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Further Variations on the Six Exponentials Theorem

Michel Waldschmidt

Abstract. Let \( \tilde{\mathcal{L}} \) denote the set of linear combinations, with algebraic coefficients, of 1 and logarithms of algebraic numbers. The Strong Six Exponentials Theorem of D. Roy gives sufficient conditions for a \( 2 \times 3 \) matrix
\[
M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}
\]
whose entries are in \( \tilde{\mathcal{L}} \) to have rank 2.

Here we give sufficient conditions so that one at least of the three
\[
\begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix}, \quad \begin{vmatrix} \Lambda_{12} & \Lambda_{13} \\ \Lambda_{22} & \Lambda_{23} \end{vmatrix}, \quad \begin{vmatrix} \Lambda_{13} & \Lambda_{11} \\ \Lambda_{23} & \Lambda_{21} \end{vmatrix}
\]
is not in \( \tilde{\mathcal{L}} \)

1. Main result

We denote by \( \mathbb{Q} \) the field of rational numbers, by \( \overline{\mathbb{Q}} \) the field of algebraic numbers (algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \)), by \( \mathcal{L} \) the \( \mathbb{Q} \)-vector space of logarithms of algebraic numbers:
\[
\mathcal{L} = \{ \lambda \in \mathbb{C} ; \ e^\lambda \in \overline{\mathbb{Q}}^\times \} = \{ \log \alpha ; \ \alpha \in \overline{\mathbb{Q}}^\times \} = \exp^{-1}(\overline{\mathbb{Q}}^\times)
\]
and by \( \tilde{\mathcal{L}} \) the \( \mathbb{Q} \)-vector subspace of \( \mathbb{C} \) spanned by \( \{1\} \cup \mathcal{L} \). Hence \( \tilde{\mathcal{L}} \) is the set of linear combinations of 1 and logarithms of algebraic numbers with algebraic coefficients:
\[
\tilde{\mathcal{L}} = \left\{ \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n ; \right. \\
\left. \quad n \geq 0, (\alpha_1, \ldots, \alpha_n) \in (\overline{\mathbb{Q}}^\times)^n, (\beta_0, \beta_1, \ldots, \beta_n) \in \overline{\mathbb{Q}}^{n+1} \right\}.
\]

Here is the so-called strong six exponentials Theorem of D. Roy ( ([5] Corollary 2 §4 p. 38; see also [7] Corollary 11.16):

\begin{verbatim}
Key words and phrases. Transcendental numbers, logarithms of algebraic numbers, four exponentials Conjecture, six exponentials Theorem, algebraic independence.

Acknowledgements: A suggestion by D. Roy in Banff in November 2004 turned out to be a key point in the proof of the main result. Thanks also to him and to Guy Diaz for their comments on previous versions of this text.
\end{verbatim}
THEOREM 1.1. Let $M$ be a $2 \times 3$ matrix with entries in $\tilde{\mathbb{C}}$:

$$M = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23}
\end{pmatrix}.$$ 

Assume that the two rows of $M$ are linearly independent over $\bar{\mathbb{Q}}$ and also that the three columns are linearly independent over $\bar{\mathbb{Q}}$. Then $M$ has rank 2.

Consider the three $2 \times 2$ determinants

$$\Delta_1 = A_{12}A_{23} - A_{13}A_{22}, \quad \Delta_2 = A_{13}A_{21} - A_{11}A_{23}, \quad \Delta_3 = A_{11}A_{22} - A_{12}A_{21}.$$ 

From the relation

$$\Delta_1 \begin{pmatrix}
A_{11} \\
A_{21}
\end{pmatrix} + \Delta_2 \begin{pmatrix}
A_{12} \\
A_{22}
\end{pmatrix} + \Delta_3 \begin{pmatrix}
A_{13} \\
A_{23}
\end{pmatrix} = 0,$$ 

it follows from the assumptions of Theorem 1.1 that one at least of the three numbers $\Delta_1, \Delta_2, \Delta_3$ is transcendental. We want to prove that one at least of these three numbers is not in $\tilde{\mathbb{C}}$.

If the five rows of the matrix $\begin{pmatrix} M & \bar{I}_3 \end{pmatrix}$ (where $I_3$ is the $3 \times 3$ identity matrix) are linearly dependent over $\bar{\mathbb{Q}}$, which means that there exists $(\gamma_1, \gamma_2) \in \bar{\mathbb{Q}}^2 \setminus \{0\}$ such that the three numbers

$$\delta_j = \gamma_1 A_{1j} + \gamma_2 A_{2j} \quad (j = 1, 2, 3)$$ 

are algebraic, then the three numbers $\Delta_1, \Delta_2, \Delta_3$ are in $\tilde{\mathbb{C}}$. Indeed, if $(j, h, k)$ denotes any of the triples $(1, 2, 3), (2, 3, 1), (3, 1, 2)$, then

$$\gamma_1 \Delta_j = \delta_h A_{2k} - \delta_k A_{2h} \quad \text{and} \quad \gamma_2 \Delta_j = \delta_h A_{1h} - \delta_k A_{1k}.$$ 

Here is the main result of this paper.

THEOREM 1.2. Let $M$ be a $2 \times 3$ matrix with entries in $\tilde{\mathbb{C}}$:

$$M = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23}
\end{pmatrix}.$$ 

Assume that the five rows of the matrix

$$\begin{pmatrix} M & \bar{I}_3 \end{pmatrix} = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & 1 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & 0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}$$ 

are linearly independent over $\bar{\mathbb{Q}}$, which means that there exists $(\gamma_1, \gamma_2, \gamma_3) \in \bar{\mathbb{Q}}^3 \setminus \{0\}$ such that the three numbers $\Delta_1, \Delta_2, \Delta_3$ are in $\tilde{\mathbb{C}}$. Indeed, if $(j, h, k)$ denotes any of the triples $(1, 2, 3), (2, 3, 1), (3, 1, 2)$, then

$$\gamma_1 \Delta_j = \delta_h A_{2k} - \delta_k A_{2h} \quad \text{and} \quad \gamma_2 \Delta_j = \delta_h A_{1h} - \delta_k A_{1k}.$$
are linearly independent over \( \overline{\mathbb{Q}} \) and that the five columns of the matrix
\[
(I_2, M) = \begin{pmatrix}
1 & 0 & A_{11} & A_{12} & A_{13} \\
0 & 1 & A_{21} & A_{22} & A_{23}
\end{pmatrix}
\]
are linearly independent over \( \overline{\mathbb{Q}} \). Then one at least of the three numbers
\[
\Delta_1 = \begin{vmatrix}
A_{12} & A_{13} \\
A_{22} & A_{23}
\end{vmatrix}, \quad \Delta_2 = \begin{vmatrix}
A_{13} & A_{11} \\
A_{23} & A_{21}
\end{vmatrix}, \quad \Delta_3 = \begin{vmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{vmatrix}
\]
is not in \( \widetilde{\mathbb{L}} \).

If \( M \) is a \( d \times \ell \) matrix of rank 1, with \( d \geq 2 \) and \( \ell \geq 2 \), whose columns are \( \overline{\mathbb{Q}} \)-linearly independent, then the \( d + \ell \) columns of the matrix \( (I_d \ M) \) are also \( \overline{\mathbb{Q}} \)-linearly independent. Hence on the one hand Theorem 1.2 generalizes Theorem 1.1. On the other hand, as noticed by G. Diaz, when one of the six numbers \( \Lambda_{ij} \) is algebraic, Theorem 1.2 reduces to the next consequence of Theorem 1.1 (further related results are given in [1] and [8]).

**Corollary 1.3.** Let \( \Lambda_1, \Lambda_2, \Lambda_3 \) be three elements of \( \widetilde{\mathbb{L}} \) Assume that \( \Lambda_1 \) is transcendental and that the three numbers \( 1, \Lambda_2, \Lambda_3 \) are \( \overline{\mathbb{Q}} \)-linearly independent. Then one at least of the two numbers \( \Lambda_1 \Lambda_2, \Lambda_1 \Lambda_3 \) is not in \( \widetilde{\mathbb{L}} \).

The simple example
\[
M = \begin{pmatrix}
0 & A_2 & A_3 \\
A_1 & 0 & 0
\end{pmatrix}
\]
shows that the assumptions of Theorem 1.2 are not sufficient to ensure that none of the three determinants is in \( \widetilde{\mathbb{L}} \).

Here is a simple result which follows from Theorem 1.2: Let \( \Lambda_1, \Lambda_2, \Lambda_3 \) be three elements in \( \widetilde{\mathbb{L}} \) such that \( 1, \Lambda_1, \Lambda_2, \Lambda_3 \) are linearly independent over \( \overline{\mathbb{Q}} \). Then one at least of the three numbers
\[
\Lambda_1^2 - \Lambda_2 \Lambda_3, \quad \Lambda_2^2 - \Lambda_3 \Lambda_1, \quad \Lambda_3^2 - \Lambda_1 \Lambda_2
\]
is not in \( \widetilde{\mathbb{L}} \).

In §3 we shall deduce from Theorem 1.2 the following corollary.

**Corollary 1.4.** Let \( M \) be a \( 2 \times 3 \) matrix with entries in \( \mathbb{L} \):
\[
M = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23}
\end{pmatrix}
\]
Assume that the two rows of \( M \) are linearly independent over \( \mathbb{Q} \) and also that the three columns of \( M \) are linearly independent over \( \mathbb{Q} \). Then one at
least of the three numbers
\( (1.5) \quad \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}, \quad \lambda_{12}\lambda_{23} - \lambda_{13}\lambda_{22}, \quad \lambda_{13}\lambda_{21} - \lambda_{11}\lambda_{23} \)
is not in \( \tilde{L} \).

The six exponentials Theorem of S. Lang ([3], Chap. II § 1) and K. Ramachandra ([4] II § 4) states that, under the assumptions of Corollary 1.4, one at least of the three numbers \( (1.5) \) is not zero.

It is expected that a result similar to Theorem 1.2 holds when \( M \) is replaced by a \( 2 \times 2 \) matrix:

**Conjecture 1.6.** Let \( M \) be a \( 2 \times 2 \) matrix with entries in \( \tilde{L} \):
\[
M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}.
\]
Assume that the four rows of the matrix
\[
\begin{pmatrix} M \\ I_2 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
are linearly independent over \( \overline{\mathbb{Q}} \) and that the four columns of the matrix
\[
\begin{pmatrix} I_2 \\ M \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}
\]
are linearly independent over \( \overline{\mathbb{Q}} \). Then the number
\[
\Delta = \left| \begin{array}{cc} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{array} \right|
\]
is not in \( \tilde{L} \).

Conjecture 1.6 follows from the conjecture (see for instance [3], Historical Note of Chapter III, [2], Chap. 6 p. 259 and [7], Conjecture 1.15 and [8] Conjecture 1.1) that \( \mathbb{Q} \)-linearly independent logarithms of algebraic numbers are algebraically independent.

**2. A consequence of the Linear Subgroup Theorem**

Let \( n \) be a positive integer and \( Y \) a \( \overline{\mathbb{Q}} \)-vector subspace of \( \mathbb{C}^n \). We define
\[
\mu(Y, \mathbb{C}^n) = \min_{V \subset \mathbb{C}^n} \frac{\dim_{\mathbb{C}}(Y/Y \cap V)}{\dim_{\mathbb{C}}(\mathbb{C}^n/V)},
\]
where \( V \) runs over the set of \( \mathbb{C} \)-vector subspaces of \( \mathbb{C}^n \) with \( V \neq \mathbb{C}^n \).
For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ we denote by $x \cdot y$ the scalar product
\[ x \cdot y = x_1y_1 + \cdots + x_ny_n. \]
For $X$ and $Y$ two subsets of $\mathbb{C}^n$, we denote by $X \cdot Y$ the set of scalar products $x \cdot y$ where $x$ ranges over the set $X$ and $y$ over $Y$.

**Theorem 2.1.** Let $X$ and $Y$ be two $\mathbb{Q}$-vector subspaces of $\mathbb{C}^n$. Assume $X$ has dimension $d$ with $d > n$. Assume further
\[ \mu(Y, \mathbb{C}^n) > \frac{d}{d - n}. \]
Then the set $X \cdot Y$ is not contained in $\tilde{\mathcal{L}}$.

**Proof.** This is essentially Proposition 6.1 of [6], where $\mathbb{Q}$ is replaced by $\mathbb{Q}$ and the $\mathbb{Q}$-vector space $\mathcal{L}$ by the $\overline{\mathbb{Q}}$-vector space $\tilde{\mathcal{L}}$. Henceforth the proof runs as follows.

Like in Lemma 5.2 of [6], one checks that if $X$ and $Y$ are two vector subspaces of $\mathbb{C}^n$ over $\overline{\mathbb{Q}}$, of dimensions $d$ and $\ell$ respectively, then there exist a positive integer $n' \leq n$ and two vector subspaces $X'$ and $Y'$ of $\mathbb{C}^{n'}$, of dimensions $d'$ and $\ell'$ respectively, such that
\[ \mu(X', \mathbb{C}^{n'}) = \frac{d'}{n'} \geq \frac{d}{n}, \quad \mu(Y', \mathbb{C}^{n'}) = \frac{\ell'}{n'} \geq \mu(Y, \mathbb{C}^n) \]
and
\[ (2.2) \quad X' \cdot Y' \subset X \cdot Y. \]
This shows that for the proof of Theorem 2.1, there is no loss of generality to assume $\mu(X, \mathbb{C}^n) = d/n$ and $\mu(Y, \mathbb{C}^n) = \ell/n$. The assumption $\mu(Y, \mathbb{C}^n) > d/(d - n)$ reduces to $\ell d > n(\ell + d)$.

Following the argument of Lemma 5.4 in [6], one proves that if $X$ and $Y$ are two vector subspaces of $\mathbb{C}^n$ over $\overline{\mathbb{Q}}$, of dimensions $d$ and $\ell$ respectively, $X_1$ a subspace of $X$ of dimension $d_1$ and $Y_1$ a subspace of $Y$ of dimension $\ell_1$ such that $X_1 \cdot Y_1 = \{0\}$, then
\[ (2.3) \quad (d - d_1)\mu(Y, \mathbb{C}^n) + (\ell - \ell_1)\mu(X, \mathbb{C}^n) \geq n\mu(X, \mathbb{C}^n)\mu(Y, \mathbb{C}^n). \]
In Lemma 5.4 in [6] an extra assumption is required, namely
\[ \mu(X, \mathbb{C}^n)\mu(Y, \mathbb{C}^n) \geq \mu(X, \mathbb{C}^n) + \mu(Y, \mathbb{C}^n), \]
but we do not need it here, since our assumption $X_1 \cdot Y_1 = \{0\}$ is stronger than the assumption in Lemma 5.4 of [6] that $X_1 \cdot Y_1$ has rank $\leq 1$. 

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Next we introduce the coefficient $\theta(M)$ attached to a $d \times \ell$ matrix $M$ with entries in $\mathbb{C}$. It is defined as follows:

$$\theta(M) = \min \frac{\ell'}{d'},$$

where $(d', \ell')$ ranges over the set of pairs of integers satisfying $0 \leq \ell' \leq \ell$, $1 \leq d' \leq d$, such that there exist a $d \times d$ regular matrix $P$ and a regular $\ell \times \ell$ regular matrix $Q$, both with entries in $\overline{\mathbb{Q}}$, with

$$PMQ = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} d^* \\ \ell^* \end{pmatrix} \begin{pmatrix} d' \\ \ell' \end{pmatrix}$$

From (2.3) with $d_1 = d'$ and $\ell_1 = \ell^*$ it follows that if

$$X = \overline{\mathbb{Q}}x_1 + \cdots + \overline{\mathbb{Q}}x_d \quad \text{and} \quad Y = \overline{\mathbb{Q}}y_1 + \cdots + \overline{\mathbb{Q}}y_\ell$$

are again two vector subspaces of $\mathbb{C}^n$ over $\overline{\mathbb{Q}}$, of dimensions $d$ and $\ell$ respectively, satisfying $\mu(X, \mathbb{C}^n) = d/n$, then the matrix

(2.4) $$M = (x_i \cdot y_j)_{1 \leq i \leq d, \ 1 \leq j \leq \ell}$$

has

$$\theta(M) \geq \frac{n}{d} \cdot \mu(Y, \mathbb{C}^n).$$

In particular if $\mu(X, \mathbb{C}^n) = d/n$ and $\mu(Y, \mathbb{C}^n) = \ell/n$, then $\theta(M) = \ell/d$.

Finally Theorem 4 in [5] (which is Proposition 11.19 or Theorem 12.19 in [7]) shows that the rank $r$ of a $d \times \ell$ matrix $M$ with entries in $\tilde{\mathbb{L}}$ satisfies

$$r \geq \frac{d\theta}{1 + \theta},$$

where $\theta = \theta(M)$. Using this result for the matrix $M$ given by (2.4) whose rank $r$ is $\leq n$, one concludes that if $\mu(X, \mathbb{C}^n) = d/n$ and $\mu(Y, \mathbb{C}^n) = \ell/n$ with $X \cdot Y \subset \tilde{\mathbb{L}}$, then

$$n \geq \frac{\ell d}{\ell + d}.$$
3. Proof of the main results

In this section we prove Theorem 1.2 and Corollary 1.4.

Proof of Theorem 1.2. Assume that the hypotheses of Theorem 1.2 are satisfied. Define elements $v_1, \ldots, v_5$ in $\tilde{L}$ by

$$v_1 = e_1, \quad v_2 = e_2, \quad v_{2+j} = (\Lambda_1, \Lambda_2), \quad (j = 1, 2, 3),$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For $v = (x, y) \in \mathbb{C}^2$, set $v' = (-y, x)$, so that $v' \cdot v = 0$. Consider the $5 \times 5$ matrix

$$A = \begin{pmatrix}
0 & 1 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\
-1 & 0 & -\Lambda_{11} & -\Lambda_{12} & -\Lambda_{13} \\
-\Lambda_{21} & \Lambda_{11} & 0 & \Delta_3 & -\Delta_2 \\
-\Lambda_{22} & \Lambda_{12} & -\Delta_3 & 0 & \Delta_1 \\
-\Lambda_{23} & \Lambda_{13} & \Delta_2 & -\Delta_1 & 0
\end{pmatrix}.$$

Let $X$ be the $\overline{\mathbb{Q}}$-vector space spanned by $v_1, \ldots, v_5$ in $\mathbb{C}^2$ and similarly let $Y$ be the subspace of $\mathbb{C}^2$ spanned by $v'_1, \ldots, v'_5$ over $\overline{\mathbb{Q}}$. We claim

$$\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) \geq 2. \quad (3.1)$$

The equality $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2)$ follows from the fact that the map $(x, y) \mapsto (-y, x)$ is an automorphism of $\mathbb{C}^2$.

Since the five columns of $\begin{pmatrix} I_2 & M \end{pmatrix}$ are linearly independent over $\overline{\mathbb{Q}}$, 

$$\dim_{\overline{\mathbb{Q}}} X = 5.$$ 

Let $V$ be a vector subspace of $\mathbb{C}^2$ of dimension 1 and let $t_1z_1 + t_2\tilde{z}_2 = 0$ be an equation of $V$ in $\mathbb{C}^2$, with $(t_1, t_2) \in \mathbb{C}^2 \setminus \{0\}$. Consider the linear map

$$p : \mathbb{C}^2 \to \mathbb{C}$$

$$(z_1, \tilde{z}_2) \mapsto t_1z_1 + t_2\tilde{z}_2$$

whose kernel is $V$. Since the five rows of $\begin{pmatrix} M & I_3 \end{pmatrix}$ are $\overline{\mathbb{Q}}$-linearly independent,

$$\dim_{\overline{\mathbb{Q}}}((X \cap V)/V) = \dim_{\overline{\mathbb{Q}}} p(X) \geq 2.$$ 

This completes the proof of (3.1).

From (3.1) we deduce that the hypothesis $\mu(Y, \mathbb{C}^2) > d/(d - n)$ of Theorem 2.1 is satisfied with $d = 5$ and $n = 2$, hence the set $X \cdot Y$ is not contained in $\tilde{L}$. Consequently one at least of the three numbers $\Delta_1, \Delta_2, \Delta_3$ is not in $\tilde{L}$.
This completes the proof of the Main Theorem 1.2. ∎

Remark. In (3.1) we may have equality: for instance if \( \Lambda_{22} = \Lambda_{23} = 0 \) then \( \mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) = 2 \).

However the proof of Theorem 2.1 shows that in the case \( \mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) < 5/2 \), Theorem 1.2 should follow from Theorem 1.1. Indeed after a change of variables rational over \( \overline{\mathbb{Q}} \) one needs only to consider a matrix

\[
M = \begin{pmatrix} 0 & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & 0 & 0 \end{pmatrix},
\]

which is the situation of Corollary 1.3. If \( X \) is the \( \overline{\mathbb{Q}} \)-subspace of \( \mathbb{C}^2 \) spanned by

\[
v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (0, \Lambda_{21}), \quad v_4 = (\Lambda_{12}, 0), \quad v_4 = (\Lambda_{13}, 0)
\]

and \( Y \) the subspace spanned by

\[
v_1' = (0,1), \quad v_2' = (-1,0), \quad v_3' = (-\Lambda_{21}, 0), \quad v_4' = (0, \Lambda_{12}), \quad v_4' = (0, \Lambda_{13}),
\]

then

\[
X' = \overline{\mathbb{Q}} + \overline{\mathbb{Q}} \Lambda_{12} + \overline{\mathbb{Q}} \Lambda_{13} \quad \text{and} \quad Y' = \overline{\mathbb{Q}} + \overline{\mathbb{Q}} \Lambda_{21}
\]

are \( \overline{\mathbb{Q}} \)-subspaces of \( \mathbb{C} \) satisfying (2.2). Here \( \mu(X', \mathbb{C}) = 3 > d/n = 5/2 \) and \( \mu(Y', \mathbb{C}) = 2 = \mu(Y, \mathbb{C}^2) \).

Proof of Corollary 1.4. From Baker’s Theorem it follows that if \( Y_0 \) is a \( \mathbb{Q} \)-vector subspace of \( \mathcal{L}^n \) of dimension \( \ell \), then the \( \overline{\mathbb{Q}} \)-vector subspace of \( \tilde{\mathcal{L}}^n \) spanned by \( \overline{\mathbb{Q}}^n \cup Y_0 \) has dimension \( \ell + n \) (see Exercise 1.5 (iii) of [7]). Taking firstly \( n = 2, \ell = 3 \), and secondly \( n = 3, \ell = 2 \), we deduce that the matrix \( M \) of corollary 1.4 satisfies the assumptions of Theorem 1.2. Corollary 1.4 follows. ∎

4. Erratum to [8]

We take the opportunity of this paper to point out a mistake in the statement of Corollary 2.12 p. 347 of [8]: the assumption that \( \Lambda_{21} \) is not zero and \( \Lambda_{11}/\Lambda_{21} \) is transcendental should be replaced by the assumption that the three numbers \( 1, \Lambda_{11} \) and \( \Lambda_{21} \) are linearly independent over the field of algebraic numbers. Otherwise a counterexample is obtained for instance with \( \Lambda_{21} = 1 \) and \( \Lambda_{2j} = 0 \) for \( 2 \leq j \leq 5 \).
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References


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