}{2}} q^{\frac{1}{8}u^2 + \frac{7}{4}uv + \frac{1}{8}v^2 + \frac{3}{4}u + \frac{5}{4}v + \frac{t}{2}(u+v)} \\
&= \Big(\sum_{\substack{n+j\geq 0 \\ n-j \geq 0}} - \sum_{\substack{n+j<0 \\ n-j<0}}\Big)(-1)^j q^{2n^2+(2+t)n-j(3j+1)/2} \\
&= \sum_{n\geq 0} q^{2n^2+(2+t)n} \sum_{j=-n}^{n} (-1)^j q^{-j(3j+1)/2} - \sum_{n<0} q^{2n^2+(2+t)n} \sum_{n<j<-n} (-1)^j q^{-j(3j+1)/2} \\
&= \sum_{n\geq 0} q^{2n^2+(2+t)n} \sum_{j=-n}^{n} (-1)^j q^{-j(3j+1)/2} \\
&\quad - \sum_{n\geq 0} q^{2(-n-1)^2+(2+t)(-n-1)} \sum_{-n-1<j<n+1} (-1)^j q^{-j(3j+1)/2} \\
&= \sum_{n\geq 0} q^{2n^2+(2+t)n} \sum_{j=-n}^{n} (-1)^j q^{-j(3j+1)/2} - \sum_{n\geq 0} q^{2n^2+(2-t)n-t} \sum_{j=-n}^{n} (-1)^j q^{-j(3j+1)/2} \\
&= \sum_{\operatorname{sg}(j)=\operatorname{sg}(n-j)} \operatorname{sg}(j)(-1)^j q^{2n^2+(2+t)n-j(3j+1)/2}(1-q^{-2tn-t}) \\
&= \mathfrak{f}_{4,4,1}(-q^{4+t}, q^{2+t}, q) - q^{-t}\mathfrak{f}_{4,4,1}(-q^{4-t}, q^{2-t}, q),
\end{aligned}$$

where the last line follows from the substitution $u = j$ and $v = n - j$. $\square$

Proposition 3.9. *For $m \in \mathbb{Z}$ and $k \in \{0, 1, 2, 3\}$, we have*

$$\begin{aligned}&\mathfrak{f}_{4,4,1}(-q^{4+4m+k}, q^{2+4m+k}, q) - q^{-4m-k}\mathfrak{f}_{4,4,1}(-q^{4-4m-k}, q^{2-4m-k}, q) \\ &\quad = \big(\mathfrak{f}_{4,4,1}(-q^{4+k}, q^{2+k}, q) - q^{-k}\mathfrak{f}_{4,4,1}(-q^{4-k}, q^{2-k}, q)\big)q^{-2m^2-2m-mk}\end{aligned} \tag{3.37}$$

Proposition 3.10. *For $m \in \mathbb{Z}$ and $k \in \{0, 1, 2\}$, we have*

$$\begin{aligned}&\mathfrak{f}_{3,3,1}(-q^{3+3m+k}, q^{2+3m+k}, q) - q^{-3m-k}\mathfrak{f}_{3,3,1}(-q^{3-3m-k}, q^{2-3m-k}, q) \\ &\quad = \Big(\mathfrak{f}_{3,3,1}(-q^{3+k}, q^{2+k}, q) - q^{-k}\mathfrak{f}_{3,3,1}(-q^{3-k}, q^{2-k}, q)\Big)q^{-\frac{3}{2}m^2-\frac{3}{2}m-mk}.\end{aligned} \tag{3.38}$$

Lemma 3.11. *If we define*

$$a_k := \mathfrak{f}_{4,4,1}(-q^{4+k}, q^{2+k}, q) - q^{-k}\mathfrak{f}_{4,4,1}(-q^{4-k}, q^{2-k}, q), \tag{3.39}$$

$$b_k := \mathfrak{f}_{3,3,1}(-q^{3+k}, q^{2+k}, q) - q^{-k}\mathfrak{f}_{3,3,1}(-q^{3-k}, q^{2-k}, q), \tag{3.40}$$

then

$$a_0 = 0,\ a_1 = -q^{-1}J_1^2,\ a_2 = 0, a_3 = q^{-3}J_1^2, \tag{3.41}$$

and

$$b_0 = 0,\ b_1 = -q^{-1}J_1^3/J_2,\ b_2 = q^{-2}J_1^3/J_2. \tag{3.42}$$

Proof of Proposition 3.9. We recall Corollary 3.5. We first show that

$$\begin{aligned}
&\mathfrak{h}_{4,4,1}(-q^{4+m}, q^{2+m}, q, -1, -1) - q^{-m}\mathfrak{h}_{4,4,1}(-q^{4-m}, q^{2-m}, q, -1, -1)\\
&\quad = j(-q^{4+m}; q^4)m\big(q, q^3, -1\big) - q^{-m}j(-q^{4-m}; q^4)m\big(q, q^3, -1\big)\\
&\quad = q^{-m}j(-q^{m}; q^4)m\big(q, q^3, -1\big) - q^{-m}j(-q^{m}; q^4)m\big(q, q^3, -1\big)\\
&\quad = 0,
\end{aligned}$$

where we have used (3.17a). As a result,

$$\begin{aligned}
&\mathfrak{f}_{4,4,1}(-q^{4+m}, q^{2+m}, q) - q^{-m}\mathfrak{f}_{4,4,1}(-q^{4-m}, q^{2-m}, q)\\
&\quad = -\sum_{d=0}^{3}\frac{q^{3\binom{d+1}{2}}j(q^{5+3d+m}; q^4)j(q^{11-3d}; q^{12})J_{12}^3 j(-q^{3+3d-3m}; q^{12})}{\overline{J}_{0,3}\overline{J}_{0,12}j(q^{2-3m}; q^{12})j(-q^{1+3d}; q^{12})}\\
&\qquad + q^{-m}\sum_{d=0}^{3}\frac{q^{3\binom{d+1}{2}}j(q^{5+3d-m}; q^4)j(q^{11-3d}; q^{12})J_{12}^3 j(-q^{3+3d+3m}; q^{12})}{\overline{J}_{0,3}\overline{J}_{0,12}j(q^{2+3m}; q^{12})j(-q^{1+3d}; q^{12})}.
\end{aligned}$$

We make the substitution $m \to 4m + k$, where $k \in \mathbb{Z}$, $0 \le k \le 3$:

$$\begin{aligned}
&\mathfrak{f}_{4,4,1}(-q^{4+4m+k}, q^{2+4m+k}, q) - q^{-4m-k}\mathfrak{f}_{4,4,1}(-q^{4-4m-k}, q^{2-4m-k}, q)\\
&\quad = -\sum_{d=0}^{3}\frac{q^{3\binom{d+1}{2}}j(q^{5+3d+4m+k}; q^4)j(q^{11-3d}; q^{12})J_{12}^3 j(-q^{3+3d-12m-3k}; q^{12})}{\overline{J}_{0,3}\overline{J}_{0,12}j(q^{2-12m-3k}; q^{12})j(-q^{1+3d}; q^{12})}\\
&\qquad + q^{-4m-k}\sum_{d=0}^{3}\frac{q^{3\binom{d+1}{2}}j(q^{5+3d-4m-k}; q^4)j(q^{11-3d}; q^{12})J_{12}^3 j(-q^{3+3d+12m+3k}; q^{12})}{\overline{J}_{0,3}\overline{J}_{0,12}j(q^{2+12m+3k}; q^{12})j(-q^{1+3d}; q^{12})}\\
&\quad = -\sum_{d=0}^{3}\frac{(-1)^m q^{-m(5+3d+k)}q^{-4\binom{m}{2}}q^{m(3+3d-3k)}q^{-12\binom{-m}{2}}}{(-1)^m q^{m(2-3k)}q^{-12\binom{-m}{2}}}\\
&\qquad \cdot\frac{q^{3\binom{d+1}{2}}j(q^{5+3d+k}; q^4)j(q^{11-3d}; q^{12})J_{12}^3 j(-q^{3+3d-3k}; q^{12})}{\overline{J}_{0,3}\overline{J}_{0,12}j(q^{2-3k}; q^{12})j(-q^{1+3d}; q^{12})}\\
&\qquad + q^{-4m-k}\sum_{d=0}^{3}\frac{(-1)^m q^{m(5+3d-k)}q^{-4\binom{-m}{2}}q^{-m(3+3d+3k)}q^{-12\binom{m}{2}}}{(-1)^m q^{-m(2+3k)}q^{-12\binom{m}{2}}}\\
&\qquad \cdot\frac{q^{3\binom{d+1}{2}}j(q^{5+3d-k}; q^4)j(q^{11-3d}; q^{12})J_{12}^3 j(-q^{3+3d+3k}; q^{12})}{\overline{J}_{0,3}\overline{J}_{0,12}j(q^{2+3k}; q^{12})j(-q^{1+3d}; q^{12})}\\
&\quad = \big(\mathfrak{f}_{4,4,1}(-q^{4+k}, q^{2+k}, q) - q^{-k}\mathfrak{f}_{4,4,1}(-q^{4-k}, q^{2-k}, q)\big)q^{-2m^2-2m-mk}.
\end{aligned}$$

□

Proof of Proposition 3.10. We recall Corollary 3.6. We first show that

$$\begin{aligned}
&\mathfrak{h}_{3,3,1}(-q^{3+m},q^{2+m},q)-q^{-m}\mathfrak{h}_{3,3,1}(-q^{3-m},q^{2-m},q)\\
&\quad = j(-q^{3+m};q^3)m\big(q,q^2,-1\big)-q^{-m}j(-q^{3-m};q^3)m\big(q,q^2,-1\big)\\
&\quad = q^{-m}j(-q^{m};q^3)m\big(q,q^2,-1\big)-q^{-m}j(-q^{m};q^3)m\big(q,q^2,-1\big)\\
&\quad = 0,
\end{aligned}$$

where we have used (3.17a). As a result, we have

$$\begin{aligned}
&\mathfrak{f}_{3,3,1}(-q^{3+m},q^{2+m},q)-q^{-m}\mathfrak{f}_{3,3,1}(-q^{3-m},q^{2-m},q)\\
&\quad = -\sum_{d=0}^{2}\frac{q^{d(d+1)}j\big(q^{4+2d+m};q^3\big)j\big(q^{5-2d};q^6\big)J_6^3 j\big(q^{1+2d-2m};q^6\big)}{4\overline{J}_{2,8}\overline{J}_{6,24}j\big(-q^{-2m};q^6\big)j\big(-q^{1+2d};q^6\big)}\\
&\qquad + q^{-m}\sum_{d=0}^{2}\frac{q^{d(d+1)}j\big(q^{4+2d-m};q^3\big)j\big(q^{5-2d};q^6\big)J_6^3 j\big(q^{1+2d+2m};q^6\big)}{4\overline{J}_{2,8}\overline{J}_{6,24}j\big(-q^{2m};q^6\big)j\big(-q^{1+2d};q^6\big)}.
\end{aligned}$$

We replace $m\to 3m+k$, where $k\in\mathbb{Z}$ and $k\in\{0,1,2\}$:

$$\begin{aligned}
&\mathfrak{f}_{3,3,1}(-q^{3+3m+k},q^{2+3m+k},q)-q^{-3m-k}\mathfrak{f}_{3,3,1}(-q^{3-3m-k},q^{2-3m-k},q)\\
&= -\sum_{d=0}^{2}\frac{q^{d(d+1)}j\big(q^{4+2d+3m+k};q^3\big)j\big(q^{5-2d};q^6\big)J_6^3 j\big(q^{1+2d-6m-2k};q^6\big)}{4\overline{J}_{2,8}\overline{J}_{6,24}j\big(-q^{-6m-2k};q^6\big)j\big(-q^{1+2d};q^6\big)}\\
&\quad + q^{-3m-k}\sum_{d=0}^{2}\frac{q^{d(d+1)}j\big(q^{4+2d-3m-k};q^3\big)j\big(q^{5-2d};q^6\big)J_6^3 j\big(q^{1+2d+6m+2k};q^6\big)}{4\overline{J}_{2,8}\overline{J}_{6,24}j\big(-q^{6m+2k};q^6\big)j\big(-q^{1+2d};q^6\big)}\\
&= -\sum_{d=0}^{2}\frac{(-1)^m q^{m(1+2d-2k)}q^{-6\binom{-m}{2}}(-1)^m q^{-m(4+2d+k)}q^{-3\binom{m}{2}}}{q^{m(-2k)}q^{-6\binom{-m}{2}}}\\
&\quad \cdot\frac{q^{d(d+1)}j\big(q^{4+2d+k};q^3\big)j\big(q^{5-2d};q^6\big)J_6^3 j\big(q^{1+2d-2k};q^6\big)}{4\overline{J}_{2,8}\overline{J}_{6,24}j\big(-q^{-2k};q^6\big)j\big(-q^{1+2d};q^6\big)}\\
&\quad + q^{-3m-k}\sum_{d=0}^{2}\frac{(-1)^m q^{-m(1+2d+2k)}q^{-6\binom{m}{2}}(-1)^m q^{m(4+2d-k)}q^{-3\binom{-m}{2}}}{q^{-m(2k)}q^{-6\binom{m}{2}}}\\
&\quad \cdot\frac{q^{d(d+1)}j\big(q^{4+2d-k};q^3\big)j\big(q^{5-2d};q^6\big)J_6^3 j\big(q^{1+2d+2k};q^6\big)}{4\overline{J}_{2,8}\overline{J}_{6,24}j\big(-q^{2k};q^6\big)j\big(-q^{1+2d};q^6\big)}\\
&= -\sum_{d=0}^{2}q^{-\frac{3}{2}m^2-\frac{3}{2}m-mk}\cdot\frac{q^{d(d+1)}j\big(q^{4+2d+k},q^3\big)j\big(q^{5-2d};q^6\big)J_6^3 j\big(q^{1+2d-2k};q^6\big)}{4\overline{J}_{2,8}\overline{J}_{6,24}j\big(-q^{-2k};q^6\big)j\big(-q^{1+2d};q^6\big)}\\
&\quad + q^{-3m-k}\sum_{d=0}^{2}q^{-\frac{3}{2}m^2+\frac{3}{2}m-mk}\cdot\frac{q^{d(d+1)}j\big(q^{4+2d-k};q^3\big)j\big(q^{5-2d};q^6\big)J_6^3 j\big(q^{1+2d+2k};q^6\big)}{4\overline{J}_{2,8}\overline{J}_{6,24}j\big(-q^{2k};q^6\big)j\big(-q^{1+2d};q^6\big)}\\
&= \Big(\mathfrak{f}_{3,3,1}(-q^{3+k},q^{2+k},q)-q^{-k}\mathfrak{f}_{3,3,1}(-q^{3-k},q^{2-k},q)\Big)q^{-\frac{3}{2}m^2-\frac{3}{2}m-mk}.
\end{aligned}$$

□

Proof of Lemma 3.11. The proofs of the evaluations of (3.41) and (3.42) are similar, so we will only do the latter. That $b_0=0$ is trivial. Using Corollay 3.6 and recalling from the proof of Proposition

3.10 that the $m(x,q,z)$ terms sum to zero, we have

$$\begin{aligned} b_1 &= -\sum_{d=0}^{2} \frac{q^{d(d+1)} j(q^{5+2d};q^3) j(q^{5-2d};q^6) J_6^3 j(q^{2d-1};q^6)}{4\overline{J}_{2,8}\overline{J}_{6,24} j(-q^{-2};q^6) j(-q^{1+2d};q^6)} \\ &\quad + q^{-1}\sum_{d=0}^{2} \frac{q^{d(d+1)} j(q^{3+2d};q^3) j(q^{5-2d};q^6) J_6^3 j(q^{3+2d};q^6)}{4\overline{J}_{2,8}\overline{J}_{6,24} j(-q^{2};q^6) j(-q^{1+2d};q^6)} \\ &= -2q^{-1}\frac{J_1 J_{1,6}^2 J_6^3}{4\overline{J}_{2,8}\overline{J}_{6,24}\overline{J}_{2,6}\overline{J}_{1,6}} - 2q^{-1}\frac{J_1 J_{3,6} J_6^3 J_{1,6}}{4\overline{J}_{2,8}\overline{J}_{6,24} j\overline{J}_{2,6}\overline{J}_{3,6}}, \end{aligned}$$

where we have used (3.17a) and simplified. Continuing with the calculation we have

$$\begin{aligned} b_1 &= -2q^{-1}\cdot \frac{J_1 J_{1,6} J_6^3}{4\overline{J}_{2,8}\overline{J}_{6,24}\overline{J}_{2,6}} \cdot \Big[\frac{J_{1,6}\overline{J}_{3,6} + J_{3,6}\overline{J}_{1,6}}{\overline{J}_{1,6}\overline{J}_{3,6}}\Big] \\ &= -2q^{-1}\cdot \frac{J_1 J_{1,6} J_6^3}{4\overline{J}_{2,8}\overline{J}_{6,24}\overline{J}_{2,6}} \cdot \Big[\frac{2J_{4,12}^2}{\overline{J}_{1,6}\overline{J}_{3,6}}\Big], \end{aligned}$$

where we used (3.18). Elementary product rearrangements yield the result. For the final evaluation we recall (3.36)

$$\begin{aligned} b_2 &= f_{1,5,1}(q^4,q^4,q) + q^5 f_{1,5,1}(q^7,q^7,q) \\ &= -q^{-1} f_{1,5,1}(q^3,q^3,q) - q^{-2} f_{1,5,1}(1,1,q) \\ &= -q^{-1} f_{1,5,1}(q^3,q^3,q) - q^{5} f_{1,5,1}(q^6,q^6,q) \\ &= -q^{-1} b_1, \end{aligned}$$

where for the second equality we used (3.28), for the third equality we used (3.29) with $(\ell,k)=(1,1)$, and for the last equality we used (3.36). □

4. Motivation

We explore the use of the heuristic in determining the modularity of triple-sums. First we recall how the heuristic is useful in studying Hecke-type double-sums, see also [HiMo14, Section 3], [Mo14]. We have [Hi88b, (1.15)]

$$\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)} \mathrm{sg}(r) c_{r,s} = \sum_{\mathrm{sg}(r)=\mathrm{sg}(s)} \mathrm{sg}(r) c_{r+\ell,s+k} + \sum_{r=0}^{\ell-1}\sum_{s\in\mathbb{Z}} c_{r,s} + \sum_{s=0}^{k-1}\sum_{r\in\mathbb{Z}} c_{r,s}. \tag{4.43}$$

Letting $\ell,k\to\infty$ in (4.43) we have

$$\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)} \mathrm{sg}(r) c_{r,s} \sim \sum_{r\geq 0}\sum_{s\in\mathbb{Z}} c_{r,s} + \sum_{s\geq 0}\sum_{r\in\mathbb{Z}} c_{r,s}, \tag{4.44}$$

where '$\sim$' again means 'mod theta'. This relation is useful in determining expansions of Hecke-type double-sums [HiMo14]. As an example, consider

$$\begin{aligned} &\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)} \mathrm{sg}(r)(-1)^{r+s} x^r y^s q^{\binom{r}{2}+2rs+\binom{s}{2}} \\ &\qquad \sim \sum_{r=0}^{\infty}\sum_{s\in\mathbb{Z}} (-1)^{r+s} x^r y^s q^{\binom{r}{2}+2rs+\binom{s}{2}} + \sum_{s=0}^{\infty}\sum_{r\in\mathbb{Z}} (-1)^{r+s} x^r y^s q^{\binom{r}{2}+2rs+\binom{s}{2}} \end{aligned}$$

$$\begin{aligned}
&\sim \sum_{r=0}^{\infty}(-1)^r x^r q^{\binom{r}{2}} \sum_{s\in\mathbb{Z}}(-1)^s y^s q^{2rs+\binom{s}{2}} + \sum_{s=0}^{\infty}(-1)^s q^{\binom{s}{2}} y^s \sum_{r\in\mathbb{Z}}(-1)^r x^r q^{\binom{r}{2}+2rs}\\
&\sim \sum_{r=0}^{\infty}(-1)^r x^r q^{\binom{r}{2}} j(yq^{2r};q) + \sum_{s=0}^{\infty}(-1)^s q^{\binom{s}{2}} y^s j(xq^{2s};q)\\
&\sim j(y;q)\sum_{r=0}^{\infty}(-1)^r x^r q^{\binom{r}{2}} y^{-2r} q^{-\binom{2r}{2}} + j(x;q)\sum_{s=0}^{\infty}(-1)^s q^{\binom{s}{2}} y^s x^{-2s} q^{-\binom{2s}{2}}\\
&\sim j(y;q)\sum_{r=0}^{\infty}(-1)^r q^{2r} x^r y^{-2r} q^{-3\binom{r+1}{2}} + j(x;q)\sum_{s=0}^{\infty}(-1)^s q^{2s} y^s x^{-2s} q^{-3\binom{s+1}{2}},
\end{aligned}$$

where in the last line we have used the elliptic transformation property (3.17a). Recalling the heuristic (3.21), we have

$$\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)} \mathrm{sg}(r)(-1)^{r+s} x^r y^s q^{\binom{r}{2}+2rs+\binom{s}{2}} \sim j(y;q)m(q^2x/y^2,q^3,*) + j(x;q)m(q^2y/x^2,q^3,*).$$

The '$\sim$' becomes an equality upon specialising the '$*$' 's and adding an appropriate theta function (2.5).

5. Triple-sum relations

We obtain an identity for triple-sums analogous to (4.43). Without loss of generality, we assume $R,S,T\geq 0$. We write

$$\begin{aligned}
&\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)=\mathrm{sg}(t)} c_{r,s,t} - \sum_{\mathrm{sg}(r)=\mathrm{sg}(s)=\mathrm{sg}(t)} c_{r+R,s+S,t+T}\\
&\qquad = \sum_{\mathrm{sg}(r)=\mathrm{sg}(s)=\mathrm{sg}(t)} c_{r,s,t} - \sum_{\mathrm{sg}(r-R)=\mathrm{sg}(s-S)=\mathrm{sg}(t-T)} c_{r,s,t}\\
&\qquad = \Big(\sum_{r,s,t\geq 0} + \sum_{r,s,t<0}\Big)c_{r,s,t} - \Big(\sum_{r-R,s-S,t-T\geq 0} + \sum_{r-R,s-S,t-T<0}\Big)c_{r,s,t}\\
&\qquad = \Big(\sum_{r,s,t\geq 0} - \sum_{r-R,s-S,t-T\geq 0}\Big)c_{r,s,t} - \Big(\sum_{r-R,s-S,t-T<0} - \sum_{r,s,t<0}\Big)c_{r,s,t}.
\end{aligned}$$

We rewrite some terms to have

$$\begin{aligned}
&\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)=\mathrm{sg}(t)} c_{r,s,t} - \sum_{\mathrm{sg}(r)=\mathrm{sg}(s)=\mathrm{sg}(t)} c_{r+R,s+S,t+T}\\
&\qquad = \Big[\sum_{\substack{0\leq r<R\\ s,t\geq 0}} + \sum_{\substack{r\geq R\\ s\geq S\\ 0\leq t<T}} + \sum_{\substack{r\geq R\\ 0\leq s<S\\ t\geq T}} + \sum_{\substack{r\geq R\\ 0\leq s<S\\ 0\leq t<T}}\Big]c_{r,s,t}\\
&\qquad\quad - \Big[\sum_{\substack{0\leq r<R\\ s<S\\ t<T}} + \sum_{\substack{r<0\\ s<0\\ 0\leq t<T}} + \sum_{\substack{r<0\\ 0\leq s<S\\ t<0}} + \sum_{\substack{r<0\\ 0\leq s<S\\ 0\leq t<T}}\Big]c_{r,s,t}\\
&\qquad = \Big[\sum_{\substack{0\leq r<R\\ s,t\geq 0}} + \Big(\sum_{\substack{r\geq 0\\ s\geq 0\\ 0\leq t<T}} - \sum_{\substack{0\leq r<R\\ s\geq S\\ 0\leq t<T}} - \sum_{\substack{r\geq R\\ 0\leq s<S\\ 0\leq t<T}} - \sum_{\substack{0\leq r<R\\ 0\leq s<S\\ 0\leq t<T}}\Big)
\end{aligned}$$

$$+\Big(\sum_{\substack{r\geq 0\\ 0\leq s<S\\ t\geq 0}}-\sum_{\substack{0\leq r<R\\ 0\leq s<S\\ t\geq T}}-\sum_{\substack{r\geq R\\ 0\leq s<S\\ 0\leq t<T}}-\sum_{\substack{0\leq r<R\\ 0\leq s<S\\ 0\leq t<T}}\Big)+\sum_{\substack{r\geq R\\ 0\leq s<S\\ 0\leq t<T}}\Big]c_{r,s,t}$$
$$-\Big[\Big(\sum_{\substack{0\leq r<R\\ s<0\\ t<0}}+\sum_{\substack{0\leq r<R\\ 0\leq s<S\\ t<0}}+\sum_{\substack{0\leq r<R\\ s<0\\ 0\leq t<T}}+\sum_{\substack{0\leq r<R\\ 0\leq s<S\\ 0\leq t<T}}\Big)+\sum_{\substack{r<0\\ s<0\\ 0\leq t<T}}+\sum_{\substack{r<0\\ 0\leq s<S\\ t<0}}+\sum_{\substack{r<0\\ 0\leq s<S\\ 0\leq t<T}}\Big]c_{r,s,t}.$$

Rearranging terms, we have

$$\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)=\mathrm{sg}(t)}c_{r,s,t}-\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)=\mathrm{sg}(t)}c_{r+R,s+S,t+T} \tag{5.45}$$
$$=\sum_{r=0}^{R-1}\sum_{\mathrm{sg}(s)=\mathrm{sg}(t)}\mathrm{sg}(s)c_{r,s,t}+\sum_{s=0}^{S-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(t)}\mathrm{sg}(r)c_{r,s,t}+\sum_{t=0}^{T-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)}\mathrm{sg}(r)c_{r,s,t}$$
$$-\sum_{r=0}^{R-1}\sum_{t=0}^{T-1}\sum_{s\in\mathbb{Z}}c_{r,s,t}-\sum_{s=0}^{S-1}\sum_{t=0}^{T-1}\sum_{r\in\mathbb{Z}}c_{r,s,t}-\sum_{r=0}^{R-1}\sum_{s=0}^{S-1}\sum_{t\in\mathbb{Z}}c_{r,s,t}.$$

Letting $R,S,T\to\infty$, and discarding the $c_{r+R,s+S,t+T}$ term as in (4.44), we arrive at

$$\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)=\mathrm{sg}(t)}c_{r,s,t} \tag{5.46}$$
$$\sim\sum_{r=0}^{\infty}\sum_{\mathrm{sg}(s)=\mathrm{sg}(t)}\mathrm{sg}(s)c_{r,s,t}+\sum_{s=0}^{\infty}\sum_{\mathrm{sg}(r)=\mathrm{sg}(t)}\mathrm{sg}(r)c_{r,s,t}+\sum_{t=0}^{\infty}\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)}\mathrm{sg}(r)c_{r,s,t}$$
$$-\sum_{r=0}^{\infty}\sum_{t=0}^{\infty}\sum_{s\in\mathbb{Z}}c_{r,s,t}-\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t\in\mathbb{Z}}c_{r,s,t}-\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r\in\mathbb{Z}}c_{r,s,t},$$

where '$\sim$' is 'mod theta' in determining the modularity of triple sums of the form (2.8).

We finish with two propositions whose proofs are straightforward.

Proposition 5.1. *We have*

$$\mathfrak{g}_{a,b,c,d,e,f}(x,y,z,q) \tag{5.47}$$
$$=(-1)^{R+S+T}x^Ry^Sz^Tq^{a\binom{R}{2}+bRS+c\binom{S}{2}+dRT+eST+f\binom{T}{2}}$$
$$\cdot\,\mathfrak{g}_{a,b,c,d,e,f}(q^{aR+bS+dT}x,q^{bR+cS+eT}y,q^{dR+eS+fT}z,q)$$
$$+\sum_{r=0}^{R-1}(-1)^rx^rq^{a\binom{r}{2}}\mathfrak{f}_{c,e,f}(q^{br}y,q^{dr}z,q)+\sum_{s=0}^{S-1}(-1)^sy^sq^{c\binom{s}{2}}\mathfrak{f}_{a,d,f}(q^{bs}x,q^{es}z,q)$$
$$+\sum_{t=0}^{T-1}(-1)^tz^tq^{f\binom{t}{2}}\mathfrak{f}_{a,b,c}(q^{dt}x,q^{et}y,q)$$
$$-\sum_{r=0}^{R-1}(-1)^rx^rq^{a\binom{r}{2}}\sum_{t=0}^{T-1}(-1)^tz^tq^{drt+f\binom{t}{2}}j(q^{br+et}y;q^c)$$
$$-\sum_{s=0}^{S-1}(-1)^sy^sq^{c\binom{s}{2}}\sum_{t=0}^{T-1}(-1)^tz^tq^{est+f\binom{t}{2}}j(q^{bs+dt}x;q^a)$$
$$-\sum_{r=0}^{R-1}(-1)^rx^rq^{a\binom{r}{2}}\sum_{s=0}^{S-1}(-1)^sy^sq^{brs+c\binom{t}{2}}j(q^{dr+es}z;q^f).$$

Proposition 5.2. *We have*

$$\mathfrak{g}_{a,b,c,d,e,f}(x,y,z,q) = -\frac{q^{a+b+c+d+e+f}}{xyz}\mathfrak{g}_{a,b,c,d,e,f}\Big(\frac{q^{2a+b+d}}{x},\frac{q^{b+2c+e}}{y},\frac{q^{d+e+2f}}{z},q\Big). \tag{5.48}$$

6. A heuristic guide to mock theta function identities

The left-hand sides of identities (2.6) and (2.7) may be viewed as specializations of

$$\mathfrak{g}_{1,2,1,2,2,1}(x,y,z,q) := \Big(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\Big)(-1)^{k+l+m}q^{\binom{k}{2}+2kl+\binom{l}{2}+2km+2lm+\binom{m}{2}}x^k y^l z^m. \tag{6.49}$$

In particular, Identities (2.6) and (2.7) may be rewritten

$$2-\frac{1}{J_1^2}\cdot \mathfrak{g}_{1,2,1,2,2,1}(q,q,q,q) = \chi_0(q), \tag{6.50}$$

$$\frac{1}{J_1^2}\cdot \mathfrak{g}_{1,2,1,2,2,1}(q^2,q^2,q^2,q) = \chi_1(q). \tag{6.51}$$

Using (2.2) and (2.3), we may rewrite the above identities as

$$2-\frac{1}{J_1^2}\cdot \mathfrak{g}_{1,2,1,2,2,1}(q,q,q,q) = 2-3m(q^7,q^{15},q^9)-3q^{-1}m(q^2,q^{15},q^4)+\frac{2J_5^2J_{2,5}}{J_{1,5}^2}, \tag{6.52}$$

$$\frac{1}{J_1^2}\cdot \mathfrak{g}_{1,2,1,2,2,1}(q^2,q^2,q^2,q) = -3q^{-1}m(q^4,q^{15},q^3)-3q^{-2}m(q,q^{15},q^2)-\frac{2J_5^2J_{1,5}}{J_{2,5}^2}. \tag{6.53}$$

In this section, we demonstrate how heuristic methods take us from the left-hand sides of (6.52) and (6.53) to their respective right-hand sides up to a theta function.

Let us apply the heuristic to (6.49). We recall the (slightly rewritten) general form (5.46):

$$\begin{aligned}&\sum_{\mathrm{sg}(k)=\mathrm{sg}(l)=\mathrm{sg}(m)} c_{k,l,m} \\ &\sim \sum_{k=0}^{\infty}\sum_{\mathrm{sg}(l)=\mathrm{sg}(m)}\mathrm{sg}(l)c_{k,l,m}+\sum_{l=0}^{\infty}\sum_{\mathrm{sg}(k)=\mathrm{sg}(m)}\mathrm{sg}(k)c_{k,l,m}+\sum_{m=0}^{\infty}\sum_{\mathrm{sg}(k)=\mathrm{sg}(l)}\mathrm{sg}(k)c_{k,l,m}\\ &\quad -\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l\in\mathbb{Z}}c_{k,l,m}-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m\in\mathbb{Z}}c_{k,l,m}-\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k\in\mathbb{Z}}c_{k,l,m}.\end{aligned} \tag{6.54}$$

We have two types of summands on the right-hand side of (6.54). We consider the first type of summand:

$$\begin{aligned}&\sum_{k\geq 0}\sum_{l\geq 0}\sum_{m\in\mathbb{Z}}(-1)^{k+l+m}q^{\binom{k}{2}+\binom{l}{2}+\binom{m}{2}+2kl+2km+2lm}x^ky^lz^m\\ &\quad\sim\sum_{k\geq 0}(-1)^kx^kq^{\binom{k}{2}}\sum_{l\geq 0}(-1)^ly^lq^{\binom{l}{2}+2kl}\sum_{m\in\mathbb{Z}}(-1)^mq^{\binom{m}{2}+2km+2lm}z^m\\ &\quad\sim\sum_{k\geq 0}(-1)^kx^kq^{\binom{k}{2}}\sum_{l\geq 0}(-1)^ly^lq^{\binom{l}{2}+2kl}j(zq^{2k+2l};q).\end{aligned} \tag{6.55}$$

We note that if we set z to be q or q^2 then $j(z;q)=0$, so we can ignore the contributions from the second line of (6.54). We consider the second type of summand:

$$\sum_{m\geq 0}\sum_{k,l}\mathrm{sg}(k,l)(-1)^{k+l+m}q^{\binom{k}{2}+\binom{l}{2}+\binom{m}{2}+2kl+2km+2lm}x^ky^lz^m \tag{6.56}$$

$$\sim \sum_{m\geq 0}(-1)^m q^{\binom{m}{2}} z^m \sum_{k,l} \mathrm{sg}(k,l)(-1)^{k+l} q^{\binom{k}{2}+\binom{l}{2}+2kl+2km+2lm} x^k y^l$$
$$\sim \sum_{m\geq 0}(-1)^m q^{\binom{m}{2}} z^m \sum_{k,l} \mathrm{sg}(k,l)(-1)^{k+l} q^{\binom{k}{2}+2kl+\binom{l}{2}} (q^{2m}x)^k (q^{2m}y)^l$$
$$\sim \sum_{m\geq 0}(-1)^m q^{\binom{m}{2}} z^m \mathfrak{f}_{1,2,1}(q^{2m}x, q^{2m}y, q).$$

We rewrite the last line of (6.56). We recall (2.5):

$$\mathfrak{f}_{1,2,1}(x,y,q) \\ = j(y;q)m\big(\tfrac{q^2x}{y^2}, q^3, -1\big) + j(x;q)m\big(\tfrac{q^2y}{x^2}, q^3, -1\big) - \frac{yJ_3^3 j(-x/y;q)j(q^2xy;q^3)}{\overline{J}_{0,3} j(-qy^2/x, -qx^2/y;q^3)}. \tag{6.57}$$

Let us consider the double-sum:

$$\mathfrak{f}_{1,2,1}(q^{2m}x, q^{2m}y, q). \tag{6.58}$$

The first thing we note is that if x and y are either q or q^2, then theta coefficients of the $m(x,q,z)$ functions in (6.57) are both zero, and both $m(x,q,z)$ functions are defined. So we only need to consider the theta function from the right-hand side of (6.57) which after $x \to q^{2m}x$ and $y \to q^{2m}y$ is

$$\mathfrak{f}_{1,2,1}(q^{2m}x, q^{2m}y, q) = -\frac{yq^{2m}J_3^3 j(-x/y;q)j(q^{2+4m}xy;q^3)}{\overline{J}_{0,3} j(-q^{1+2m}y^2/x, -q^{1+2m}x^2/y;q^3)}. \tag{6.59}$$

In order to use the elliptic transformation property (3.17a), we will have to consider m mod three, so let $m \to 3m+a$. We first consider the case $a=0$

$$-\sum_{m\geq 0}(-1)^m q^{\binom{3m}{2}} z^{3m} \frac{yq^{6m}J_3^3 j(-x/y;q)j(q^{2+12m}xy;q^3)}{\overline{J}_{0,3} j(-q^{1+6m}y^2/x, -q^{1+6m}x^2/y;q^3)}$$
$$\sim -\frac{yJ_3^3 j(-x/y;q)j(q^2xy;q^3)}{\overline{J}_{0,3} j(-qy^2/x, -qx^2/y;q^3)} \cdot \sum_{m\geq 0}(-1)^m q^{\binom{3m}{2}} z^{3m} q^{6m} \frac{(q^2xy)^{-4m} q^{-3\binom{4m}{2}}}{(-qy^2/x)^{-2m}(-qx^2/y)^{-2m} q^{-6\binom{2m}{2}}}$$
$$\sim -\frac{yJ_3^3 j(-x/y;q)j(q^2xy;q^3)}{\overline{J}_{0,3} j(-qy^2/x, -qx^2/y;q^3)} \sum_{m\geq 0}(-1)^m \Big(\frac{z^3}{x^2y^2}\Big)^m q^{-15m^2/2+m/2}$$
$$\sim -\frac{yJ_3^3 j(-x/y;q)j(q^2xy;q^3)}{\overline{J}_{0,3} j(-qy^2/x, -qx^2/y;q^3)} m\Big(\frac{q^8z^3}{x^2y^2}, q^{15}, *\Big).$$

We consider $m \to 3m+1$. Then

$$\sum_{m\geq 0}(-1)^m q^{\binom{3m+1}{2}} z^{3m+1} \frac{yq^{6m+2}J_3^3 j(-x/y;q)j(q^{6+12m}xy;q^3)}{\overline{J}_{0,3} j(-q^{3+6m}y^2/x, -q^{3+6m}x^2/y;q^3)}$$
$$\sim \frac{yJ_3^3 j(-x/y;q)j(xy;q^3)}{\overline{J}_{0,3} j(-y^2/x, -x^2/y;q^3)} \cdot \sum_{m\geq 0}(-1)^m q^{\binom{3m+1}{2}} z^{3m+1} q^{6m+2} \frac{(xy)^{-4m-2} q^{-3\binom{4m+2}{2}}}{(-y^2/x)^{-2m-1}(-x^2/y)^{-2m-1} q^{-6\binom{2m+1}{2}}}$$
$$\sim \frac{yJ_3^3 j(-x/y;q)j(xy;q^3)}{\overline{J}_{0,3} j(-y^2/x, -x^2/y;q^3)} \frac{z}{xy} \sum_{m\geq 0}(-1)^m \Big(\frac{z^3}{x^2y^2}\Big)^m q^{-15m^2/2-9m/2-1}$$

$$\sim \frac{yJ_3^3 j(-x/y;q)j(xy;q^3)}{\overline{J}_{0,3}j(-y^2/x,-x^2/y;q^3)}\cdot\frac{z}{xyq}\cdot m\Big(\frac{q^3z^3}{x^2y^2},q^{15},*\Big).$$

We let $m\to 3m+2$. We have

$$-\sum_{m\geq 0}(-1)^{3m+2}q^{\binom{3m+2}{2}}z^{3m+2}\frac{yq^{6m+4}J_3^3 j(-x/y;q)j(q^{12m+10}xy;q^3)}{\overline{J}_{0,3}j(-q^{6m+5}y^2/x,-q^{6m+5}x^2/y;q^3)}$$
$$\sim -\frac{yJ_3^3 j(-x/y;q)j(qxy;q^3)}{\overline{J}_{0,3}j(-q^2y^2/x,-q^2x^2/y;q^3)}$$
$$\cdot\sum_{m\geq 0}(-1)^m q^{\binom{3m+2}{2}}z^{3m+2}q^{6m+4}\frac{(-1)^{4m+3}q^{-3\binom{4m+3}{2}}(qxy)^{-(4m+3)}}{q^{-6\binom{2m+1}{2}}(q^2x^2/y)^{-(2m+1)}(q^2y^2/x)^{-(2m+1)}}$$
$$\sim \frac{yJ_3^3 j(-x/y;q)j(qxy;q^3)}{\overline{J}_{0,3}j(-q^2y^2/x,-q^2x^2/y;q^3)}\cdot\frac{z^2}{x^2y^2}\cdot\sum_{m\geq 0}(-1)^m\Big(\frac{z^3}{x^2y^2}\Big)^m q^{-15m^2/2-19m/2-3}$$
$$\sim \frac{yJ_3^3 j(-x/y;q)j(qxy;q^3)}{\overline{J}_{0,3}j(-q^2y^2/x,-q^2x^2/y;q^3)}\cdot\frac{z^2}{x^2y^2q^3}\cdot m\Big(\frac{z^3}{q^2x^2y^2},q^{15},*\Big).$$

For x,y,z integral powers of q, we suspect that we have

$$\begin{aligned}\mathfrak{g}_{1,2,1,2,2,1}&(x,y,z,q)\\ &\sim -y\cdot\frac{J_3^3 j(-x/y;q)j(q^2xy;q^3)}{\overline{J}_{0,3}j(-qy^2/x,-qx^2/y;q^3)}\cdot m\Big(\frac{q^8z^3}{x^2y^2},q^{15},*\Big)+idem(z;x,y)\\ &\quad+\frac{z}{xq}\cdot\frac{J_3^3 j(-x/y;q)j(xy;q^3)}{\overline{J}_{0,3}j(-y^2/x,-x^2/y;q^3)}\cdot m\Big(\frac{q^3z^3}{x^2y^2},q^{15},*\Big)+idem(z;x,y)\\ &\quad+\frac{z^2}{x^2yq^3}\cdot\frac{J_3^3 j(-x/y;q)j(qxy;q^3)}{\overline{J}_{0,3}j(-q^2y^2/x,-q^2x^2/y;q^3)}\cdot m\Big(\frac{z^3}{q^2x^2y^2},q^{15},*\Big)+idem(z;x,y),\end{aligned}\tag{6.60}$$

where '$\sim$' means modulo a theta function and $idem(z;x;y)$ means the preceding term is repeated twice–once with z and x swapped and once with z and y swapped.

Let us try the values for $\chi_0(q)$, i.e. $x=y=z=q$. In the third summand on the the right-hand side, we have a $j(q^3;q^3)=0$ in the numerator of the quotient of theta functions, so only the first two summands contribute. Using (2.6) gives

$$\begin{aligned}2-\chi_0(q)&=\frac{1}{J_1^2}\cdot\mathfrak{g}_{1,2,1,2,2,1}(q,q,q,q)\\ &\sim -3q\frac{J_3^3 j(-1;q)j(q^4;q^3)}{J_1^2\overline{J}_{0,3}j(-q^2;q^3)^2}m(q^7,q^{15},*)+3q^{-1}\frac{J_3^3 j(-1;q)j(q^2;q^3)}{J_1^2\overline{J}_{0,3}j(-q;q^3)^2}m(q^2,q^{15},*)\\ &\sim 3\frac{J_3^3\overline{J}_{0,1}J_1}{J_1^2\overline{J}_{0,3}\overline{J}_{1,3}^2}m(q^7,q^{15},*)+3q^{-1}\frac{J_3^3\overline{J}_{0,1}J_1}{J_1^2\overline{J}_{0,3}\overline{J}_{1,3}^2}m(q^2,q^{15},*)\\ &\sim 3m(q^7,q^{15},*)+3q^{-1}m(q^2,q^{15},*),\end{aligned}$$

where we have used (3.17a) and product rearrangements. This agrees with (2.2) up to a theta function.

Let us the values for $\chi_0(q)$, i.e. $x=y=z=q^2$. In the first summand on the the right-hand side, we have a $j(q^6;q^3)=0$ in the numerator of the quotient of theta functions, so only the second and third summands contribute. Using (2.7), we arrive at

$$\begin{aligned}\chi_1(q)=\frac{1}{J_1^2}\cdot\mathfrak{g}_{1,2,1,2,2,1}(q^2,q^2,q^2,q)&\sim -3q^{-2}m(q,q^{15},*)-3q^{-5}m(q^{-4},q^{15},*)\\ &\sim -3q^{-2}m(q,q^{15},*)-3q^{-1}m(q^4,q^{15},*),\end{aligned}$$

where we have used (3.19b). This agrees with (2.3) up to a theta function.

7. A heuristic guide to false theta function identities

We recall a theorem of Kim and Lovejoy:

Theorem 7.1. *[KiLo17, Theorem 1.1] We have*

$$\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}=(1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{\frac{3}{2}+rs+\frac{1}{2}r+s}$$
$$+\frac{(q)_\infty}{(aq,q/a)_\infty}\sum_{r\geq 0}(-1)^r a^{2r+1}q^{3r(r+1)/2}, \tag{7.61}$$
$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(q)_nq^n}{(aq,q/a)_n}=(1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{rs+\frac{1}{2}r+\frac{1}{2}s}$$
$$+\frac{(q)_\infty}{(aq,q/a,-q)_\infty}\sum_{r\geq 0}(-1)^r a^{3r+1}q^{3r^2+2r}(1+aq^{2r+1}). \tag{7.62}$$

We collect the $a=1$ specializations in the following corollary:

Corollary 7.2. *We have*

$$\sum_{n=0}^{\infty}\frac{(q;q)_{2n}q^n}{(q;q)_n^2}=\frac{1}{J_1}\sum_{r=0}^{\infty}(-1)^r q^{3r(r+1)/2}, \tag{7.63}$$
$$\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^n}{(q;q)_n}=\frac{1}{J_2}\sum_{r=0}^{\infty}(-1)^r q^{3r^2+2r}(1+q^{2r+1}). \tag{7.64}$$

We point out that (7.64) is the $a=-1$ specialization of [Wa03, (5.2)]. In [KiLo17], Kim and Lovejoy also show how to write [KiLo17, Theorem 1.1] in terms of Hecke-type triple-sums.

Corollary 7.3. *[KiLo17, Propositions 5.1, 5.2] We have*

$$\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}=\frac{1}{(q,aq,q/a;q)_\infty}\Big(\mathfrak{g}_{1,7,1,1,1,1}(aq^2,q^3,q,q) \tag{7.65}$$
$$+q^4\mathfrak{g}_{1,7,1,1,1,1}(aq^6,q^7,q^2,q)\Big),$$
$$\sum_{n\geq 0}\frac{(q;q^2)_n(q)_nq^n}{(aq,q/a)_n}=\frac{1}{(q,aq,q/a;q)_\infty}\Big(\mathfrak{g}_{1,5,1,1,1,1}(aq^2,q^2,q,q) \tag{7.66}$$
$$+q^3\mathfrak{g}_{1,5,1,1,1,1}(aq^5,q^5,q^2,q)\Big).$$

In this section, we will use the triple-sum relation (5.46) and the Hecke-type double-sum expansions of [HiMo14] in order to guide us from the $a=1$ specialization of Corollary 7.3 to the false theta functions of Corollary 7.2.

7.A. On the triple-sum for the false theta function Identity (7.63)

We apply our heuristic methods to

$$\mathfrak{g}_{1,7,1,1,1,1}(q^2,q^3,q,q)+q^4\mathfrak{g}_{1,7,1,1,1,1}(q^6,q^7,q^2,q).$$

We consider the contributions of $\mathfrak{g}_{1,7,1,1,1,1}(q^2, q^3, q, q)$ to the top row of (5.45). We have

$$\begin{aligned}\sum_{r=0}^{R-1}\sum_{\mathrm{sg}(s)=\mathrm{sg}(t)} \mathrm{sg}(s)c_{r,s,t} &= \sum_{r=0}^{R-1}(-1)^r q^{2r} q^{\binom{r}{2}} \sum_{s,t} \mathrm{sg}(s,t)(-1)^{s+t} q^{3s} q^t q^{7rs+\binom{s}{2}+rt+st+\binom{t}{2}}\\ &= \sum_{r=0}^{R-1}(-1)^r q^{2r} q^{\binom{r}{2}} \mathfrak{f}_{1,1,1}(q^{3+7r}, q^{1+r}, q)\\ &= 0,\end{aligned}$$

where for the last equality we used Proposition 3.7. Similarly, we have

$$\sum_{s=0}^{S-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(t)} \mathrm{sg}(r)c_{r,s,t} = \sum_{s=0}^{S-1}(-1)^s q^{3s} q^{\binom{s}{2}} \mathfrak{f}_{1,1,1}(q^{2+7s}, q^{1+s}, q) = 0.$$

For the final piece in the top row of (5.45), we have

$$\sum_{t=0}^{T-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)} \mathrm{sg}(r)c_{r,s,t} = \sum_{t=0}^{T-1}(-1)^t q^t q^{\binom{t}{2}} \mathfrak{f}_{1,7,1}(q^{2+t}, q^{3+t}, q).$$

We consider the bottom row of (5.45). We have

$$\begin{aligned}\sum_{r=0}^{R-1}\sum_{s=0}^{S-1}\sum_{t\in\mathbb{Z}} c_{r,s,t} &= \sum_{r=0}^{R-1}\sum_{s=0}^{S-1}(-1)^{r+s} q^{2r} q^{3s} q^{\binom{r}{2}+7rs+\binom{s}{2}} \sum_{t\in\mathbb{Z}}(-1)^t q^{(1+r+s)t} q^{\binom{t}{2}}\\ &= \sum_{r=0}^{R-1}\sum_{s=0}^{S-1}(-1)^{r+s} q^{2r} q^{3s} q^{\binom{r}{2}+7rs+\binom{s}{2}} j(q^{1+r+s}; q)\\ &= 0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{r=0}^{R-1}\sum_{t=0}^{T-1}\sum_{t\in\mathbb{Z}} c_{r,s,t} &= \sum_{r=0}^{R-1}\sum_{t=0}^{T-1}(-1)^{r+t} q^{2r} q^t q^{\binom{r}{2}+rt+\binom{t}{2}} j(q^{3+t+7r}; q) = 0,\\ \sum_{s=0}^{S-1}\sum_{t=0}^{T-1}\sum_{r\in\mathbb{Z}} c_{r,s,t} &= \sum_{s=0}^{S-1}\sum_{t=0}^{T-1}(-1)^{s+t} q^{3s} q^t q^{\binom{s}{2}+st+\binom{r}{2}} j(q^{2+7s+t}; q) = 0.\end{aligned}$$

We proceed to work on the second sum $\mathfrak{g}_{1,7,1,1,1,1}(q^6, q^7, q^2, q)$. We consider the contributions to the top row of (5.45). Arguing as above we have

$$\begin{aligned}\sum_{r=0}^{R-1}\sum_{\mathrm{sg}(s)=\mathrm{sg}(t)} \mathrm{sg}(s)c_{r,s,t} &= \sum_{r=0}^{R-1}(-1)^r q^{6r} q^{\binom{r}{2}} \mathfrak{f}_{1,1,1}(q^{7+7r}, q^{2+r}, q) = 0,\\ \sum_{s=0}^{S-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(t)} \mathrm{sg}(r)c_{r,s,t} &= \sum_{s=0}^{S-1}(-1)^s q^{7s} q^{\binom{s}{2}} \mathfrak{f}_{1,1,1}(q^{6+7s}, q^{2+s}, q) = 0,\\ \sum_{t=0}^{T-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)} \mathrm{sg}(r)c_{r,s,t} &= \sum_{t=0}^{T-1}(-1)^t q^{2t} q^{\binom{t}{2}} \mathfrak{f}_{1,7,1}(q^{6+t}, q^{7+t}, q).\end{aligned}$$

For the bottom row, we have

$$\sum_{r=0}^{R-1}\sum_{s=0}^{S-1}\sum_{t\in\mathbb{Z}} c_{r,s,t} = \sum_{r=0}^{R-1}\sum_{s=0}^{S-1}(-1)^{r+s}q^{6r}q^{7s}q^{\binom{r}{2}+7rs+\binom{s}{2}}j(q^{2+r+s};q) = 0,$$
$$\sum_{r=0}^{R-1}\sum_{t=0}^{T-1}\sum_{t\in\mathbb{Z}} c_{r,s,t} = \sum_{r=0}^{R-1}\sum_{t=0}^{T-1}(-1)^{r+t}q^{6r}q^{2t}q^{\binom{r}{2}+rt+\binom{t}{2}}j(q^{6+7r+t};q) = 0,$$
$$\sum_{s=0}^{S-1}\sum_{t=0}^{T-1}\sum_{r\in\mathbb{Z}} c_{r,s,t} = \sum_{s=0}^{S-1}\sum_{t=0}^{T-1}(-1)^{s+t}q^{7s}q^{2t}q^{\binom{s}{2}+st+\binom{t}{2}}j(q^{6+7s+t};q) = 0.$$

Combining terms and letting $R,S,T\to\infty$ as in (5.46), gives

$$\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(q)_n^2} \sim \frac{1}{J_1^3}\Big[\sum_{t=0}^{\infty}(-1)^t q^{\binom{t+1}{2}}\Big(\mathfrak{f}_{1,7,1}(q^{2+t},q^{3+t},q)+q^{4+t}\mathfrak{f}_{1,7,1}(q^{6+t},q^{7+t},q)\Big)\Big].$$

Using Identity (3.35) gives

$$\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(q)_n^2} \sim \frac{1}{J_1^3}\Big[\sum_{m=0}^{\infty}(-1)^m q^{\binom{m+1}{2}}\Big(\mathfrak{f}_{4,4,1}(-q^{4+m},q^{2+m},q)-q^{-m}\mathfrak{f}_{4,4,1}(-q^{4-m},q^{2-m},q)\Big)\Big].$$

Making the substitution $m\to 4m+k$, $k\in\{0,1,2,3\}$, and using Proposition 3.9 yields

$$\begin{aligned}
&\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(q)_n^2}\\
&\quad\sim \frac{1}{J_1^3}\sum_{k=0}^{3}\Big(\mathfrak{f}_{4,4,1}(-q^{4+k},q^{2+k},q)-q^{-k}\mathfrak{f}_{4,4,1}(-q^{4-k},q^{2-k},q)\Big)\\
&\qquad\cdot\Big[\sum_{m=0}^{\infty}(-1)^{4m+k}q^{\binom{4m+k+1}{2}}q^{-2m^2-2m-mk}\Big]\\
&\quad\sim \frac{1}{J_1^3}\sum_{k=0}^{3}\Big[(-1)^k q^{\binom{k+1}{2}}\Big(\mathfrak{f}_{4,4,1}(-q^{4+k},q^{2+k},q)-q^{-k}\mathfrak{f}_{4,4,1}(-q^{4-k},q^{2-k},q)\Big)\cdot\sum_{m=0}^{\infty}q^{6m^2+3mk}\Big].
\end{aligned}$$

We recall the evaluations found in (3.41). If we define

$$a_k := \mathfrak{f}_{4,4,1}(-q^{4+k},q^{2+k},q)-q^{-k}\mathfrak{f}_{4,4,1}(-q^{4-k},q^{2-k},q),$$

then $a_0=0$, $a_1=-q^{-1}J_1^2$, $a_2=0$, and $a_3=q^{-3}J_1^2$. Finally, we arrive at Identity (7.63):

$$\begin{aligned}
\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(q)_n^2} &\sim \frac{1}{J_1^3}\Big[0-q\cdot(-q^{-1}J_1^2)\sum_{m=0}^{\infty}q^{6m^2+3m}+0-q^6(q^{-3}J_1^2)\sum_{m=0}^{\infty}q^{6m^2+9m}\Big]\\
&\sim \frac{1}{J_1}\Big[\sum_{m=0}^{\infty}q^{6m^2+3m}-q^3\sum_{m=0}^{\infty}q^{6m^2+9m}\Big]\\
&\sim \frac{1}{J_1}\sum_{m=0}^{\infty}(-1)^m q^{3m(m+1)/2}.
\end{aligned}$$

7.B. On the triple-sum for the false theta function Identity (7.64)

We recall our triple-sum

$$\mathfrak{g}_{1,5,1,1,1,1}(q^2,q^2,q,q)+q^3\mathfrak{g}_{1,5,1,1,1,1}(q^5,q^5,q^2,q).$$

We consider the contributions of $\mathfrak{g}_{1,5,1,1,1,1}(q^2,q^2,q,q)$ to the top row of (5.45). Arguing as in the previous subsection, we have

$$\sum_{r=0}^{R-1}\sum_{\mathrm{sg}(s)=\mathrm{sg}(t)}\mathrm{sg}(s)c_{r,s,t}=\sum_{r=0}^{R-1}(-1)^r q^{2r}q^{\binom{r}{2}}\mathfrak{f}_{1,1,1}(q^{2+5r},q^{1+r},q)=0,$$
$$\sum_{s=0}^{S-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(t)}\mathrm{sg}(r)c_{r,s,t}=\sum_{s=0}^{S-1}(-1)^s q^{2s}q^{\binom{s}{2}}\mathfrak{f}_{1,1,1}(q^{2+5s},q^{1+s},q)=0,$$
$$\sum_{t=0}^{T-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)}\mathrm{sg}(r)c_{r,s,t}=\sum_{t=0}^{T-1}(-1)^t q^{t}q^{\binom{t}{2}}\mathfrak{f}_{1,5,1}(q^{2+t},q^{2+t},q).$$

For the bottom row, we have

$$\sum_{r=0}^{R-1}\sum_{t=0}^{T-1}\sum_{s\in\mathbb{Z}}c_{r,s,t}=\sum_{r=0}^{R-1}\sum_{t=0}^{T-1}(-1)^{r+t}q^{2r}q^{t}q^{\binom{r}{2}+rt+\binom{t}{2}}j(q^{2+5r+t};q)=0,$$
$$\sum_{s=0}^{S-1}\sum_{t=0}^{T-1}\sum_{r\in\mathbb{Z}}c_{r,s,t}=\sum_{s=0}^{S-1}\sum_{t=0}^{T-1}(-1)^{st}q^{2s}q^{t}q^{\binom{s}{2}+st+\binom{t}{2}}j(q^{2+5s+t};q)=0,$$
$$\sum_{r=0}^{R-1}\sum_{s=0}^{S-1}\sum_{t\in\mathbb{Z}}c_{r,s,t}=\sum_{r=0}^{R-1}\sum_{s=0}^{S-1}(-1)^{r+s}q^{2r}q^{2s}q^{\binom{r}{2}+5rs+\binom{s}{2}}j(q^{1+r+s};q)=0.$$

We consider the contributions of $\mathfrak{g}_{1,5,1,1,1,1}(q^5,q^5,q^2,q)$ to the top row of (5.45). Computing the summands yields

$$\sum_{r=0}^{R-1}\sum_{\mathrm{sg}(s)=\mathrm{sg}(t)}\mathrm{sg}(s)c_{r,s,t}=\sum_{r=0}^{R-1}(-1)^r q^{5r}q^{\binom{r}{2}}\mathfrak{f}_{1,1,1}(q^{5+5r},q^{2+r},q)=0,$$
$$\sum_{s=0}^{S-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(t)}\mathrm{sg}(r)c_{r,s,t}=\sum_{s=0}^{S-1}(-1)^s q^{5s}q^{\binom{s}{2}}\mathfrak{f}_{1,1,1}(q^{5+5s},q^{2+s},q)=0,$$
$$\sum_{t=0}^{T-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)}\mathrm{sg}(r)c_{r,s,t}=\sum_{t=0}^{T-1}(-1)^t q^{2t}q^{\binom{t}{2}}\mathfrak{f}_{1,5,1}(q^{5+t},q^{5+t},q).$$

For the bottom row, we have as before

$$\sum_{r=0}^{R-1}\sum_{s=0}^{S-1}\sum_{t\in\mathbb{Z}}c_{r,s,t}=\sum_{r=0}^{R-1}\sum_{s=0}^{S-1}(-1)^{r+s}q^{5r}q^{5s}q^{\binom{r}{2}+5rs+\binom{s}{2}}j(q^{2+r+s};q)=0,$$
$$\sum_{r=0}^{R-1}\sum_{t=0}^{T-1}\sum_{s\in\mathbb{Z}}c_{r,s,t}=\sum_{r=0}^{R-1}\sum_{t=0}^{T-1}(-1)^{r+t}q^{5r}q^{2t}q^{\binom{r}{2}+rt+\binom{t}{2}}j(q^{5+5r+t};q)=0,$$
$$\sum_{s=0}^{S-1}\sum_{t=0}^{T-1}\sum_{r\in\mathbb{Z}}c_{r,s,t}=\sum_{s=0}^{S-1}\sum_{t=0}^{T-1}(-1)^{s+t}q^{5s}q^{2t}q^{\binom{s}{2}+st+\binom{t}{2}}j(q^{5+5s+t};q)=0.$$

Combining terms and letting $R, S, T \to \infty$ as in (5.46), gives

$$\sum_{n\geq 0} \frac{(q;q^2)_n q^n}{(q)_n} \sim \frac{1}{J_1^3}\Big[\sum_{t=0}^{\infty}(-1)^t q^{\binom{t+1}{2}}\Big(\mathfrak{f}_{1,5,1}(q^{2+t},q^{2+t},q) + q^{3+t}\mathfrak{f}_{1,5,1}(q^{5+t},q^{5+t},q)\Big)\Big].$$

Using Identity (3.36) gives

$$\sum_{n\geq 0} \frac{(q)_{2n} q^n}{(q)_n^2} \sim \frac{1}{J_1^3}\Big[\sum_{m=0}^{\infty}(-1)^m q^{\binom{m+1}{2}}\Big(\mathfrak{f}_{3,3,1}(-q^{3+m},q^{2+m},q) - q^{-m}\mathfrak{f}_{3,3,1}(-q^{3-m},q^{2-m},q)\Big)\Big].$$

Making the substitution $m \to 3m+k$, $k \in \{0,1,2\}$, and using Proposition 3.10 yields

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(q)_n} &\sim \frac{1}{J_1^3}\sum_{k=0}^{2}\Big(f_{3,3,1}(-q^{3+k},q^{2+k},q) - q^{-k}f_{3,3,1}(-q^{3-k},q^{2-k},q)\Big)\\
&\qquad \cdot \Big[\sum_{m=0}^{\infty}(-1)^{3m+k}q^{\binom{3m+k+1}{2}}q^{-\frac{3}{2}m^2-\frac{3}{2}m-mk}\Big]\\
&\sim \frac{1}{J_1^3}\sum_{k=0}^{2}\Big[(-1)^k q^{\binom{k+1}{2}}\Big(f_{3,3,1}(-q^{3+k},q^{2+k},q) - q^{-k}f_{3,3,1}(-q^{3-k},q^{2-k},q)\Big)\\
&\qquad \cdot \sum_{m=0}^{\infty}(-1)^m q^{3m^2+2mk}\Big].
\end{aligned}$$

We recall the evaluations found in (3.42). If we define

$$b_k := \mathfrak{f}_{3,3,1}(-q^{3+k},q^{2+k},q) - q^{-k}\mathfrak{f}_{3,3,1}(-q^{3-k},q^{2-k},q),$$

then $b_0 = 0$, $b_1 = -q^{-1}J_1^3/J_2$, and $b_2 = q^{-2}J_1^3/J_2$. Finally, we arrive at Identity (7.64):

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(q)_n}\\
&\quad \sim \frac{1}{J_1^3}\Big[0 - q(-q^{-1}J_1^3/J_2)\sum_{m=0}^{\infty}(-1)^m q^{3m^2+2m} + q^3(q^{-2}J_1^3/J_2)\sum_{m=0}^{\infty}(-1)^m q^{3m^2+4m}\Big]\\
&\quad \sim \frac{1}{J_2}\sum_{m=0}^{\infty}(-1)^m q^{3m^2+2m}(1+q^{2m+1}).
\end{aligned}$$

8. Proof of Corollary 2.1

We present some corollaries to identities (2.6) and (2.7). Identity (2.11) follows from (5.48). For Identity (2.12), we employ (5.47) with $(R,S,T) = (0,0,1)$ to obtain

$$\mathfrak{g}_{1,2,1,2,2,1}(q,q,q^2,q) = -q^2\mathfrak{g}_{1,2,1,2,2,1}(q^3,q^3,q^3,q) + \mathfrak{f}_{1,2,1}(q,q,q) = \mathfrak{f}_{1,2,1}(q,q,q) = J_1^2,$$

where we used (2.11) for the second equality and (2.5) for the third equality. For Identity (2.13), we use (5.47) with $(R,S,T) = (0,1,1)$ to have

$$\begin{aligned}
\mathfrak{g}_{1,2,1,2,2,1}(q,q^2,q^2,q) &= q^6\mathfrak{g}_{1,2,1,2,2,1}(q^5,q^5,q^5,q) + 2\mathfrak{f}_{1,2,1}(q,q^2,q)\\
&= -\mathfrak{g}_{1,2,1,2,2,1}(q,q,q,q) + 2\mathfrak{f}_{1,2,1}(q,q^2,q)\\
&= -\mathfrak{g}_{1,2,1,2,2,1}(q,q,q,q) + 2J_1^2,
\end{aligned}$$

where the second equality follows from (5.48) and the third equality from (2.5). Comparing the last equality with (2.6) gives the result. for Identity (2.14), we again use (5.47) but with $(R,S,T)=(0,0,2)$ to obtain

$$\begin{aligned}\mathfrak{g}_{1,2,1,2,2,1}(q,q,q^3,q) &= q^7\mathfrak{g}_{1,2,1,2,2,1}(q^5,q^5,q^5,q)+\mathfrak{f}_{1,2,1}(q,q,q)-q^3\mathfrak{f}_{1,2,1}(q^3,q^3,q)\\ &= -q\mathfrak{g}_{1,2,1,2,2,1}(q,q,q,q)+\mathfrak{f}_{1,2,1}(q,q,q)-q^3\mathfrak{f}_{1,2,1}(q^3,q^3,q)\\ &= -q\mathfrak{g}_{1,2,1,2,2,1}(q,q,q,q)+\mathfrak{f}_{1,2,1}(q,q,q)+q\mathfrak{f}_{1,2,1}(q,q,q)\\ &= qJ_1^2(\chi_0(q)-2)+J_1^2+qJ_1^2\\ &= qJ_1^2\chi_0(q)+J_1^2-qJ_1^2,\end{aligned}$$

where the second equality follows from (5.48), the third equality from (3.28), the fourth equality from (2.5), and the last equality from (2.6). For the final Identity (2.15) we use (5.47) with $(R,S,T)=(-1,0,1)$ to have

$$\begin{aligned}&\mathfrak{g}_{1,2,1,2,2,1}(q,q^2,q^3,q)\\ &\qquad = q\mathfrak{g}_{1,2,1,2,2,1}(q^2,q^2,q^2,q)-q^{-1}\mathfrak{f}_{1,2,1}(1,q,q)+\mathfrak{f}_{1,2,1}(q,q,q)+q^{-1}j(1;q).\end{aligned}$$

Using (2.5) twice gives

$$\mathfrak{g}_{1,2,1,2,2,1}(q,q^2,q^3,q)=q\mathfrak{g}_{1,2,1,2,2,1}(q^2,q^2,q^2,q)+J_1^2=qJ_1^2\chi_1(q)+J_1^2,$$

where the last equality followed from (2.7) and (2.5).

9. Obtaining Identity (2.16)

We consider

$$\mathfrak{g}_{1,3,1,3,3,1}(q,q,q,q). \tag{9.67}$$

We recall the specialization $(n,p)=(1,2)$ of [HiMo14, Proposition 8.1] where $\ell=1$:

$$\begin{aligned}\mathfrak{f}_{1,3,1}(x,y,q) &= j(y;q)m\Big(-\frac{q^5x}{y^3},q^8,\frac{q^2y}{x}\Big)+j(x;q)m\Big(-\frac{q^5y}{x^3},q^8,\frac{x}{q^2y}\Big)\\ &\quad+\frac{q^5x^2yJ_{2,4}J_{8,16}j(q^7xy;q^8)j(q^{22}x^2y^2;q^{16})}{j(-q^5x^2;q^8)j(-q^9y^2;q^8)}.\end{aligned} \tag{9.68}$$

Following the reasoning in Section 6., we only need to consider terms of the form

$$\sum_{m\geq 0}(-1)^mq^{\binom{m}{2}}z^m\mathfrak{f}_{1,3,1}(q^{3m}x,q^{3m}y,q). \tag{9.69}$$

Because the theta coefficients of the two Appell–Lerch terms vanish and because the Appell–Lerch terms are defined, we can write the above as

$$\mathfrak{f}_{1,3,1}(q^{3m}x,q^{3m}y,q)=\frac{q^{5+9m}x^2yJ_{2,4}J_{8,16}j(q^{7+6m}xy;q^8)j(q^{22+12m}x^2y^2;q^{16})}{j(-q^{5+6m}x^2;q^8)j(-q^{9+6m}y^2;q^8)}. \tag{9.70}$$

We will need to consider $m\to 4m+a$, $a\in\{0,1,2,3\}$. In general, (9.69) reads

$$\sum_{m\geq 0}(-1)^aq^{\binom{4m+a}{2}}z^{4m+a}\frac{q^{5+36m+9a}x^2yJ_{2,4}J_{8,16}j(q^{7+24m+6a}xy;q^8)j(q^{22+48m+12a}x^2y^2;q^{16})}{j(-q^{5+24m+6a}x^2;q^8)j(-q^{9+24m+6a}y^2;q^8)}.$$

For the case $a=0$

$$\sum_{m\geq 0} q^{\binom{4m}{2}} z^{4m} \frac{q^{5+36m}x^2yJ_{2,4}J_{8,16}j(q^{7+24m}xy;q^8)j(q^{22+48m}x^2y^2;q^{16})}{j(-q^{5+24m}x^2;q^8)j(-q^{9+24m}y^2;q^8)}$$

$$\sim \sum_{m\geq 0} q^{\binom{4m}{2}} z^{4m} \frac{q^{5+36m}x^2yJ_{2,4}J_{8,16}j(q^{8\cdot 3m}q^7xy;q^8)j(q^{16\cdot(3m+1)}q^6x^2y^2;q^{16})}{j(-q^{8\cdot 3m}q^5x^2;q^8)j(-q^{8\cdot(3m+1)}qy^2;q^8)}$$

$$\sim \sum_{m\geq 0} q^{\binom{4m}{2}} z^{4m} \frac{q^{5+36m}x^2yJ_{2,4}J_{8,16}j(q^7xy;q^8)j(q^6x^2y^2;q^{16})}{j(-q^5x^2;q^8)j(-qy^2;q^8)}$$

$$\cdot \frac{(-1)^{3m}q^{-8\binom{3m}{2}}(q^7xy)^{-3m}(-1)^{3m+1}q^{-16\binom{3m+1}{2}}(q^6x^2y^2)^{-(3m+1)}}{q^{-8\binom{3m}{2}}(q^5x^2)^{-3m}q^{-8\binom{3m+1}{2}}(qy^2)^{-(3m+1)}}$$

$$\sim -y\cdot \frac{J_{2,4}J_{8,16}j(q^7xy;q^8)j(q^6x^2y^2;q^{16})}{j(-q^5x^2;q^8)j(-qy^2;q^8)} \sum_{m\geq 0}\Big(\frac{z^4}{x^3y^3}\Big)^m q^{-28m^2+m}$$

$$\sim -y\cdot \frac{J_{2,4}J_{8,16}j(q^7xy;q^8)j(q^6x^2y^2;q^{16})}{j(-q^5x^2;q^8)j(-qy^2;q^8)} \sum_{m\geq 0}(-1)^m\Big(-\frac{q^{29}z^4}{x^3y^3}\Big)^m q^{-56\binom{m+1}{2}}$$

$$\sim -y\cdot \frac{J_{2,4}J_{8,16}j(q^7xy;q^8)j(q^6x^2y^2;q^{16})}{j(-q^5x^2;q^8)j(-qy^2;q^8)} m\Big(-\frac{q^{29}z^4}{x^3y^3},q^{56},*\Big).$$

For the case $a=1$

$$-\sum_{m\geq 0} q^{\binom{4m+1}{2}} z^{4m+1} \frac{q^{14+36m}x^2yJ_{2,4}J_{8,16}j(q^{13+24m}xy;q^8)j(q^{34+48m}x^2y^2;q^{16})}{j(-q^{11+24m}x^2;q^8)j(-q^{15+24m}y^2;q^8)}$$

$$\sim -\sum_{m\geq 0} q^{\binom{4m+1}{2}} z^{4m+1} \frac{q^{14+36m}x^2yJ_{2,4}J_{8,16}j(q^{8\cdot(3m+1)}q^5xy;q^8)j(q^{16\cdot(3m+2)}q^2x^2y^2;q^{16})}{j(-q^{8\cdot(3m+1)}q^3x^2;q^8)j(-q^{8\cdot(3m+1)}q^7y^2;q^8)}$$

$$\sim -\sum_{m\geq 0} q^{\binom{4m+1}{2}} z^{4m+1} \frac{q^{14+36m}x^2yJ_{2,4}J_{8,16}j(q^5xy;q^8)j(q^2x^2y^2;q^{16})}{j(-q^3x^2;q^8)j(-q^7y^2;q^8)}$$

$$\cdot \frac{(-1)^{3m+1}q^{-8\binom{3m+1}{2}}(q^5xy)^{-(3m+1)}(-1)^{3m+2}q^{-16\binom{3m+2}{2}}(q^2x^2y^2)^{-(3m+2)}}{q^{-8\binom{3m+1}{2}}(q^3x^2)^{-(3m+1)}q^{-8\binom{3m+1}{2}}(q^7y^2)^{-(3m+1)}}$$

$$\sim \frac{z}{xy^2}\cdot \frac{J_{2,4}J_{8,16}j(q^5xy;q^8)j(q^2x^2y^2;q^{16})}{j(-q^3x^2;q^8)j(-q^7y^2;q^8)} \sum_{m\geq 0}\Big(\frac{z^4}{x^3y^3}\Big)^m q^{-28m^2-13m-1}$$

$$\sim \frac{z}{xy^2q}\cdot \frac{J_{2,4}J_{8,16}j(q^5xy;q^8)j(q^2x^2y^2;q^{16})}{j(-q^3x^2;q^8)j(-q^7y^2;q^8)} \sum_{m\geq 0}(-1)^m\Big(-\frac{q^{15}z^4}{x^3y^3}\Big)^m q^{-56\binom{m+1}{2}}$$

$$\sim \frac{z}{xy^2q}\cdot \frac{J_{2,4}J_{8,16}j(q^5xy;q^8)j(q^2x^2y^2;q^{16})}{j(-q^3x^2;q^8)j(-q^7y^2;q^8)} m\Big(-\frac{q^{15}z^4}{x^3y^3},q^{56},*\Big).$$

For the case $a=2$

$$\sum_{m\geq 0} q^{\binom{4m+2}{2}} z^{4m+2} \frac{q^{23+36m}x^2yJ_{2,4}J_{8,16}j(q^{19+24m}xy;q^8)j(q^{46+48m}x^2y^2;q^{16})}{j(-q^{17+24m}x^2;q^8)j(-q^{21+24m}y^2;q^8)}$$

$$\sim \sum_{m\geq 0} q^{\binom{4m+2}{2}} z^{4m+2} \frac{q^{23+36m}x^2yJ_{2,4}J_{8,16}j(q^{8\cdot(3m+2)}q^3xy;q^8)j(q^{16\cdot(3m+2)}q^{14}x^2y^2;q^{16})}{j(-q^{8\cdot(3m+2)}qx^2;q^8)j(-q^{8\cdot(3m+2)}q^5y^2;q^8)}$$

$$\sim \sum_{m\geq 0} q^{\binom{4m+2}{2}} z^{4m+2} \frac{q^{23+36m}x^2yJ_{2,4}J_{8,16}j(q^3xy;q^8)j(q^{14}x^2y^2;q^{16})}{j(-qx^2;q^8)j(-q^5y^2;q^8)}$$

$$\cdot \frac{(-1)^{3m+2}q^{-8\binom{3m+2}{2}}(q^3xy)^{-(3m+2)}(-1)^{3m+2}q^{-16\binom{3m+2}{2}}(q^{14}x^2y^2)^{-(3m+2)}}{q^{-8\binom{3m+2}{2}}(qx^2)^{-(3m+2)}q^{-8\binom{3m+2}{2}}(q^5y^2)^{-(3m+2)}}$$
$$\sim \frac{z^2}{y}\cdot\frac{J_{2,4}J_{8,16}j(q^3xy;q^8)j(q^{14}x^2y^2;q^{16})}{j(-qx^2;q^8)j(-q^5y^2;q^8)}\sum_{m\geq 0}\Big(\frac{z^4}{x^3y^3}\Big)^m q^{-28m^2-27m-6}$$
$$\sim \frac{z^2}{yq^6}\cdot\frac{J_{2,4}J_{8,16}j(q^3xy;q^8)j(q^{14}x^2y^2;q^{16})}{j(-qx^2;q^8)j(-q^5y^2;q^8)}\sum_{m\geq 0}(-1)^m\Big(-\frac{qz^4}{x^3y^3}\Big)^m q^{-56\binom{m+1}{2}}$$
$$\sim \frac{z^2}{yq^6}\cdot\frac{J_{2,4}J_{8,16}j(q^3xy;q^8)j(q^{14}x^2y^2;q^{16})}{j(-qx^2;q^8)j(-q^5y^2;q^8)}m\Big(-\frac{qz^4}{x^3y^3},q^{56},*\Big).$$

For the case $a=3$

$$-\sum_{m\geq 0}q^{\binom{4m+3}{2}}z^{4m+3}\frac{q^{32+36m}x^2yJ_{2,4}J_{8,16}j(q^{25+24m}xy;q^8)j(q^{58+48m}x^2y^2;q^{16})}{j(-q^{23+24m}x^2;q^8)j(-q^{27+24m}y^2;q^8)}$$
$$\sim -\sum_{m\geq 0}q^{\binom{4m+3}{2}}z^{4m+3}\frac{q^{32+36m}x^2yJ_{2,4}J_{8,16}j(q^{8\cdot(3m+3)}qxy;q^8)j(q^{16\cdot(3m+3)}q^{10}x^2y^2;q^{16})}{j(-q^{8\cdot(3m+2)}q^7x^2;q^8)j(-q^{8\cdot(3m+3)}q^3y^2;q^8)}$$
$$\sim -\sum_{m\geq 0}q^{\binom{4m+3}{2}}z^{4m+3}\frac{q^{32+36m}x^2yJ_{2,4}J_{8,16}j(qxy;q^8)j(q^{10}x^2y^2;q^{16})}{j(-q^7x^2;q^8)j(-q^3y^2;q^8)}$$
$$\cdot\frac{(-1)^{3m+3}q^{-8\binom{3m+3}{2}}(qxy)^{-(3m+3)}(-1)^{3m+3}q^{-16\binom{3m+3}{2}}(q^{10}x^2y^2)^{-(3m+3)}}{q^{-8\binom{3m+2}{2}}(q^7x^2)^{-(3m+2)}q^{-8\binom{3m+3}{2}}(q^3y^2)^{-(3m+3)}}$$
$$\sim -\frac{z^3}{x^3y^2}\cdot\frac{J_{2,4}J_{8,16}j(qxy;q^8)j(q^{10}x^2y^2;q^{16})}{j(-q^7x^2;q^8)j(-q^3y^2;q^8)}\sum_{m\geq 0}\Big(\frac{z^4}{x^3y^3}\Big)^m q^{-28m^2-41m-15}$$
$$\sim -\frac{z^3}{x^3y^2q^{15}}\cdot\frac{J_{2,4}J_{8,16}j(qxy;q^8)j(q^{10}x^2y^2;q^{16})}{j(-q^7x^2;q^8)j(-q^3y^2;q^8)}\sum_{m\geq 0}(-1)^m\Big(-\frac{z^4}{q^{13}x^3y^3}\Big)^m q^{-56\binom{m+1}{2}}$$
$$\sim -\frac{z^3}{x^3y^2q^{15}}\cdot\frac{J_{2,4}J_{8,16}j(qxy;q^8)j(q^{10}x^2y^2;q^{16})}{j(-q^7x^2;q^8)j(-q^3y^2;q^8)}m\Big(-\frac{z^4}{q^{13}x^3y^3},q^{56},*\Big).$$

Modulo a theta function, our heuristic methods suggest

$$\begin{aligned}&\mathfrak{g}_{1,3,1,3,3,1}(q,q,q,q)\\ &\sim 3\frac{J_{2,4}J_{8,16}J_{1,8}J_{6,16}}{\overline{J}_{1,8}\overline{J}_{5,8}}m\Big(-q^{27},q^{56},*\Big)+3q^{-2}\frac{J_{2,4}J_{8,16}J_{1,8}J_{6,16}}{\overline{J}_{1,8}\overline{J}_{5,8}}m\Big(-q^{13},q^{56},*\Big)\\ &\quad -3q^{-7}\frac{J_{2,4}J_{8,16}J_{3,8}J_{2,16}}{\overline{J}_{1,8}\overline{J}_{5,8}}m\Big(-q^{-1},q^{56},*\Big)-3q^{-16}\frac{J_{2,4}J_{8,16}J_{3,8}J_{2,16}}{\overline{J}_{1,8}\overline{J}_{5,8}}m\Big(-q^{-15},q^{56},*\Big)\\ &\sim 3J_{1,2}\overline{J}_{3,8}m\Big(-q^{27},q^{56},*\Big)+3q^{-2}J_{1,2}\overline{J}_{3,8}m\Big(-q^{13},q^{56},*\Big)\\ &\quad -3q^{-7}J_{1,2}\overline{J}_{1,8}m\Big(-q^{-1},q^{56},*\Big)-3q^{-16}J_{1,2}\overline{J}_{1,8}m\Big(-q^{-15},q^{56},*\Big),\end{aligned}$$

which leads us to Identity (2.16).

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Eric T. Mortenson
Saint Petersburg State University
7/9 Universitetskaya Emb.
Saint Petersburg, 199034, Russia
e-mail: etmortenson@gmail.com