Hardy-Ramanujan Journal
Vol. 3 (1980) 1—24

# SOME REMARKS ON THE MEAN VALUE OF THE RIEMANN ZETA-FUNCTION AND OTHER DIRICHLET SERIES - II

By K. RAMACHANDRA

To the Academician Ivan Matveevich Vinogradov
A Humble Dedication on his Ninetieth Birthday

## § 1. Introduction

The object of this note is to prove the following (all constants in the course of this work are effective positive constants) theorem.

**Theorem 1 :**

*Let* k *and* m *be any two fixed natural numbers and let* $100 \leqslant (\log \mathrm{T})^{\frac{1}{\mathrm{m}}} \leqslant \mathrm{H} \leqslant \mathrm{T}$. *Then for any non-negative integer* $l$ *and all* $\mathrm{T} \geqslant \mathrm{T}_0(\mathrm{k}, l, \mathrm{m})$ *we have,*

$$\frac{1}{H}\int_{\mathrm{T}}^{T+H}\left|\frac{d^l}{ds^l}\left((\zeta(s))^k\right)\right|_{s=\frac{1}{2}+it} dt > C_{k,\,l}(\log H)^{\lambda}, \qquad (1)$$

*where* $\lambda = l + \frac{k^2}{4}$, and $\mathrm{C}_{\mathrm{k},\,l}$ *is a positive constant depending only on* k *and* $l$. *As simple special cases, we have,*

$$\frac{1}{H} \int_{T}^{T+H} \left| \zeta\left(\tfrac{1}{2} + it\right) \right| dt \gg (\log H)^{\frac{1}{4}}, \tag{2}$$

*and*

$$\frac{1}{H} \int_{T}^{T+H} \left| \zeta'\left(\tfrac{1}{2} + it\right) \right| dt \gg (\log H)^{\frac{5}{4}}. \tag{3}$$

*where, as usual,*

$$\zeta\left(\tfrac{1}{2} + it\right) = \left(1 - 2^{\frac{1}{2} - it}\right)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\frac{1}{2} + it}}. \tag{4}$$

**It must be remarked that these results generalise easily to zeta functions of algebraic number fields and so on. To enable this we state our result in the following form.**

**Theorem 2 :**

*Let* $\{a_n\}$ *be a sequence of complex numbers, and* $A_1, A_2, \ldots\ldots A_7$, *be positive constants satisfying the following conditions.*

(i) *For* $1 < \sigma < 2$, *the series* $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{\sigma}}$ *converges and lies between* $A_1\, B(\sigma)$ *and* $A_2\, B(\sigma)$ *where* $B(\sigma) = \left(\frac{1}{\sigma - 1}\right)^{\mu}$ *for some positive constant* $\mu$ *(we can also work with more general functions* $B(\sigma)$ *as can be easily seen).*

(ii) *Let* $A_3 \leqslant \lambda_1 < \lambda_2 < \lambda_3 < \ldots\ldots$ *where, for* $n \geqslant 1$, $\lambda_{n+1} - \lambda_n \geqslant A_4$ *and* $\lambda_n \leqslant A_5\, n$.

(iii) *Let* $F(s) = \left(\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^{\,s}}\right)^2$ *(which is a regular function* *of* $s=\sigma+it$ *by assumption* (i) *in* $\sigma>1$) *admit an analytic continuation in* $t \geqslant A_6 \geqslant 10, \sigma \geqslant \frac{1}{2}$ *and there* $|F(s)| \leqslant \operatorname{Exp}((\log t)^{A_7})$. *Then for any non-negative integer* $l$ *and all* $T \geqslant T_0(l, m, A_1 \ldots A_7)$ (*with* m, H *as in the statement of Theorem 1*), *we have,*

$$\frac{1}{H}\int_T^{T+H} |F^{(l)}(\tfrac{1}{2}+it)|\, dt \gg B\left(1+\frac{1}{\log H}\right)(\log H)^l \qquad (5)$$

*where the constant implied by the Vinogradov symbol* $\gg$ *is independent of* m, T *and* H.

**Remark 1 :** As a historical introduction to our theorems we mention that ours is an attempt to get a lower bound for

$$\frac{1}{H}\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^k\, dt$$

where $H$ and $T$ are positive. So far the only result known in this direction is an old result due to E. C. Titchmarsh and his result (see Theorem 7. 19 of his famous book [ 6 ] ), asserts that if $0 < \delta \leqslant \frac{1}{2}$ then

$$\int_0^{\infty} e^{-\delta t} |\zeta(\tfrac{1}{2}+it)|^k\, dt \gg \frac{1}{\delta}\left(\log \frac{1}{\delta}\right)^{k^2/4} \qquad (6)$$

where $k$ is *an even positive integer.* Given (6) for any (positive real) $k$ we can deduce for the same $k$ (as far as I am aware) only

$$\limsup_{T \to \infty} \left( \left( \frac{1}{T}\int_T^{2T} \left|\zeta(\tfrac{1}{2}+it)\right|^k dt \right) (\log T)^{-\frac{k^2}{4}} \right) > 0. \qquad (7)$$

It is also obvious that given (7) with "lim sup" replaced by "lim inf" we can recover (6) for the same $k$. But (7) does not seem to imply (6). Because of some difficulties of principle, Titchmarsh had to restrict his proof of (6) only to even positive integers $k$. By certain improvements on the method of Titchmarsh we have achieved in this paper the proof of (6) for all positive integers $k$ and not only that. But we establish (7) with "lim sup" replaced by "lim inf". Yet another point which I have to say is that we can handle things like

$$\frac{1}{H}\int_{T}^{T+H} |\zeta(\tfrac{1}{2}+it)|^{k}\, dt \text{ and } \frac{1}{H}\int_{T}^{T+H} \left|\frac{d^{l}}{dt^{l}}(\zeta(\tfrac{1}{2}+it))^{k}\right| dt$$

and get "optimal lower bounds" provided that $\dfrac{\log H}{\log\log T}$ is bounded below by a positive constant. All these things are clear from the statement of our Theorem 1. We have used the phrase "optimal lower bounds" to mean "what is believed to be optimal". In a forthcoming paper "Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series III" I show actually that

$$\frac{1}{H}\int_{T}^{T+H} \left|\frac{d^{l}}{dt^{l}}(\zeta(\tfrac{1}{2}+it))^{k}\right| dt = O\left((\log H)^{\frac{k^{2}}{4}+l}\right)$$

under certain conditions on $H$ and $k$ and in particular when $0 \leqslant k \leqslant 4$, $k$ is real, $0 \leqslant l \leqslant k$ (this restriction on $l$ is unnecessary if $k$ is an integer) and $H = T$. As is well known we do not need Riemann hypothesis when $k = 0$, 2 and 4. But I prove this result also when $k = 1$. However when $0 < k < 4$ and $k \neq 1, 2$, I need the assumption of a hypothesis which is slightly weaker than Riemann hypothesis. Every expert believes that this is true at least for all integers $k > 0$ provided $H = T$; but nobody appears to have proved this result say even for $k = 6$ on the assumption of Riemann hypothesis.

**Remark 2 :** In a previous paper [4] with the same title as the present one, I considered (1) where $k$ is any real number $\geqslant 1$ and obtained an imperfect result where $(\log H)^{\lambda}$ is replaced by $(\log H)^{\lambda}$ $(\log\log H)^{-C}$ where $C$ was a positive constant depending only on $k$ and $l$. In this greater generality it remains now to improve this result by knocking off the factor $(\log\log H)^{-C}$.

**Remark 3 :** I take this opportunity to explain a certain term "Frequency of Titchmarsh's phenomenon" with reference to mean value of Dirichlet series. Since all these ideas stem from the work of Titchmarsh I prefer to define a class of Dirichlet series (which I prefer to call Titchmarsh series and for some reasons I feel it better to call it Kummer-Dirichlet-Titchmarsh series or briefly K. D. T. series) which I call K. D. T. series. Let $1=\lambda_1<\lambda_2<\lambda_3<\ldots\ldots\ldots\ldots$ where $\frac{1}{A}\leqslant\lambda_{n+1}-\lambda_n\leqslant A$ where $A$ is a positive constant $>1$. Let $\{a_n\}$ be a sequence of complex numbers which may depend on a parameter $H>10$, with the properties $a_1=1$ and $|a_n|\leqslant(\lambda_n H)^A$. Surely $F(s)=\sum_{n=1}^{\infty}(a_n\lambda_n^{-s})$ converges absolutely and hence uniformly in a half plane. $F(s)$ is called a KDT series if there exists a family of rectangles $R(T,H)=\{\sigma\geqslant\frac{1}{2},T\leqslant t\leqslant T+H\}$ where $10\leqslant(\log T)^{\frac{1}{A}}\leqslant H\leqslant T$ and $T\to\infty$, such that $F(s)$ admits an analytic continuation into every one of them and there $|F(s)|\leqslant\text{Exp}((\log T)^A)$. Let $L(T,H)$ denote the left hand side of the rectangle $R(T,H)$. I start by proposing the

**Conjecture :**

$$\frac{1}{H}\int_{L(T,H)}|F(\tfrac{1}{2}+it)|^2\,dt>C_A\sum_{\lambda_n\leqslant X}\frac{|a_n|^2}{\lambda_n},$$

where $X=2+10^{-8A}H$ and $C_A>0$ depends only on $A$.

For a long time I have many results with me in the direction of this conjecture. I take this opportunity to state one or two of them. (The others along with these and detailed proofs will appear elsewhere). One of my theorems is as follows :

$$\left(\frac{1}{H}\int_{L(T,H)} |F(\tfrac{1}{2}+it)|^2\,dt\right)+1>C_A\left((\log H)^{-2E_2}\right.$$

$$\left.\sum_{\lambda_n<X} |a_n|^2 \lambda_n^{-1}\left(\frac{\log\lambda_n}{\log H}\right)^{2E_2}\left(\log\frac{X}{\lambda_n}\right)^{2E_2}\right) \qquad (8)$$

where as before $C_A > 0$ depends only on $A$, and $E_2$ is a positive integer (depending only on $A$), and $X=2+10^{-8A}H$. The result (8) is interesting first as a general result and next for the fact that it covers two important results as corollaries.

For if $(\zeta(s))^k = \sum_1^\infty \frac{d_k(n)}{n^s}$, we have from (8),

$$\frac{1}{H}\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt > C_A\left((\log H)^{-2E_2}\right.$$

$$\left.\sum_{n\leq X} \frac{(d_k(n))^2}{n}\left(\frac{\log n}{\log X}\right)^{2E_2}\left(\log\frac{X}{n}\right)^{2E_2}\right)$$

$= C_A R$, say.

While well-known results show that $R \gg (\log H)^{k^2}$, we also have

$$\max_{t\,\varepsilon\,L(T,H)} |\zeta(\tfrac{1}{2}+it)| > (\tfrac{1}{2}C_A R)^{\frac{1}{2k}} \qquad (9)$$

provided $k \leqslant \log H$. The authors of [3] have shown that maximum of the $RHS$ of (9) as $k$ varies is attained in $k\leqslant\log H$ and that if

$$Q=\left(\frac{\log\log H}{\log H}\right)^{\frac{1}{2}}\log\left(\max_k(\tfrac{1}{2}C_A R)^{\frac{1}{2k}}\right) \qquad (10)$$

then $Q$ is bounded below by a positive constant. But Dr. R. Balasubramanian has shown (subsequently) by an ingeneous reasoning that lim $Q$ exists as $H \to \infty$ and is equal to

$$\frac{1}{2} \max_{l > 0} \left( \left( e^{-2l} + \int_{2l}^{\infty} \frac{e^{-\theta}}{\theta} d\theta \right) (2\, l\, e^{2l})^{\frac{1}{2}} \right) \quad (11)$$

Professor H. E. Richert took a keen interest in computing this constant His computer calculations show that it is 0.75 ..... Accordingly

$$\max_{T \leqslant t \leqslant T+H} | \zeta(\tfrac{1}{2} + it) | > \operatorname{Exp}\left( \frac{3}{4} \left( \frac{\log H}{\log \log H} \right)^{\frac{1}{2}} \right) \quad (12)$$

where $100000 \leqslant (\log T)^{\frac{1}{100000}} \leqslant H \leqslant T$. In earlier papers than my joint paper [ 3 ], I dealt with problems considered here and some other problems. I use the word Titchmarsh's phenomenon to those problems considered here and some other problems considered in some other papers of mine viz. [ 1 ], [ 2 ], [ 3 ] and [ 4 ]. Sometimes as in [ 4 ] I deal with real $k$ as well. In addition to (8) I quote another result (in the direction of my conjecture) of mine namely

$$\frac{1}{H} \int_{L(T,H)} | F(\tfrac{1}{2} + it) |^2 dt + 1 > C_A (\log \log H)^{-2} \left( \sum_{n \leqslant X} \frac{|a_n|^2}{\lambda_n} \left( \frac{\log \lambda_n}{\log H} \right)^{2E_2} \right) \quad (8)'$$

which is nearly contained in a result included at my request by Dr. R. Balasubramanian in pages 571-575 of his paper in Proceedings of the London Math. Society (3) 36 (1978) 540-576. Results like (8) or (8)' or even some of my earlier results are enough to prove (12). But we have always to use the ingeneous limit evaluation (11) of Dr. R Balasubramanian to prove such a powerful result as (12). (8) and (8)' have been improved by me to

$$> C_A \sum_{n \leqslant X} \frac{|a_n|^2}{\lambda_n} \left( 1 - \frac{\log \lambda_n}{\log H} + \frac{1}{\log \log H} \right) \quad (8)''$$

**Acknowledgements**

I am extremely thankful to my colleague Dr. R. Balasubramanian for his readiness to listen to my expositions of this work, many useful suggestions and much encouragement by way of appreciation. Apart from this indebtedness, I am extremely thankful to Professors M. Jutila, A. Baker, F. R. S., and D. R. Heath-Brown for their encouragement.

The proofs of Theorems 1 and 2 depend on an idea (embodied in Theorem 4 below) which seems to be new. For convenience we split up the proof into six sections (sections § 2 to § 7). The proof is self contained and a beginner of complex function theory (of the Hardy-Titchmarsh style) should have no difficulty in verifying the steps involved in the proof.

## § 2. A Preliminary Lemma

With any complex valued function $f(x)$ where $x$ is real or complex and with any real $d > 0$ and any integer $l \geqslant 0$ we introduce the function $\Delta_{d,l} f(x)$ defined by

$$\Delta_{d,l} f(x) = f(x) - \binom{l}{1} f(x+d) + \binom{l}{2} f(x+2d) - \dots + \dots + (-1)^l f(x+ld),$$

$f(x)$ being defined in the necessary range. Under the assumption that $f(x)$ is (as a function of the Real part of $x$) $l$ times continuously differentiable, we have,

**Lemma 1 :**

$$\Delta_{d,l}(f(x)) =$$

$$(-1)^l \int_0^d \dots \int_0^d f^{(l)}(x + u_1 + u_2 + \dots + u_l) du_1 \dots du_l,$$

*Proof.* Follows by induction.

## § 3. A Few Definitions and Notation

Let $\delta$ be a positive constant satisfying $0 < \delta < 1$. We start with four quantities $Y$, $X$, $H$ and $T$ such that $10 < (\log T)^{\delta} \leqslant H \leqslant T$, $10 \leqslant Y = X^{\frac{1}{10}} < X \leqslant H$. $C_1, C_2, \ldots, C_5$ are positive constants independent of another set of positive constants $K_1, K_2, \ldots, K_6$. The latter will be chosen to suit our convenience. Throughout, the variable $T$ will be supposed to exceed an effective positive constant depending upon $\delta$, $C_1, \ldots, C_5$, $K_1, \ldots, K_6$ and $l$. (This $\delta$ need not be confused with the $\delta$ in equation (6) of the introduction).

Let $0 < \lambda_1 < \lambda_2 < \ldots\ldots$; $\lambda_{n+1} - \lambda_n \gg 1$, and $\lambda_n = O(n)$. Let $\{a_n\}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$ converges absolutely somewhere in the complex plane.

We write

$$\psi_1(s) = \left( \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} e^{-\frac{\lambda_n}{X}} \right)^2,$$

$$\psi_2(s) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m a_n}{(\lambda_m \lambda_n)^s} e^{\frac{-\lambda_m \lambda_n}{X^2}},$$

$$\theta_1(s) = \Delta_{D,\,l}\,(\psi_1'(s) + 2\,\psi_1(s) \log X),$$

$$\theta_2(s) = \Delta_{D,\,l}\,(\psi_2'(s) + 2\,\psi_2(s) \log X),$$

($D$ is a real number to be fixed later) and set out to prove that $I_2(\sigma)$ defined by

$$I_2(\sigma) = \frac{1}{H}\int_T^{T+H} \left( |\theta_2(s)|_{s=\sigma+it} \right) dt,$$

is large in some sense. We also write

$$I_1(\sigma) = \frac{1}{H}\int_T^{T+H} \left( |\theta_1(s)|_{s=\sigma+it} \right) dt.$$

§ 4. **Study of $I_1(\sigma)$ and $I_2(\sigma)$.** We begin with the remark that $I_1(\sigma)$ and $I_2(\sigma)$ alter only by $o(1)$ when we change the end points of the interval $[T, T+H]$ of integration by quantities which are $O(H^{\frac{1}{2}})$. We now start with the integral

$$\frac{1}{2\pi i}\int_R \frac{(\theta_1(w) - \theta_2(w))\, Y^{w-s}\, e^{(w-s)^{4a+2}}}{(w-s)^2}\, dw$$

where $w = u+iv$, $a = a$ large positive integer constant, and the path of integration is the boundary of the rectangle $R$ with corners $(\sigma_1 - i\infty, a_1 - i\infty, a_1 + i\infty, \sigma_1 + i\infty, \sigma_1 - i\infty)$. We have assumed that $\sigma_1 < \sigma < a_1$, ($a_1$ is a large constant) and indicated five points (instead of four corners) in order to specify the orientation of the path (it is anticlockwise). From this integral one can deduce that,

$$\frac{1}{H}\int_T^{T+H} |\theta_1'(s) - \theta_2'(s) + \log Y(\theta_1(s) - \theta_2(s))|\, dt$$

$$= O\left((\sigma-\sigma_1)^{-1}\, Y^{-(\sigma-\sigma_1)}\left(\frac{1}{H}\int_T^{T+H} |\theta_1(s) - \theta_2(s)|_{\sigma=\sigma_1}\, dt\right)\right) + O\left(X^{-\frac{1}{100}}\right). \quad (13)$$

To prove this we argue with the limits T, $T+H$ (of integration) on the left replaced by $T+H^{\frac{1}{2}}$, $T+H-H^{\frac{1}{2}}$. Next we change back to the original limits.

Let us now make the assumption that

$$\max_{\sigma_1 - \frac{1}{\log X} \leqslant \sigma \leqslant \sigma_2 + \frac{1}{\log X}} \left( \frac{1}{H} \int_T^{T+H} |\theta_2(s)| \, dt \right)$$

$$= o\left( \frac{1}{H} \int_T^{T+H} |\theta_1(s)|_{\sigma=\sigma_1} dt \right). \qquad (14)$$

where $\sigma_2 > \sigma_1 + \frac{2}{\log X}$.

From (14) and Cauchy's theorem it follows that

$$\max_{\sigma_1 \leqslant \sigma \leqslant \sigma_2} \left( \frac{1}{H} \int_T^{T+H} |\theta'_2(s)| \, dt \right)$$

$$= o\left( \frac{\log X}{H} \int_T^{T+H} |\theta_1(s)|_{\sigma=\sigma_1} dt \right). \qquad (15)$$

From this and (13) it follows that if $\sigma_1 < \sigma < \sigma_2$, then,

$$\frac{1}{H} \int_T^{T+H} |\theta_1'(s) + \theta_1(s) \log Y| \, dt$$

$$= \left( \frac{1}{H} \int_T^{T+H} |\theta_1(s)|_{\sigma=\sigma_1} dt \right)$$

$$\left(O\left((\sigma-\sigma_1)^{-1} Y^{-(\sigma-\sigma_1)}\right) + o(\log X)\right). \qquad (16)$$

The left hand side in (16) is $>$

$$\frac{\log Y}{H} \int_{T}^{T+H} \left| \theta_1 (s) \right| dt - \frac{1}{H} \int_{T}^{T+H} | \theta_1' (s) | \, dt. \tag{17}$$

Note that

$$| \theta_1 (s) | > 2 \log X | \psi_1 (s) | - | \psi_1' (s) | - \sum_{\nu=1}^{l} \binom{l}{\nu} \Big(2 \log X \left| \psi_1 (s+\nu D) \right| + | \psi_1' (s+\nu D) | \Big), \tag{18}$$

$$| \theta_1 (s) | < 2 \log X | \psi_1 (s) | + | \psi_1' (s) | + \sum_{\nu=1}^{l} \binom{l}{\nu} \Big(2 \log X \left| \psi_1 (s+\nu D) \right| + | \psi_1' (s+\nu D) | \Big), \tag{19}$$

and

$$| \theta_1'(s) | < | \psi_1'' (s) | + 2 \log X | \psi_1' (s) | + \sum_{\nu=1}^{l} \binom{l}{\nu} \Big(2 \log X \left| \psi_1' (s+\nu D) \right| + | \psi_1'' (s+\nu D) | \Big). \tag{20}$$

We can now integrate from $T$ to $T+H$ in (18), (19) and (20) and we get lower and upper bounds for $\frac{1}{H} \int_{T}^{T+H} | \theta_1 (s) | \, dt$ and an upper bound for $\frac{1}{H} \int_{T}^{T+H} | \theta_1' (s) | \, dt$ valid in $\sigma_1 < \sigma < \sigma_2$.

To facilitate computations we use the following special case of a theorem of H. L. Montgomery and R. C. Vaughan as a lemma in the course of our work.

**Theorem 3 :** *We have*

$$\frac{1}{H}\int_{T}^{T+H}\left|\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n^{s}}e^{-\frac{\lambda_n}{X}}\right|^2 dt$$

$$=\sum_{n=1}^{\infty}\left(1+O\left(\frac{n}{H}\right)\right)\frac{|a_n|^2}{\lambda_n^{2\sigma}}e^{-\frac{2\lambda_n}{X}},$$

*where* $\lambda_n = O(n)$ *and* $\lambda_{n+1}-\lambda_n$ *are bounded below, and* $0<\lambda_1<\lambda_2<\ldots\ldots$

**Remark :** This special case admits of a simple proof (see [5] ).

Before proceeding further we record a few simple lemmas. (Also to simplify matters we assume condition (i) of Theorem 2),

**Lemma 2 :** *If* $\frac{1}{2}<\sigma<1$ *and* $j=1, 2, 3, \ldots\ldots$, *we have* (*of course, not uniformly in* $j$)

$$\sum_{n=1}^{\infty}\frac{|a_n|^2}{\lambda_n^{2\sigma}}(\log\lambda_n)^j = O\left(\left(\sigma-\tfrac{1}{2}\right)^{-j-\mu}\right).$$

*Proof.*

$$\text{L. H. S.} = \frac{(-2)^{-j}j!}{2\pi i}\int\left(\sum_{n=1}^{\infty}\frac{|a_n|^2}{\lambda_n^{2w}}\right)\frac{dw}{(w-\sigma)^{j+1}}$$

the path of integration being the circle $|w-\sigma| = \frac{\sigma-\frac{1}{2}}{100}$, traversed in the anti-clockwise direction.

**Lemma 3:** *We have*

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{\lambda_n^{2\sigma}} e^{-\frac{2\lambda_n}{X}}$$

$$\geqslant e^{-1} \sum_{n=1}^{\infty} \frac{|a_n|^2}{\lambda_n^{2\sigma}} + O\left(\frac{X^{-(\sigma-\frac{1}{2})}}{(\sigma-\frac{1}{2})^{\mu}}\right)$$

*Proof* We have only to observe that $e^{-\frac{2\lambda_n}{X}} \geqslant e^{-1}$ if $\lambda_n \leqslant \frac{X}{2}$ and also if $\lambda_n \geqslant \frac{X}{2}$, we have

$$\frac{1}{\lambda_n^{2\sigma}} \leqslant \frac{1}{\lambda_n^{2(\sigma-\frac{1}{2}(\sigma-\frac{1}{2}))}} \left(\frac{1}{\left(\frac{X}{2}\right)^{\sigma-\frac{1}{2}}}\right).$$

**Lemma 4:** *We have,*

$$\frac{1}{H}\int_{T}^{T+H} \left| \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^{s}} e^{-\frac{\lambda_n}{X}} \right|^2 dt = O(B(2\sigma)).$$

*Also the quantity on the left is*

$$> C_1 B(2\sigma) + O(e^{-\frac{K_1}{2}} B(2\sigma)), \tag{21}$$

provided $1 > \sigma \geqslant \frac{1}{2} + \frac{K_1}{\log X}$.

*Proof* In Theorem 3 we use $\frac{n}{H} e^{-\frac{2\lambda_n}{X}} \leqslant \frac{X_n}{H\lambda_n}$ $= O(1)$ and this gives the first part. To prove the second part we use the following facts. In $1 + O\left(\frac{n}{H}\right)$ we need consider only terms for which $n$ exceeds a small positive constant multiple of $H$. In this portion of the sum,

$$\sum_n \frac{n}{H} \frac{|a_n|^2}{\lambda_n^{2\sigma}} e^{-\frac{2\lambda_n}{X}}$$

$$= O\left(\frac{1}{H} \sum_n \frac{n\,|a_n|^2 e^{-\frac{2\lambda_n}{X}}}{\lambda_n^{2(\sigma-\phi+\phi)}}\right)$$

where $\phi = \left(\frac{\sigma - \frac{1}{2}}{4}\right)$ and so $2\phi > \frac{1}{2}(\sigma - \frac{1}{2}) > \frac{K_1}{2 \log X}$. We next use $e^{-\frac{2\lambda_n}{X}} < \frac{X}{2\lambda_n}$ and that $\lambda_n \gg n$. This completes the proof of Lemma 4

**Lemma 5 :** *We have*

$$\frac{1}{H} \int_T^{T+H} \left|\psi'_1(s)\right| dt = O\left(\frac{B(2\sigma)}{\sigma - \frac{1}{2}}\right), \quad (22)$$

*and*

$$\frac{1}{H} \int_T^{T+H} \left|\psi''_1(s)\right| dt = O\left(\frac{B(2\sigma)}{(\sigma - \frac{1}{2})^2}\right), \quad (23)$$

*Proof.* Follows from Holder's inequality and Theorem 3.

We now assume

$\frac{1}{2} + \frac{K_1}{\log X} < \sigma < \frac{1}{2} + \frac{K_2}{\log X}$ where $K_1$ is a large constant and $K_2 > K_1$. We now set $\sigma_1 = \frac{1}{2} + \frac{K_3}{\log X'}$, $\sigma = \frac{1}{2} + \frac{K_3 + K_4}{\log X}$,

$\sigma_2 = \frac{1}{2} + \frac{K_3 + K_4 + K_5}{\log X}$ and $D = \frac{K_6}{\log X}$.

From equations (16) to (23) we get

$$\frac{C_1 - C_2 e^{-\frac{K_3}{2}}}{10\,(K_3 + K_4)^{\mu}} - \frac{C_3}{(K_3 + K_4)^{\mu+1}} - \frac{2^{l}\, C_4}{(K_3 + K_4 + K_6)^{\mu}} \leq \frac{2^{l}\, C_5\, e^{-\frac{K_4}{10}}}{K_4\, K_3^{\mu}} \qquad (24)$$

(the real parts of all the numbers $s$, $s + \nu D$ which appear in the definition of $\theta_1(s)$ and $\theta_2(s)$ when we put $\sigma$ = the numbers above involving $K_3$, $K_4$, $K_5$ are such that they can be accommodated in the inequality involving $K_1$, $K_2$; $K_1$, $K_2$ will be chosen in this manner).

To contradict (24 it suffices to put $K_3 = K_4$ and $K_6 = K_3^2$ and then choose a large constant $K_3$. This contradiction proves that (14) is false. Since our computations show also that

$$\frac{1}{H}\int_{T}^{T+H} \left| \theta_1(s) \right|_{\sigma = \sigma_1} dt \gg (\log X)^{\mu+1},$$

We must therefore have

$$\max_{\sigma_1 - \frac{1}{\log X} \leqslant \sigma \leqslant \sigma_2 + \frac{1}{\log X}} \left( \frac{1}{H} \int_T^{T+H} \left| \theta_2(s) \right| dt \right) \gg (\log X)^{\mu+1}.$$

## § 5 Main Result

We now collect the result proved so far. It is our first main result and runs as follows.

**Theorem 4 :**

*Let* $10 \leqslant X \leqslant H$, $0 < \lambda_1 < \lambda_2 < \ldots$ *where* $\lambda_n = O(n)$ *and* $\lambda_{n+1} - \lambda_n$ *is bounded below. Let* $\{a_n\}$ *be a sequence of complex numbers such that* $\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$ *is convergent at least at one complex* $s$. *Put*

$$\psi_2(s) = \sum_{\substack{m \geqslant 1 \\ n \geqslant 1}} \frac{a_m a_n}{(\lambda_m \lambda_n)^s} e^{-\frac{\lambda_m \lambda_n}{X^2}},$$

$$\psi_2^*(s) = \psi_2'(s) + 2\psi_2(s) \log X,$$

*and for* $D > 0$ *and non-negative integers* $l$, *define*

$$\theta_2(s) = \psi_2^*(s) - \binom{l}{1} \psi_2^*(s+D) + \binom{l}{2} \psi_2^*(s+2D) - + \ldots + (-1)^l \binom{l}{l} \psi_2^*(s+lD).$$

*Then with $a = \frac{1}{2}$, we have, given any positive constant $K_7$, constants $K_6$ and $K_8$ (which are effective and positive) such that $K_8 > K_7$ and if $D = \frac{K_6}{\log X}$,*

$$\max_{a + \frac{K_7}{\log X} \leq \sigma \leq a + \frac{K_8}{\log X}} \left( \frac{1}{H} \int_T^{T+H} |\theta_2(s)| \, dt \right) \gg \log X \sum_{n=1}^{\infty} \frac{|a_n|^2}{\lambda_n^{2a}} e^{-\frac{2\lambda_n}{X}}, \tag{25}$$

*and moreover the constant implied by the Vinogradov symbol $\gg$ is effective.*

*(We have proved (25) under the assumption that if $\sigma > \frac{1}{2}$, $B(2\sigma)$ is both $\gg$ and $\ll (\sigma - \frac{1}{2})^{-\mu}$ where $\mu$ is a positive constant, independent of $\sigma$).*

**Remark :** Some drastic restrictions seem to be necessary on $B(2\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|^2}{\lambda_n^{2\sigma}}$ and this can be seen as follows.

Suppose $a_n$ to be defined by

$\zeta(s) - 2^{-s} 10^{100} = \left( \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right)^2$ where $a_1 = 1$. In this case $\lambda_n = n$ and one can see that (25) is false say for $a = \frac{1}{2}$.

## § 6. A Corollary to Theorem 4.

From Theorem 4 (and using lemma 1) we can deduce the following corollary :

**Theorem 5 :**

*Let* $10 \leqslant X \leqslant H$, $0 < \lambda_1 < \lambda_2 < \dots$, $\lambda_n = O(n)$ *and* $\lambda_{n+1} - \lambda_n$ *be bounded below. Suppose that the sequence of complex numbers* $a_n$ *is such that* $B(2\sigma)$ *defined as* $\sum_{n=1}^{\infty} \frac{|a_n|^2}{\lambda_n^{2\sigma}}$ *(the assumption involves the convergence of this series) is both* $\gg$ *and* $\ll \left(\frac{1}{\sigma - \frac{1}{2}}\right)^{\mu} + 1$ *where* $\sigma > \frac{1}{2}$ *and* $\mu$ *is a positive constant independent of* $\sigma$. *Put*

$$\psi_2(s) = \sum_{\substack{m \geqslant 1 \\ n \geqslant 1}} \frac{a_m a_n}{(\lambda_m \lambda_n)^s} e^{-\frac{\lambda_m \lambda_n}{X^2}} \quad \text{and}$$

$$\psi_2^*(s) = \psi_2'(s) + 2\,\psi_2(s) \log X.$$

*Then given any positive constant* $K_9$, *there exists an effective positive constant* $K_{10} > K_9$ *such that*

$$\max_{\frac{1}{2} + \frac{K_9}{\log X} \leqslant \sigma \leqslant \frac{1}{2} + \frac{K_{10}}{\log X}} \left( \frac{1}{H} \int_T^{T+H} \left| \frac{d^l}{ds^l} \left( \psi_2^*(s) \right) \right| dt \right) \gg (\log X)^{l+1} B\left(1 + \frac{1}{\log X}\right)$$

*where as usual the implied constant is effective.*

## § 7. Application of Theorem 5 to the Final Result :

We now assume that the function $F(s) = \left( \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \right)^2$

can be continued in $\sigma \geqslant \frac{1}{2}$, $t \geqslant t_0$ and there satisfies $|F(s)| \leqslant \operatorname{Exp}((\log t)^A)$ where $A$ is a positive constant, and $a_n$ and $\lambda_n$ satisfy the same conditions as in Theorem 5. We now prove

**Theorem 6 :**

*Let* $0 < \delta \leqslant 1$ *and* $100 \leqslant (\log T)^{\delta} \leqslant H \leqslant T$. *Then for any non-negative integer* $l$, *we have*

$$\frac{1}{H} \int_{T}^{T+H} \left| F^{(l)} (\tfrac{1}{2} + it) \right| dt \gg (\log H)^{l} \; B \left(1 + \frac{1}{\log H}\right)$$

*the constant implied is independent of H and T and is effective.*

**Remark 1 :** A similar result also holds with $\frac{1}{2} + it$ replaced by $\sigma + it$ where $\sigma > \frac{1}{2}$ and $\sigma - \frac{1}{2} = O\left(\frac{1}{\log H}\right)$. This is clear from our proof.

**Remark 2 :** Theorem 2 follows from this by putting $\delta = \frac{1}{m}$.

Theorem 1 follows by writing $F(s) = \left( (\sqrt{\zeta(s)})^k \right)^2$ in Theorem 2.

**Proof of Theorem 6 :**

By Cauchy's Theorem it follows that in $t \geqslant t_0$, $\sigma \geqslant \frac{1}{2} + \frac{1}{\log H}$ we have (for $0 \leqslant \nu \leqslant l$), $|F^{(\nu)}(s)| \leqslant \operatorname{Exp}((\log t)^{2A})$. Let us assume the theorem to be false. Then it follows that in $(\sigma > \frac{1}{2} + \frac{1}{\log H}, T + H^{\frac{1}{2}} \leqslant t \leqslant T+H-H^{\frac{1}{2}})$ we have $|F^{(\nu)}(s)| \lessapprox H^{100}$ for $0 \leqslant \nu \leqslant l + 3$. To see this we start with

$$\frac{F^{(\nu)}(s)}{\nu!} = \frac{1}{2\pi i}\int \frac{F(w)\, e^{(w-s)^{4a+2}}}{(w-s)^{\nu+1}}\, dw \quad (a, \text{ large}$$

positive integer constant) where the path of integration is the boundary of a rectangle, in the anticlockwise direction. The rectangle is assumed to contain $s$ in it. By choosing a suitable rectangle we get the assertion made. Next we start with

$$\left(\frac{d^{l+1}}{ds^{l+1}}(\Psi_2^*(s))\right) X^{2s} = \frac{d}{ds}(X^{2s}\,\Psi_2^{(l+1)}(s))$$

$$= \frac{1}{2\pi i}\int_{Re\, w = 10} \phi'(w+s)\,\Gamma(w)\, dw$$

(where $\phi(s) = F^{(l+1)}(s)\, X^{2s}$ and $\frac{1}{2} < \sigma < 1$).

$$= \frac{d}{ds}(F^{(l+1)}(s) X^{2s}) + \frac{1}{2\pi i}\int_{Re\, w = -\frac{1}{\log H}} \phi'(w+s)\,\Gamma(w)\, dw$$

$$= \frac{d}{ds}(F^{(l+1)}(s) X^{2s}) + \frac{1}{2\pi i}\int_{Re\, w = -\frac{1}{\log X}} \phi(w+s)\,\Gamma'(w)\, dw$$

$$= \frac{d}{ds}(F^{(l+1)}(s) X^{2s})$$

$$+ \frac{1}{2\pi i}\int_{v=-\infty}^{\infty} \phi\left(\sigma - \frac{1}{\log H} + iv\right)\Gamma'\left(-\frac{1}{\log H} + iv - it\right) i\,dv$$

where we have assumed that $\sigma > \frac{1}{2} + \frac{2}{\log H}$. Let I denote the interval $T + H^{\frac{1}{2}} \leqslant t \leqslant T + H - H^{\frac{1}{2}}$. We now choose that $\sigma$ which determines the maximum in the inequality of theorem 5, By Theorem 5 with $X = H$, we have,

$$B\left(1 + \frac{1}{\log H}\right)(\log H)^{l+2} \ll \frac{1}{H}\int_I \left|\frac{d^{l+1}}{ds^{l+1}}(\Psi_2^*(s))\right| dt$$

$$\ll \frac{\log H}{H}\int_I \left|F^{(l+1)}(s)\right| dt + \frac{1}{H}\int_I \left|F^{(l+2)}(s)\right| dt$$

$$+ \frac{\log H}{H}\int_I \left|F^{(l+1)}\left(s - \frac{1}{\log H}\right)\right| dt.$$

Next we apply Cauchy's theorem to

$$\frac{1}{2\pi i}\int_{R_1} F^{(l)}(w)\, e^{(w-s)^{4a+2}} \frac{dw}{(w-s)^2}$$

where a is a large positive integer constant and $R_1$ is the boundary of a suitable rectangle which contains $s$ in its interior. Let $R_1$ be the rectangle with the sides bounded by $Re\, w = \sigma - \frac{1}{\log H}$, $Re\, w = 100$, $Im\, w = t + \frac{1}{2} H^{\frac{1}{2}}$, $Im\, w = t - \frac{1}{2} H^{\frac{1}{2}}$ and let us limit $s$ by saying $t$ should lie in $I$.

Let $J$ denote the interval $T + \frac{1}{2} H^{\frac{1}{2}} \leqslant t \leqslant T + H - \frac{1}{2} H^{\frac{1}{2}}$. Then we get

$$\frac{1}{H}\int_I \left|F^{(l+2)}(s)\right| dt$$

$$\ll \frac{\log H}{H}\int_J \left|F^{(l+1)}\left(s - \frac{1}{\log H}\right)\right| dt$$

so that

$$B\left(1+\frac{1}{\log H}\right)(\log H)^{l+1} \ll \frac{1}{H}\int_J \left| F^{(l+1)}(s) \right| dt + \frac{1}{H}\int_J \left| F^{(l+1)}\left(s-\frac{1}{\log H}\right) \right| dt,$$

Again by a suitable application of Cauchy's theorem we pass to the critical line $\sigma = \frac{1}{2}$ by repeating the arguments just used.

We obtain,

$$B\left(1+\frac{1}{\log H}\right)(\log H)^{l} \ll \frac{1}{H}\int_T^{T+H} \left| F^{(l)}(\tfrac{1}{2}+it) \right| dt,$$

and this proves Theorem 6.

**A Remark :** Theorem 6 (and hence theorem 2), as is clear from our arguments, is valid under the following conditions (instead of (i) in Theorem 2).

(1) $\sum_{n=1}^{\infty} \frac{|a_n|^2}{\lambda_n^{2\sigma}}$ is convergent in $\sigma > \frac{1}{2}$. Denoting this sum by $B(2\sigma)$ we need that $B(2\sigma)$ is both $\gg$ and $\ll g(\sigma - \frac{1}{2})$ in $\frac{1}{2} < \sigma < 1$ and $g(x)$ $(x > 0)$ has the following properties.

(2) $g(x) \to \infty$ as $x \to 0$.

(3) As $x \to 0$ we have for every positive constant $\lambda$, $g(\lambda x) \sim G(\lambda)\, g(x)$, where $G(\lambda)$ depends only on $\lambda$.

(4) $\frac{G(\lambda)}{G(2\lambda)}$ is both $\gg$ and $\ll 1$.

(5) $G(\lambda) \to 0$ as $\lambda \to \infty$.

## References


1. **K. Ramachandra,** *On the frequency of Titchmarsh's phenomenon for* $\zeta(s)$, J. London Math. Soc., 8 (1974), 683 - 690.

2. **K. Ramachandra,** Same title II, Acta Math. Acad. Sci. Hungaricae, Tomus 30 (1-2) (1977), 7-13.

3. **R. Balasubramanian** and **K. Ramachandra,** Same title III, Proc. Ind. Acad. Sci., 86A, 4 (1977), 341-351.

4. **K. Ramachandra,** *Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series*-I, Hardy-Ramanujan Journal, Vol- 1 (1978).

5. **K. Ramachandra,** *Some remarks on a theorem of Montgomery*, Journal of Number Theory, Vol. 11 (1979) 465-471.

6. **E. C. Titchmarsh,** *The Theory of the Riemann zeta-function*, Oxford (1951).



*(Manuscript Completed in Final Form on 4-3-1979)*



*School of Mathematics*
*Tata Institute of Fundamental Research*
*Homi Bhabha Road, Bombay 400 005.*
*(India)*