SOME REMARKS ON THE MEAN VALUE OF THE RIEMANN ZETA-FUNCTION AND OTHER DIRICHLET SERIES—I

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§ 1. Introduction

In the last section of my paper [2], I raised some questions on the mean value of generalised Dirichlet series. It looks too ambitious unless we limit ourselves to Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ $(s = \sigma + it)$ and that too with some restrictions. We do not claim to solve all the problems raised. Let $F(s)$ be convergent absolutely somewhere in the complex plane and let $F(s)$ admit an analytic continuation in $\sigma > \frac{1}{2}$, $t > \tau_0$ and there $F(s) = O(t^A)$. If σ_1 is large enough, $F(s)$ is zero free in $\sigma > \sigma_1$ and we define $(F(s))^{2k}$ (for any positive real number $2k$) as the analytic continuation of $(F(s))^{2k}$ $(\sigma > \sigma_1)$ along lines parallel to the real axis. (If such lines contain a zero of $F(s)$ we do not define $(F(s))^{2k}$ on such lines). Let δ be a positive constant not exceeding $\frac{1}{10}$ and m a non-negative integer. Let $T > T_0(\delta)$ be a real variable and H a real variable subject to $(\log T)^{\delta} < H < T$. Imposing the conditions $\delta < 2k < \delta^{-1}$ and $0 < m < \delta^{-1}$ we define for $\sigma > \frac{1}{2}$,

$$Q(\sigma) = \frac{1}{H} \int_{T}^{T+H} \left| \frac{d^m}{ds^m} (F(s))^{2k} \right| dt.$$

Our main problem is to study lower bounds for $Q(\sigma)$. The only known progress so far, in this direction is a theorem of Ingham which gives for fixed $\sigma > \frac{1}{2}$ and fixed $2k$ $(0 < 2k < 4$, $m = 0)$ an asymptotic formula for $Q(\sigma)$ in the special case $F(s) = \zeta(s)$, $H = T$. Ingham's proof was complicated and

Davenport gave a simpler proof of Ingham's Theorem (fo. references see [5]). Our first object in this note is to give in § 3, a satisfactory lower bound for max $Q(\sigma)$ as σ runs over all real numbers $\geqslant \frac{1}{2} + \frac{\log \log H}{\log H}$. However our proof of lower bound depends very much on the existence of an "Euler product" for $F(s)$. (The Euler product condition is general enough to include the case when $F(s)$ is the zeta-function and Hecke L-series of algebraic number fields). From such a theorem we can also get satisfactory lower bounds for $Q(\frac{1}{2})$ as will be seen. Two simple samples of our general results (subject to the condition $2k \geqslant 1$) are

$$(*) \quad \frac{1}{H} \int_{T}^{T+H} |\zeta(\tfrac{1}{2}+it)|^{2k} dt > (\log H)^{k^2} (\log \log H)^{-C}$$

$$(**) \quad \frac{1}{H} \int_{T}^{T+H} |\zeta'(\tfrac{1}{2}+it)| \, dt > (\log H)^{\frac{5}{4}} (\log \log H)^{-C}$$

where C is a constant depending only on δ. Our next object is to deal (in § 4 and § 5) with the case when $F(s)$ has no Euler product. Here we are forced, for lack of better ideas, to limit ourselves to the case $2k = 1$. To compensate for the generality the lower bounds we obtain for $Q(\sigma)$ are not so satisfactory, but they are still, I hope, of some interest.

The problem of upper bounds for $Q(\sigma)$ seems hopelessly difficult. Even with the assumption of Riemann hypothesis it is not clear how to improve the trivial inequality

$$(***) \quad \frac{1}{T} \int_{T}^{2T} |\zeta(\tfrac{1}{2}+it)| \, dt = O((\log T)^{\frac{1}{2}}).$$

Foot-Note : The publication of this paper which was ready by the middle of 1977 was delayed due to various reasons. In the meanwhile I have replaced $(\log \log H)^{-C}$ in (*) and (**) by $\frac{1}{C}$. Next I have replaced $O((\log T)^{\frac{1}{2}})$ by $O((\log T)^{\frac{1}{4}})$ in (***) unconditionally. These (and other) results will appear in papers II and III with the same title as the present paper.

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§ 2. Notation

The letter C with or without subscripts denotes a positive constant depending only on δ. The letter K with or without subscripts denotes constants to be chosen later appropriately in a proof. Also the constant C may not be the same at each occurrence.

§ 3. The case of Euler product

Let in $\sigma > 1$, $F(s)$ be defined by

$$F(s) = \prod_{p} \left(1 - \frac{x(p)}{p^s}\right)^{-a_p} \quad (p \text{ runs over all primes})$$

where $\{ x(p) \}$ and $\{ a_p \}$ are bounded sequences of complex numbers, and $| x(p) | < p^{\frac{1}{2}}$ for every prime p. We assume either of the following conditions on $F(s)$.

(i) There exists a constant p_0 such that whenever $a_p \neq 0$ and $p > p_0$ we have $| a_p | \geqslant a$, where a is a positive constant. Further the function $\prod_{p} \left(1 - \frac{| x(p) |^2}{p^s}\right)^{-|a_p|}$ which is analytic in $\sigma > 1$ can be continued analytically in a neighbourhood of $s = 1$ and has a simple pole at $s = 1$.

(ii) Suppose that as $\sigma \to \frac{1}{2} + 0$, $\left(\prod_{p} \left(1 + \frac{| x(p) a_p |^2}{p^{2\sigma}}\right)\right)$ $(\sigma - \frac{1}{2})^{a}$ lies between $\left(\log \frac{1}{\sigma - \frac{1}{2}}\right)^{b_1}$ and $\left(\log \frac{1}{\sigma - \frac{1}{2}}\right)^{b_2}$ where $a > 0$ and b_1, b_2 are three real constants.

We can now state

Theorem I

With the notation explained already,

$$\max_{\sigma \geq \frac{1}{2} + \frac{\log\log H}{\log H}} Q(\sigma) > (\log H)^{ak^2+m_1} (\log\log H)^{-C}$$

where C *is a positive constant, and* $m_1 = m$ *or* $\min(m, 1)$ *according as we have* (ii) *or* (i).

As a corollary we can deduce the results for $\zeta(\frac{1}{2} + it)$ quoted in the introduction. More generally we have

Theorem 2

Let L (s) *denote either the Dedekind zeta-function or the Hecke* L-*series of an algebraic number field of degree* n *and according as it is Galois or not we put* $a = n$ *or* 1 *and also* $m_1 = m$ *or* $\min(1, m)$. *Then we have with* $2k \geq 1$,

$$\frac{1}{H}\int_T^{T+H} | L(\tfrac{1}{2}+it|^{2k} \, dt > (\log H)^{ak^2} (\log\log H)^{-C}$$

$$\frac{1}{H}\int_T^{T+H} |L^{(m)}(\tfrac{1}{2}+it)| \, dt > (\log H)^{\frac{a}{4}+m_1} (\log\log H)^{-C}$$

where C *is a positive constant. Further the condition* $2k \geq 1$ *is unnecessary if we assume the hypothesis that* $L(s) \neq 0$ *for* $\sigma > \frac{1}{2}$.

Remark : Theorem 2 gives an improvement on theorem 3 of my paper [3]. It will be seen later that m_1 can also be replaced by $m_2 = \max(m_1, mn^{-1})$.

Deduction of Theorem 2 from Theorem I

Let $F(s) = L(s)$. We put $s = \sigma + it$, $w = u + iv$, where σ is the number at which $\max Q(\sigma)$ is attained. We limit ourselves to the case $2k > 1$, $m = 0$ or $2k = 1$, $m = 1$. The proof in both the cases are similar and we consider the first case $2k \geq 1$, $m = 0$. We impose $T + \frac{H}{4} \leq t \leq T + \frac{3H}{4}$.
If now b is an odd positive integer which is fixed to be a large integer depending on δ, we have by Cauchy's Theorem

$$F(s) = \frac{1}{2\pi i} \int_R F(W) \, e^{(W-s)^{2b}} \frac{dW}{W-s}$$

where R is the rectangle with corners $\frac{1}{2} + i(T+H)$, $\frac{1}{2} + iT$, $2 + iT$, $2 + i(T+H)$. Because b is large we can check that the horizontal portions contribute a bounded quantity to the integral. The same is trivially true of the line $u = 2$. Thus

$$|F(s)|^{2k} = O\left(\left(\int_{u=\frac{1}{2}} \left| F(W)\, e^{(W-s)^{2b}} \frac{dW}{W-s} \right| \right)^{2k} + 1\right)$$

$$= O\left(1 + \int_{u=\frac{1}{2}} \left| (F(W))^{2k}\, e^{(W-s)^{2b}} \frac{dW}{W-s} \right| \left(\int_{u=\frac{1}{2}} \left| e^{(W-s)^{2b}} \frac{dW}{W-s} \right| \right)^{2k-1}\right)$$

Theorem 2 now follows on integrating with respect to s and also using the fact that $\sigma - \frac{1}{2} > \frac{\log\log H}{\log H}$. We have to remember that if the field is Galois, condition (ii) is satisfied; otherwise the condition (i) is satisfied.

To prove Theorem 1, we assume that it is false with $C = 1$.

We now proceed to prove by a series of lemmas the truth of the theorem for some $C > 1$. We can certainly choose the latter constant and this would prove the Theorem 1. Accordingly we begin with

Lemma 1: *We have*

$$\max_{\sigma \geqslant \frac{1}{2} + \frac{\log\log H}{\log H}} \frac{1}{H} \int_T^{T+H} |F(\sigma+it)|^{2k} dt < (\log H)^{ak^2+m} \log\log H.$$

Proof: Trivial since, for

$$0 \leqslant j \leqslant \delta^{-1}, \quad \left| \frac{d^j}{dt^j} (F(2+it))^{2k} \right| \text{ is bounded.}$$

Lemma 2: *The maximum of* $|F(\sigma+it)|$ *taken over all*

$\sigma > \frac{1}{2} + \frac{2\log\log H}{\log H}$, $T+1 \leqslant t \leqslant T+H-1$, *does not exceed* H^2.

Proof : Follows from the fact $|F(s)|^{2k}$ is subharmonic. However we supply a proof. If $F(s) \neq 0$ in $|s - s_0| \leqslant r$

we have $\log |F(s_0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(s_0 + re^{i\theta})| \, d\theta.$

The first quantity is less than the second if $F(s_0) = 0$ Assuming now $F(s_0) \neq 0$ and defining

$$\phi(s) = F(s) \pi \left(\frac{r^2 - (\bar{\rho} - \bar{s}_0)(s - s_0)}{(s - \rho) r} \right)$$

where ρ runs over all the zeros of $F(s)$ in $|s - s_0| \leqslant r$, we see that $\log |F(s_0)| \leqslant \log |\phi(s_0)|$ and that on $|s - s_0| = r$ we have $|F(s)| = |\phi(s)|$. This proves

$$\log |F(s_0)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log |F(s_0 + r e^{i\theta})| \, d\theta.$$

This gives easily

$$\log |F(s_0)| \leqslant \frac{1}{\pi r^2} \int_{|s - s_0| \leqslant r} \log |F(s)| \, dv$$

where dv is the element of area of the disc. We multiply this by $2k$ and apply the arithmetico-geometric inequality (in the limiting form to suit integrals) and we obtain

$$|F(s_0)|^{2k} \leqslant \frac{1}{\pi r^2} \int_{|s - s_0| \leqslant r} |F(s)|^{2k} \, dv.$$

By taking a suitable radius say $\frac{\log \log H}{\log H}$ we get the lemma.

Lemma 3 : *Let* $N(\alpha, T_1, T_2)$ *denote the number of zeros of* $F(s)$ *in* $\sigma \geqslant \alpha$ $\left(\alpha > \frac{1}{2} + \frac{3 \log \log H}{\log H}\right)$ and $T_1 \leqslant t \leqslant T_2$. *Then if* $T + (\log H)^8 \leqslant t \leqslant T + H - (\log H)^8$,

$$N(\alpha, t, t + 1) \leqslant (\log H)^4.$$

Proof: Follows from Jensen's inequality (see page 126 of [6])

$$\int_0^r \frac{n(x)}{x}\, dx = \frac{1}{2\pi}\int_0^{2\pi} \log |F(2+it+re^{i\theta})|\, d\theta - \log |F(2+it)|,$$

where $n(x)$ denotes the number of zeros of $F(s)$ in a disc of radius x with centre $2+it$. We have to select a suitable r and use Lemma 2. Note that if $0 < x_1 < r$ we have

$$\int_0^r \frac{n(x)}{x}\, dx \geqslant n(x_1) \log \frac{r}{x_1}\,.$$

Lemma 4: *For*

$$\alpha \geqslant \tfrac{1}{2} + \frac{3 \log \log H}{\log H},\ T_1 = T + (\log H)^8,$$
$$T_2 = T + H - (\log H)^8,$$

we have $N(\alpha, T_1, T_2) < H^{1 - C_1(\alpha - \frac{1}{2})} (\log H)^{C_2}$.

Proof: We select a "well-spaced" system of zeros connected in $N(\alpha, T_1, T_2)$ and proceed to estimate their number by the zero detecting function $F(s)\, M_H(s) - 1$ where $M_H(s)$ is the sum of the first H terms of the Dirichlet series for $(F(s))^{-1}$. Note that if $\phi_1(s)$ is the zero detecting function in question and $Re\ s = 1$ then

$$\frac{1}{2\pi i} \int_{Re\, W=2} \phi_1(s+W)\, \Gamma(W)\, X^W\, dW$$

is a very good approximation to $\phi_1(s)$ if we set $X = H^{C_3}$ where C_3 is a large constant. Fairly routine considerations lead to the lemma. (An excellent reference to our ideas of deducing "density estimates" which was also found by Gallagher in a more perfect form is Gallagher's paper [1]). In the proof we have of course to use Lemma 1.

Lemma 5 : *Let now* $\alpha \geqslant \alpha_0 = \frac{1}{2} + \frac{K_1 \log\log H}{\log H}$ *where* K_1 *is a large constant. Then* $N(\alpha, T_1, T_2) < H.(\log H)^{-K_2}$ *where* $K_2 \to \infty$ *as* $K_1 \to \infty$. *(Hereafter* T_1, T_2 *will be as in Lemma 4 and* α_0 *as in this Lemma).*

Proof : This lemma is a Corollary to Lemma 4.

We next divide the t interval $T_1 \leqslant t \leqslant T_2$ into equal intervals of length $(\log H)^{K_3}$ $(K_3 \geqslant 10)$ ignoring a small bit at one end. Let I_1 run through those intervals which do not contain a zero in $\sigma \geqslant \frac{1}{2} + \frac{K_1 \log\log H}{\log H}$ and I_2 the rest of the intervals. Let I_3 run through the intervals I_1 with t intervals of length $(\log H)^2$ removed both above and below. Plainly the intervals I_3 cover the interval $T_1 \leqslant t \leqslant T_2$ except certain bits of total length not exceeding $H(\log H)^{-K_2} + H(\log H)^{2-K_3}$. We put $s_0 = \alpha + it$ $\left(\alpha \text{ fixed and } \geqslant \alpha_0 + \frac{\log\log H}{\log H}\right)$ and set out to obtain an asymptotic formula for

$$Q_1(\alpha) = \sum_{I_3} \int_{I_3} | F(s_0) |^{2k} dt.$$

We prove

Lemma 6 : *Let* $K_2 = K_3$ *and* $\alpha > \alpha_0 + \frac{\log\log H}{\log H}$ *where* $\alpha_0 = \frac{1}{2} + \frac{K_1 \log\log H}{\log H}$. *Then we have,*

$$Q_1(\alpha) = H \sum_{n=1}^{\infty} | d_k(n) |^2 n^{-2\alpha} + O(H(\log H)^{C_4 - \frac{1}{2}K_2})$$
$$+ O(H(\log H)^{C_4} H^{-\frac{1}{16}(\alpha - \alpha_0)}),$$

where $d_k(n)$ *are defined by the expansion* $(F(s))^k = \sum_{n=1}^{\infty} d_k(n) n^{-s}$ *valid in* $\sigma > 1$. *Also* $K_2 \to \infty$ *as* $K_1 \to \infty$.

Proof: The proof of this lemma is nearly standard. We will merely sketch the proof. For t in I_3 we have with $X = H^{\frac{1}{4}}$ and the notation introduced already,

$$F(s_0) = P(t) + E,\ P(t) = \sum_{n=1}^{\infty} d_k(n)\, n^{-s_0} e^{-\frac{n}{X}}$$

where $$E = \frac{1}{2\pi i}\int_{\substack{|Im W| \leqslant (\log H)^2,\ Re(W+\alpha) = \alpha_0}} (F(s_0+W))^k\, \Gamma(W+1)\, X^W \frac{dW}{W} + O(H^{-10}).$$

From Lemma 1, $\sum_{I_3}\int_{I_3} |E|^2 dt = O(H(\log H)^{C_5} X^{-\frac{1}{2}(\alpha-\alpha_0)}.)$

Next we note that $|(F(s_0))^k|^2 = |P(t)|^2 + O(|(F(s_0))^k - P(t)|^2) + O(|(F(s_0)) - P(t))P(t)|)$.

Hence

$$\sum_{I_3}\int_{I_3} |F(s_0)|^{2k}\, dt = \sum_{I_3}\int_{I_3} |P(t)|^2\, dt + E' \text{ where}$$

$$E' = O\left(\sum_{I_3}\int_{I_3} |E|^2 dt + \left(\int_{T}^{T+H} |E|^2 dt\right)^{\frac{1}{2}} \left(\sum_{I_3}\int_{I_3} |P(t)|^2\, dt\right)^{\frac{1}{2}}\right).$$

It is easy to see that $\int_{T}^{T+H} |P(t)|^2\, dt = O(H(\log H)^{C_5})$, and

$\int_{T}^{T+H} |P(t)|^4\, dt = O(H(\log H)^{C_5})$ and so the O-term is

$O(H(\log H)^{C_4} X^{-\frac{1}{4}(\alpha-\alpha_0)})$. Also

$$\sum_{I_3}\int_{I_3} |P(t)|^2\, dt = \int_{T}^{T+H} |P(t)|^2\, dt + O(H(\log H)^{C_4-\frac{1}{2}K_2}),$$

and

$$\int_{T}^{T+H} |P(t)|^2\, dt = \int_{T}^{T+H} \Big|\sum_{n \leqslant X(\log X)^2} d_k(n)\, n^{-s_0}\Big|^2 dt + O(H^{-10})$$

and this by standard arguments (see for instance Lemma 9 below) is

$$(H+O(X(\log X)^3))\sum_{n\leqslant X(\log X)^2}|d_k(n)|^2 n^{-2\alpha}e^{-\frac{2n}{X}}+O(H^{10})$$

Let $S=\sum_{n=1}^{\infty}|d_k(n)|^2 n^{-2\alpha}$ and $S_1=\sum_{n\leqslant X(\log X)^2}|d_k(n)|^2 n^{-2\alpha}e^{-\frac{2n}{X}}$

Then using $1-e^{-\frac{n}{X}}=O\left(\frac{n}{X}\right)$ for $n\leqslant X$ and $O(1)$ for $n>X$ we have

$$S-S_1=O\left(\frac{1}{X}\sum_{n\leqslant X}|d_k(n)|^2 n^{1-2\alpha}+\sum_{n\geqslant X}|d_k(n)|^2 n^{-2\alpha}\right)$$

$$=O\left(X^{1-2\alpha}(\log H)^{C_4}\right).$$

This proves Lemma 6.

Lemma 7: *In case* (i) *we have* $\left|\frac{d^m}{d\alpha^m}S\right|\gg\left(\frac{1}{\alpha-\frac{1}{2}}\right)^{ak^2+m}$, *and in case* (ii) *we have* $S\,(\alpha-\frac{1}{2})^{ak^2}$ *is* $\gg\left(\log\frac{1}{\alpha-\frac{1}{2}}\right)^{b_2k^2}$ *and* $\ll\left(\log\frac{1}{\alpha-\frac{1}{2}}\right)^{b_1k^2}$.

Proof: We leave this as an exercise to the reader.

Lemma 8: *Let $f(x)$ be m times continuously differentiable in the interval of the integration below. Then for any positive number d we have,*

$$f(x)-\binom{m}{1}f(x+d)+\binom{m}{2}f(x+2d)+\cdots\cdots$$
$$+(-1)^m\binom{m}{m}f(x+md)$$

$$=(-1)^m\int_0^d\int_0^d\int_0^d\cdots\int_0^d f^{(m)}(x+u_1+u_2+\ldots\ldots u_m)\,du_1\ldots du_m$$

where $\binom{m}{1},\binom{m}{2},\ldots\ldots$ *are binomial coefficients.*

Proof: This can be proved by induction on m. Details are left to the reader.

From Lemmas 6, 7 and 8 we can deduce Theorem 1 as follows. We first assume condition (ii). Define $\alpha_1 = \frac{1}{2} + \frac{K_4 \log \log H}{\log H}$, and $\alpha_2, \ldots\ldots, \alpha_{m+1}$ by $\alpha_j = \alpha_1 + \frac{(j-1)(\log \log H)^{C_6}}{\log H}$ $(j = 2, \ldots, m+1)$ where C_6 is a large constant. Taking K_4 a large constant we see that $\frac{1}{H} Q_1(\alpha_1) > (\log H)^{ak^2+m} (\log \log H)^{-C_7}$ and this leads to Theorem 1 with $C = \max(1, C_7)$, in case $m = 0$. If $m \geqslant 1$, we see that $Q_1(\alpha_1)$ dominates $Q_1(\alpha_j)$, $(j \geqslant 2)$.

So taking $f(\alpha) = (F(\alpha + it))^{2k}$, $x = \alpha_1$, $d = \frac{(\log \log H)^{C_6}}{\log H}$ we see that

$$\sum_{I_3} \int_{I_3} \left| f(\alpha_1) - \binom{m}{1} f(\alpha_2) + \binom{m}{2} f(\alpha_3) + \ldots\ldots + (-1)^m \binom{m}{m} f(\alpha_{m+1}) \right| dt$$

$$< \int_0^d \ldots\ldots \int_0^d \Big(\sum_{I_3} \int_{I_3} | f^{(m)} (\alpha_1 + u_1 + \ldots + u_m) | \, dt \Big) \, du_1 \ldots du_m.$$

This shows that the max $Q_1(\sigma) H^{-1}$ and so of $Q(\sigma)$ in $\sigma \geqslant \frac{1}{2} + \frac{\log \log H}{\log H}$ exceeds $(\log H)^{ak^2+m} (\log \log H)^{-C_8}$ and this proves Theorem 1 with $C = \max(1, C_8)$.

Next we prove Theorem 1 subject to the condition (i). The case $m = 0$ can be disposed off as before. Let $m > 1$ and the theorem be false with a large constant C. Then in addition to Lemma 1 we also have

$$\max_{\sigma > \frac{1}{2} + \frac{\log \log H}{\log H}} \frac{1}{H} \int_T^{T+H} \left| \frac{d}{ds} (F(s))^{2k} \right| dt = O\left((\log H)^{ak^2+m_1} (\log \log H)^{-C} \right).$$

We choose α_1 and α_2 as before and we have

$$\sum_{I_3} \int_{I_3} (|f(\alpha_1)| - |f(\alpha_2)|)\, dt \leqslant \sum_{I_3} \int_{I_3} |f(\alpha_1) - f(\alpha_2)|\, dt$$

Here the left side exceeds $H (\log H)^{ak^2+m_1-1} (\log \log H)^{-C_9}$

while the right side is $O(H(\log H)^{ak^2+m_1-1} (\log \log H)^{C_6-C})$

which is a contradiction if we choose $C = C_6 + C_9 + 1$. This proves Theorem 1 with $C = C_6 + C_9 + 1$.

Before leaving this section we remark that if instead of condition (ii) we are given that the product

$\prod_p \left(1 + \frac{|\chi(p)a_p|^2}{p^{2\sigma}}\right)$ lies between $(\sigma - \frac{1}{2})^{-a'} \left(\log \frac{1}{\sigma - \frac{1}{2}}\right)^{b_1}$

and $(\sigma - \frac{1}{2})^{-a} \left(\log \frac{1}{\sigma - \frac{1}{2}}\right)^{b_2}$, where

$a' > a > 0$, b_1, b_2 are four real constants then we can conclude

$$\max_{\sigma \geqslant \frac{1}{2} + \frac{\log \log H}{\log H}} Q(\sigma) > (\log H)^{ak^2 + ma(a')^{-1}} (\log \log H)^{-C}.$$

This explains the second part of our remark below Theorem 2 since for non-Galois number fields we can take $a' = n$.

§ 4. **The case of no Euler product** : In this and the next section we assume instead of the Euler product a condition of the type, $\sum_{n \leqslant x} |a_n|^8 = O(x (\log x)^{k'})$ where k' is a positive constant and $x \geqslant 2$. (The condition can be relaxed, but we do not want to go into such questions). The main result of this section is

Theorem 3 *Suppose that as* $\sigma \to \frac{1}{2} + 0$, $|F(2\sigma)|$ *exceeds* $\left(\frac{1}{\sigma - \frac{1}{2}}\right)^a \left(\log \frac{1}{\sigma - \frac{1}{2}}\right)^{b_3}$ *where* $a > 1$ *and* b_3 *are two real constants. Then with* $2k = 1$, $m = 0$, *we have,*

$$\frac{1}{H} \int_T^{T+H} |F(\tfrac{1}{2} + it)|\, dt = Q(\tfrac{1}{2}) > (\log H)^{a-1} (\log \log H)^{-C}.$$

Proof: It suffices to prove that max $Q(\sigma)$ for $\sigma \geqslant \frac{1}{2} + \frac{\log\log H}{\log H}$, exceeds $(\log H)^{a-1} (\log\log H)^{-C_1}$ and Theorem 3 follows from this just as Theorem 2 follows from Theorem 1. (The constants C, K with or without subscripts should not be confused with the earlier constants. Also to save space the proof will be only sketchy). We assume that this is false. Lemma 2 has its analogue without modification.

We put $\alpha_1 = \frac{1}{2} + \frac{2\log\log H}{\log H}$. Let $\alpha \geqslant \alpha_1 + \frac{\log\log H}{\log H}$, and put $s_0 = \alpha + it$. We divide the interval $T_1 = T + (\log H)^8 \leqslant t \leqslant T_2 = T + H - (\log H)^8$ into equal intervals J of length $(\log H)^{K_1}$ ignoring a bit at one end. We put

$$\zeta_Y(s) = \sum_{n \leqslant Y} n^{-s}, \quad \zeta^{-1}_Y(s) = \sum_{n \leqslant Y} \mu(n) n^{-s}$$

where $Y = H^{\frac{1}{4}}$. It is not hard to prove that if $\phi(s) = \zeta_Y(s)\,\zeta^{-1}_Y(s) - 1$ and $M(J)$ denotes the maximum of $|\phi(s_0)|$ for t in J,

$$\sum_J (M(J))^2 = O(H(\log H)^{10} Y^{1-2\alpha})$$

We omit those intervals J for which $M(J) \geqslant \frac{1}{2}$ and denote the rest of the intervals by I. The number of intervals J which are excluded is $O(HY^{1-2\alpha}(\log H)^{10})$. If K is a sufficiently large constant and $\alpha = \frac{1}{2} + \frac{K\log\log H}{\log H}$ we proceed to prove that $\sum_I \int_I |F(s_0)|\,dt > H(\log H)^{a-1}(\log\log H)^{-C_2}$.

We write $F(s_0) = F_Y(s_0) + E_Y(s_0)$ where $F_Y(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0} e^{-\frac{n}{Y}}$,

and we have $\int_T^{T+H} |E_Y|\,dt = O(H(\log H)^{C_3} Y^{\alpha - \alpha_1})$.

We can replace the integrand $|F(s_0)|$ by $|F_Y(s_0)\,\zeta^{-1}_Y(s_0)\,\zeta_Y(s_0)|$ without disturbing the left side very much. Then we replace

$\sum_I \int_I |\ldots| \, dt$ by $\int_T^{T+H} |\ldots| \, dt$ without much error. Next we

use the fact that the last integral is bounded below by

$$\left| \int_T^{T+H} F_Y(s_0) \zeta_Y^{-1}(s_0) \overline{\zeta_Y(s_0)} \, dt \right|.$$

To see that this is $\gg H \, | F(2\alpha) | \, (\alpha - \frac{1}{2})$, we use the following lemma and some simple computations and this would complete the proof of Theorem 3.

Lemma 9 : *If $\{ x_n \}$ and $\{ y_n \}$ are two sequences of complex numbers, then,*

$$\int_0^T \left(\sum_{n=1}^{\infty} x_n n^{-it} \right) \left(\sum_{n=1}^{\infty} \bar{y}_n n^{it} \right) dt$$

$$= T \sum_{n=1}^{\infty} x_n \bar{y}_n + O\left(\left(\sum_{n=1}^{\infty} n \, |x_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n \, |y_n|^2 \right)^{\frac{1}{2}} \right)$$

Remark. For a simple proof of this lemma see [4].

§ 5. Balasubramanian's remark

Theorem 4. *In the case $2k = 1$, $m = 0$ as before, we have,*

$$\frac{1}{H} \int_T^{T+H} | F(\tfrac{1}{2}+it) | \, dt = Q(\tfrac{1}{2}) \gg \max \left(1, S_2^{\frac{3}{2}} S_3^{-1} (\log \log H)^{-1}\right)$$

where $S_2 = \sum_{n=1}^{\infty} | a_n |^2 n^{-2\alpha}$, $S_3 = \sum_{n=1}^{\infty} | a_n |^2 d(n) n^{-2\alpha}$,

$\alpha = \frac{1}{2} + \dfrac{C \log \log H}{\log H}$ *(where C is a large positive constant) and $d(n)$ is the usual divisor function.*

Proof: As before we assume $\max Q(\sigma)$ for $\sigma \geqslant \frac{1}{2} + \dfrac{\log \log H}{\log H}$ does not exceed $S_2^{\frac{3}{2}} S_3^{-1}$ (Note that this is

$O((\log H)^{C_4})$. Next we write $s_0 = \alpha + it$, $F(s_0) = F_Y(s_0) + E_Y$, where $Y = H^{\frac{1}{4}}$, $F_Y(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0} e^{-\frac{n}{Y}}$. We have an asymptotic formula for $\int_T^{T+H} |F_Y(s_0)|^2 \, dt$, and also a good upper bound for $\int_T^{T+H} |F_Y(s_0)|^4 \, dt$. Using

$$\int_T^{T+H} |F_Y(s_0)|^2 \, dt \leqslant \left(\int_T^{T+H} |F_Y(s_0)| \, dt\right)^{\frac{2}{3}} \left(\int_T^{T+H} |F_Y(s_0)|^4 \, dt\right)^{\frac{1}{3}}$$

we are led to the theorem. The details are left to the reader.

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