Hardy-Ramanujan Journal
Vol. 3 (1980) 25—40

ONE MORE PROOF OF SIEGEL'S THEOREM

By K. RAMACHANDRA

To the Academician Ivan Matveevich Vinogradov
A Humble Dedication on his Ninetieth Birthday

§ 1. Introduction

The object of this note is to give a trivial proof of the following theorem and apply it to obtain slightly a sharper version of Siegel's Theorem on $L(1, \chi)$ for real characters χ mod k. See Theorem 2 at the end of this section.

Theorem 1

Let $a_1, a_2, \ldots$ be an infinite sequence of complex numbers satisfying $\left| \sum_{n \leq x} a_n \right| \leq C x^{\phi}$, where C and ϕ are positive constants satisfying $C > 0$, $0 < \phi < 1$ Let $\phi < s < 1$.

Then for $x > 1$,

$$\sum_{n \leq x} \Big(n^{-s} \sum_{m \leq \frac{x}{n}} a_m m^{-s} \Big) = \frac{x^{1-s}}{1-s} f(1)$$

$$+ \zeta(s) f(s) + 10^3 C\theta E(1-\phi)^{-1},$$

where $|\theta| \leq 1$, $f(s) = \sum_{n=1}^{\infty} (a_n n^{-s})$,

$$\zeta(s) = \sum_{n=1}^{\infty} \left(n^{-s} - \int_n^{n+1} \frac{du}{u^s} \right) + \frac{1}{s-1},$$

$$and\ E = x^{\lambda}\ 10^{-3} \left(1160 + \frac{96\,(1-\phi)}{(s-\phi)\,x^{s-\phi+\lambda}} + \frac{192\,(\log(x+3))\,(1-\phi)}{(s-\phi)^2\,x^{s-\phi+\lambda}} \right)$$

λ *being* $\frac{\phi+1}{2} - s$.

This theorem will be proved in § 2 in a simple way. We now state a lemma (to be proved in § 3 in a trivial way).

Lemma 1: *Let* $3 \leqslant k_1 \leqslant k_2$ *where* k_1 *and* k_2 *are two integers. Let* χ_1 *and* χ_2 *be two non-principal real characters* mod k_1 *and* mod k_2 *respectively such that the character* $\chi_3 = \chi_1\,\chi_2$ *defined by* $\chi_3(n) = \chi_1(n)\,\chi_2(n)$ *is non principal (we can verify that* χ_3 *is actually a character mod* $k_3 = k_1 k_2$*).*

Then for $x \geqslant 1$ *we have,*

$$\Big| \sum_{n \leqslant x} \chi_1(n) \Big| < k_1 \quad and$$

$$\Big| \sum_{lmn < x} \chi_1(l)\,\chi_2(m)\,\chi_3(n) \Big| \leqslant 25\, x^{\frac{3}{4}}\,(k_1\,k_2)^2 .$$

Remark. The lemma is true for complex characters as well. Also the estimate can be improved by the use of the Polya-Vinogradov inequality.

Lemma 2: *We have, under the assumptions of Lemma 1,*

$$\sum_{mn \leqslant x} (\chi_1(n)\,(mn)^{-s}) = \frac{x^{1-s}}{1-s} L_1 + \zeta(s)\,L(s,\chi_1) + 1000\,k_1\,E_1\,\theta, \quad (0 < s < 1)$$

and

$$\sum_{lmn\nu \leqslant x} \chi_1(l)\,\chi_2(m)\,\chi_3(n)\,(lmn\nu)^{-s}$$

$$= \frac{x^{1-s}}{1-s} L_1\,L_2\,L_3 + \zeta(s)\,L(s,\chi_1)$$

$$L(s, \chi_2)\, L(s, \chi_3) + 200000\, (k_1\, k_2)^2\, E_2\, \theta, \quad (\tfrac{3}{4} < s < 1),$$

where $L(s, \chi_j) = \sum_{n=1}^{\infty} (\chi_j(m)\, n^{-s})$ *and*

$$L(1, \chi_j) = L_j, \quad (j = 1, 2, 3).$$

Also $E_1 = 10^{-3}\, x^{\lambda_1} \left(1160 + \frac{96}{s\, x^{\frac{1}{8}}} + \frac{192 \log (x+3)}{s^2\, x^{\frac{1}{8}}} \right),$

$$\lambda_1 = \tfrac{1}{8} - s,$$

and $E_2 = 10^{-3}\, x^{\lambda_2} \left(1160 + \frac{96}{(4s-3)\, x^{\frac{1}{8}}} + \frac{768 \log (x+3)}{(4s-3)^2\, x^{\frac{1}{8}}} \right),$

$$\lambda_2 = \frac{7}{8} - s.$$

Proof: We have only to put $\phi = 0$ and $\frac{3}{4}$ in Theorem 1 and use Lemma 1. This proves the lemma.

From these two lemmas we can deduce in a simple way the sharper version of Siegel's theorem, as follows.

Lemma 3: *Let* ρ *be the greatest real zero of* $L(s, \chi_1)$. *If there is no real zero at all or if* $\rho \leqslant \frac{7}{8}$ *we take* s *in the range* $\frac{7}{8} \leqslant s < 1$ *In the other case we take* s *in the range* $\rho \leqslant s < 1$.

Put $x = (10000\, k_1)^4$. *Then we have*

$$1 \leqslant \frac{x^{1-s}}{1-s} L(1, \chi_1) + (1000\, k_1)^{-\frac{1}{8}}.$$

Hence $L(1, \chi_1) \neq 0$.

Proof: Follows from Lemma 2 on using $\zeta(s) L(s, \chi_1) \leq 0$.

Lemma 4: *Let* $L(1, \chi_1) < (60000)^{-8} (\log k_1)^{-1}$. *Then there exist real zeros and their maximum* ρ *satisfies* $1 - \rho < (16 \log k_1)^{-1}$ *and further*

$$\frac{1}{2} \leq (10000\, k_1)^{4(1-\rho)} \frac{L(1, \chi_1)}{1-\rho}.$$

Hence $1 - \rho \leq 2(10000\, k_1)^{4(1-\rho)} L(1, \chi_1)$

$$\leq 400\, L(1, \chi_1).$$

Proof: Follows from lemma 3 on putting $s = 1 - (16 \log k)^{-1}$ *and assuming* $1 - \rho > (16 \log k_1)^{-1}$, and next putting $s = \rho$.

Lemma 5: *Let* $L(1, \chi_1) < (60000)^{-8} (\log k_1)^{-1}$. *Then if* χ_2 *is a non principal real character* mod k_2 $(3 \leq k_1 \leq k_2)$ *such that* $\chi_3 = \chi_1 \chi_2$ *is non principal, we must have*

$$\frac{1}{2} \leq \frac{x_o^{1-\rho}}{1-\rho} L(1, \chi_1) L(1, \chi_2) L(1, \chi_1 \chi_2)$$

where $x_o = (860000)^{32} (k_1 k_2)^{32}$ *and* ρ *is given by lemma* 4.

Proof: In the second inequality of lemma 2 we take $x = x_o$ and $s = \rho$. Plainly $\zeta(\rho) L(\rho, \chi_1) L(\rho, \chi_2) L(\rho, \chi_1 \chi_2) = 0$. The term involving θ is easily seen to be less than $\frac{1}{2}$. Hence lemma 5 is proved.

Lemma 6: *Let* $L(1, \chi_1) \leq (60000)^{-8} (\log k_1)^{-1}$ *and let* χ_1 *and* χ_2 *be as in lemma* 5. *Then we have*

$$\frac{1}{2} < (4\,(k_1\;k_2\,)^{20000\,L\,(1,\,\chi_1\,)}) \left(\frac{L(1,\chi_1\,)-L(\rho,\chi_1\,)}{1-\rho}\right) L\,(1,\chi_2\,)\;L\,(\,1,\chi_1\,\chi_2\,)\,.$$

Proof: Follows by lemma 5 on using

$$x_0^{\,1-\rho} < x_o^{\,400\,L\,(\,1,\,\chi_1\,)}.$$

Using $L\,(\,1,\,\chi_1\,\chi_2\,) < 6\log\,(\,k_1\,k_2\,)$ and

$$\frac{L\,(\,1,\chi_1\,)\,-\,L\,(\,\rho,\,\chi_1\,)}{1\,-\,\rho} < 40\,(\,\log k_1\,)^2$$ and $k_1 \leqslant k_2$ we state now our main result.

Theorem 2.

Let $3 \leqslant k_1 \leqslant k_2$ *where* k_1 *and* k_2 *are two integers. Let* χ_1 *and* χ_2 *be two real non principal characters* mod k_1 *and* k_2 *respectively such that there exists an integer* $n > 0$ *for which* $\chi_1(n)\,\chi_2(n) = -1$. *Put* $L_1 = \sum_{n=1}^{\infty}\,(\,\chi_1\,(n)\,n^{-1}\,)$,

$$L_2 = \sum_{n=1}^{\infty}\,(\,\chi_2\,(n)\,n^{-1}\,).$$

If $L_1 \leqslant 10^{-40}\,(\,\log k_1\,)^{-1}$, *then, we must have necessarily,*

$$L_2 > (\,\log k_2\,)^{-1}\;\{\;10^{-4}\,(\,\log k_1\,)^{-2}\;k_2^{-40000\,L_1}\;\}\,.$$

As a corollary we have immediately the following result due to T. TATUZAWA, which is an improvement of a result of C. L. SIEGEL.

Theorem 3.

Given any ε, $0 < \varepsilon < \frac{1}{2}$ *the inequality*

$\sum_{n=1}^{\infty} (\chi(n)\, n^{-1}) < k^{-\varepsilon}$ *where* χ *is a real non principal character mod* k *has only finitely many solutions. Moreover all exceptions to this inequality "can be determined effectively" with (essentially) at most one possible exception.*

Remark : The first part of Theorem 3 is due to Siegel and the second part due to Tatuzawa. We have not bothered to economize the constants in Theorem 2.

§ 2. **Proof of Theorem I.** The proof is based on

Lemma I : $\left| \sum_{n=1}^{N} b_n c_n \right| \leqslant$

$$3 \left(\max_{1 \leqslant n \leqslant N} \left| \sum_{m=1}^{n} b_m \right| \right) \max_{1 \leqslant n \leqslant N} | c_n |,$$

where $\{ b_n \}$ *is a finite sequence of complex numbers and* $\{ c_n \}$ *a finite monotonic sequence of real numbers. The constant 3 can be improved to 2 if all the* c_n *are of the same sign.*

Proof: Writing $B_0 = 0$, $B_n = \sum_{n=1}^{n} b_n$ we have

$$\sum_{n=1}^{N} b_n c_n = \sum_{n=1}^{N} (B_n - B_{n-1}) c_n$$

$$= \sum_{n=1}^{N-1} B_n (c_n - c_{n+1}) + B_N c_N .$$

This proves the lemma.

Lemma 2 *Let* $0 < s < 1$ *and* $x > 0$. *Then*

$$\sum_{1 \leqslant n \leqslant x} n^{-s} = \frac{x^{1-s}}{1-s} + \zeta(s) + E(x),$$

where $\zeta(s) = \sum_{n=1}^{\infty} u_n + \frac{1}{s-1}$, $u_n = \frac{1}{n^s} - \int_n^{n+1} \frac{du}{u^s}$

and $E(x) = E(x, s) = \frac{([x]+1)^{1-s} - x^{1-s}}{1-s}$

$- \sum_{n > [x]+1} u_n$. *Further* $|E(x)| \leqslant 2x^{-s}$.

Proof: *L H S*

$$= \sum_{1 \leqslant n \leqslant x} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{du}{u^s} \right) + \int_1^{[x]+1} \frac{du}{u^s}$$

and here the first term is $\sum_{n=1}^{\infty} u_n - \sum_{n \geqslant [x]+1} u_n$.

Note that $\frac{([x]+1)^{1-s} - x^{1-s}}{1-s}$

$= \int_x^{[x]+1} v^{-s} \, dv \leqslant x^{-s}$ and

$$\sum_{n > [x]+1} u_n < \sum_{n > [x]+1} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \leqslant x^{-s}.$$

This proves the lemma.

Lemma 3. *Let $\phi < s < 1$ and $x \geqslant 1$. Then, we have,*

$$\sum_{mn \leqslant x} a_n (mn)^{-s} =$$

$$\frac{x^{1-s}}{1-s} \sum_{n=1}^{\infty} \frac{a_n}{n} + \zeta(s) \sum_{n=1}^{\infty} \frac{a_n}{n^s} + \sum_{n=1}^{4} T_n ,$$

where $T_1 = \dfrac{x^{1-s} - 1}{s-1} \sum_{n>x} \dfrac{a_n}{n}$,

$$T_2 = \frac{1}{s-1} \int_s^1 \left(\sum_{n>x} - \frac{a_n \log n}{n^u} \right) du ,$$

$$T_3 = \left(-\zeta(s) + \frac{1}{s-1} \right) \sum_{n>x} (a_n n^{-s})$$

and $T_4 = \sum_{n \leqslant x} a_n n^{-s} E\left(\dfrac{x}{n}\right)$,

$E(x)$ being defined in lemma 2. Also $|E(x)| \leqslant 2x^{-s}$.

Proof: $L H S = \sum_{n \leqslant x} (a_n n^{-s} \sum_{m \leqslant x/n} m^{-s})$

$$= \sum_{n \leqslant x} a_n n^{-s} \left(\frac{(n^{-1} x)^{1-s}}{1-s} + \zeta(s) + E\left(\frac{x}{n}\right)\right)$$

by lemma 2

$$= \frac{x^{1-s}}{1-s} \left(\sum_{n=1}^{\infty} \frac{a_n}{n} - \sum_{n>x} \frac{a_n}{n} \right)$$

$$+ \zeta(s) \left(\sum_{n=1}^{\infty} \frac{a_n}{n^s} - \sum_{n>x} \frac{a_n}{n^s} \right) + T_4$$

$$\text{Next,}\ -\frac{x^{1-s}}{1-s}\sum_{n>x}\frac{a_n}{n}-\zeta(s)\sum_{n>x}\frac{a_n}{n^s}$$
$$=T_1+T_2+T_3.$$

This proves lemma 3.

Lemma 4. *We have,*

$$|T_1|\leqslant 6C\left(\frac{x^{1-s}-1}{1-s}\right)(1-2^{-(1-\phi)})x^{\phi-1},$$

$$|T_2|\leqslant\frac{12C\log(x+3)}{(1-2^{\phi-s})^2}x^{\phi-s},$$

$$|T_3|\leqslant\frac{6Cx^{\phi-s}}{(1-2^{\phi-s})}.$$

Proof: Follows from lemma 1 since

$$|T_1|\leqslant 3\left(\frac{x^{1-s}-1}{1-s}\right)\sum_{n=0}^{\infty}\frac{C(2^{n-1}x)^{\phi}}{2^n x},$$

$$|T_2|\leqslant\frac{3}{1-s}\int_s^1\sum_{n=0}^{\infty}\frac{C(2^{n+1}x)^{\phi}}{(2^n x)^u}\log(2^{n+1}x)\,du$$

$$\leqslant 6C\sum_{n=0}^{\infty}\frac{(n+1)\log 2+\log(x+3)}{(2^n x)^{s-\phi}}$$

$$=6C\left(\frac{\log(x+3)}{(1-2^{\phi-s})}+\frac{\log 2}{(1-2^{\phi-s})^2}\right)x^{\phi-s},$$

$$|T_3|\leqslant 3C\sum_{n=0}^{\infty}\frac{(2^{n+1}x)^{\phi}}{(2^n x)^s}$$

$$< \frac{6\,C\,x^{\phi - s}}{(1-2^{\phi - s})},$$

$$\sum_{n=1}^{\infty} u_n < \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = 1.$$

Lemma 5. *We have,*

$$|T_4| < 32\left(9 + \frac{6}{1-2^{\phi-1}} \right) C\, x^{\frac{1}{2}(1 + \phi - 2s)}$$

Proof :

$$|T_4| < \sum_{2^m < x} \quad \sum_{2^m \leq n < 2^{m+1}} \left(a_n\, n^{-s}\, E\left(\frac{x}{n}\right)\right)$$

$$= \sum_{2^m < x} S(U) \text{ say, where } U = 2^m$$

We have trivially

$$|S(U)| < \sum_{U \leq n < 2U} \left(2\,(2U)^{\phi}\, C\, U^{-s}\, 2\left(\frac{x}{2U}\right)^{-s} \right)$$

$$= 4\,C\,x^{-s}\, 2^{s+\phi}\, U^{1+\phi}.$$

Also $E\left(\frac{x}{n}\right)$ is monotonic except for those n for which $\left[\frac{x}{n}\right]$ has a change. Hence $\left| \sum_{U_1 \leq n < U_2} a_n\, n^{-s}\, E\left(\frac{x}{n}\right) \right|$

does not exceed

$$3\left(\max_{U_1 \leq n < U_2} \left| E\left(\frac{x}{n}\right)\right| \max_{U_3} \left| \sum_{U_1 \leq n \leq U_3 < 2U} a_n\, n^{-s} \right| \right)$$

where $U_1 \leqslant n < U_2$

is an interval contained in $U \leqslant n < 2U$ over which $\left[\frac{x}{n}\right]$ does not change. This in turn does not exceed (we have used lemma 1 above and we use it again)

$$6 \max_{U \leqslant n < 2U} \left| E\left(\frac{x}{n}\right) \right| (2\, C\, (2U)^{\phi})\, U^{-s}$$

$$\leqslant 24\, (2^{s+\phi}\, x^{-s}\, U^{\phi})\, C.$$

Since there are not more than

$$\frac{x}{U} - \frac{x}{2U} + 1 = \frac{x}{2U} + 1 \text{ intervals we have}$$

$$|S(U)| \leqslant \left(\min \left(U^{1+\phi}, \frac{12\, x}{U^{1-\phi}}\right)\right) 4C\, 2^{s+\phi}\, x^{-s}.$$

Hence

$$|T_4| \leqslant \left\{ \sum_{2^m \leqslant (12\,x)^{\frac{1}{2}}} 2^{m\,(1+\phi)} + \sum_{2^m > (12\,x)^{\frac{1}{2}}} \frac{12\,x}{2^{m\,(1-\phi)}} \right\} 4C\, 2^{s+\phi}\, x^{-s}$$

$$\leqslant \left\{ (12\,x)^{\frac{1+\phi}{2}} \sum_{m=0}^{\infty} 2^{-m\,(1+\phi)} + (12\,x)^{\frac{1+\phi}{2}} \sum_{m=0}^{\infty} 2^{-m\,(1-\phi)} \right\} 4C\, 2^{s+\phi}\, x^{-s}$$

$$\leqslant (12)^{\frac{\phi+1}{2}} C \left\{ \frac{1}{1-2^{-(1+\phi)}} + \frac{1}{1-2^{-(1-\phi)}} \right\}$$

$$\times 4C\, 2^{s+\phi}\, x^{\frac{1}{2}(1+\phi-2s)}$$

$$= 32\, C\, x^{\frac{1}{2}(1+\phi-2s)} \left\{ \frac{2^{s+1+\phi+\phi-3}\, 3^{\frac{\phi+1}{2}}}{1-2^{-(1+\phi)}} + \frac{2^{s+2\phi-2}\, 3^{\frac{\phi+1}{2}}}{1-2^{-(1-\phi)}} \right\}$$

Lemma 6. *We have,*

$$\left| T_1+T_2+T_3+T_4 \right| \leqslant \left\{ 1160 + \frac{96\,(1-\phi)}{(s-\phi)\, x^{s-\phi+\lambda}} + \frac{192\,(\log (x+3))\,(1-\phi)}{(s-\phi)^2\, x^{s-\phi+\lambda}} \right\} \frac{C x}{1-\phi},$$

where $\lambda = \frac{1}{2}(1+\phi-2s)$.

Proof: We have only to verify that

$$T_4' + 6\left(\frac{x^{1-s}-1}{1-s} \right) \left(\frac{x^{\phi-1}}{1-2^{-(1-\phi)}} \right) + \frac{12 \log (x+3)}{(1+2^{\phi-s})^2}\, x^{\phi-s} + \frac{6\, x^{\phi-s}}{(1-2^{\phi-s})}$$

$$< \left\{ 640 + \frac{96\,(1-\phi)}{(s-\phi)\, x^{s-\phi+\lambda}} + \frac{192\,(\log (x+3))\,(1-\phi)}{(s-\phi)^2\, x^{s-\phi+\lambda}} \right\} \frac{x^{\lambda}}{1-\phi},$$

where $T_4' = 32\left(9 + \frac{6}{1-2^{(1-\phi)}}\right)x^{\lambda}$ and

$$\lambda = \tfrac{1}{2}(1 + \phi - 2s).$$

(i e:) (since $x^{1-s} - 1 \leqslant 1 - s$),

$$288 + \frac{192}{1-2^{-(1-\phi)}} + \frac{6\, x^{\phi-1-\lambda}}{1-2^{-(1-\phi)}}$$

$$+ \frac{12\log(x+3)\, x^{\phi-s-\lambda}}{(1-2^{\phi-s})^2} + \frac{6\, x^{\phi-s-\lambda}}{1-2^{\phi-s}}$$

$$< \frac{1160}{1-\phi} + \frac{96}{(s-\phi)\, x^{s-\phi+\lambda}} + \frac{192\log(x+3)}{(s-\phi)^2\, x^{s-\phi+\lambda}}.$$

This is true since

$$283 + \frac{192}{1-2^{-(1-\phi)}} + \frac{6}{1-2^{-1+\phi}} \leqslant \frac{1160}{1-\phi} \quad \text{and}$$

$$\frac{1}{1-2^{\phi-s}} < \frac{3}{s-\phi}.$$

This completes the proof of theorem 1 and its corollary, viz. theorem 2 assuming the truth of Lemma 1.

§ 3. *Proof of lemma 1 :*

It is clear that $\left|\sum_{n=1}^{k_1} \chi_1(n)\right| = 0$ and so

$$\left|\sum_{n \leqslant x} \chi_1(n)\right| < \max_{1 \leqslant n \leqslant k_1} \left|\sum_{m=1}^{n} \chi_1(m)\right| < \tfrac{5}{6}k_1$$

(it is well known, due to I. M. V. and G. P., that this sum is in fact $O(k^{\frac{1}{2}}\log k)$). Next $\left|\sum_{mn \leqslant x} \chi_1(m)\chi_2(n)\right|$ is by a familiar argument

$$\left| \sum_{m \leq \sqrt{x}} \ldots + \sum_{n \leq \sqrt{x}} \ldots - \left(\sum_{n \leq \sqrt{x}} \chi_1(n) \right) \left(\sum_{n \leq \sqrt{x}} \chi_2(n) \right) \right|$$

and so does not exceed $\frac{5}{3} x^{\frac{1}{2}} \left(\frac{k_1 + k_2}{2} \right) + \frac{25}{36} k_1 k_2$.

(Here χ_1 and χ_2 are any two non-principal characters.)

We can prove lemma 1 by an extension of this argument as follows. We have

$$\left| \sum_{l m n \leq x} \chi_1(l) \chi_2(m) \chi_3(n) \right|$$

$$= \left| \sum_{l \leq x^{\frac{1}{3}}} \ldots + \sum_{m \leq x^{\frac{1}{3}}} \ldots + \sum_{n \leq x^{\frac{1}{3}}} \ldots + J \right|$$

where $J = \sum_{l \leq x^{\frac{1}{3}},\ m \leq x^{\frac{1}{3}}} - (\ldots) + \sum_{m \leq x^{\frac{1}{3}},\ n \leq x^{\frac{1}{3}}} - (\ldots)$

$$+ \sum_{n \leq x^{\frac{1}{3}},\ l \leq x^{\frac{1}{3}}} - (\ )$$

$$+ \sum_{l \leq x^{\frac{1}{3}},\ m \leq x^{\frac{1}{3}},\ n \leq x^{\frac{1}{3}}} \ldots,$$

(This is done by counting the (net) number of times a lattice point (l, m, n) appears in these sums. For example a lattice point with precisely one co-ordinate say l satisfying $l \leq x^{\frac{1}{3}}$. Next two co-ordinates and next three co-ordinates) and so a bound for the LHS of the second inequality of lemma I is

$$\sum_{l \leqslant x^{\frac{1}{3}}} \left(\frac{5}{3} \left(\frac{x}{l}\right)^{\frac{1}{3}} \left(\frac{k_2+k_3}{2}\right) + \frac{25}{36} k_2 k_3 \right) + 2 \text{ other symmetric terms}$$

$$+ \sum_{l \leqslant x^{\frac{1}{3}}} \sum_{m \leqslant x^{\frac{1}{3}}} \left(\frac{5}{6} k_3\right) + 2 \text{ other symmetric terms}$$

$$+ \frac{125}{216} k_1 k_2 k_3.$$

For $x > 1$ we have $\sum_{n \leqslant x} n^{-\frac{1}{2}} \leqslant 1 + \int_1^x \frac{du}{u^{\frac{1}{2}}} = 2x^{\frac{1}{2}} - 1$

and hence the bound above does not exceed

$$\frac{125}{216} k_1 k_2 k_3 + \frac{5}{6} (k_1 + k_2 + k_3)\, x^{\frac{2}{3}}$$

$$+ \frac{25}{36} (k_1 k_2 + k_2 k_3 + k_3 k_1)\, x^{\frac{1}{3}}$$

$$+ \frac{5}{3} x^{\frac{1}{2}} (2 x^{\frac{1}{6}} - 1)(k_1 + k_2 + k_3)$$

$$\leqslant x^{\frac{2}{3}} (k_1 k_2)^2 \left\{ \frac{125}{216} x^{-\frac{2}{3}} + \frac{5}{2} (k_1 k_2)^{-1} \right.$$

$$\left. + \frac{25}{12} x^{-\frac{1}{3}} k_1^{-1} + 10 (k_1 k_2)^{-1} \right\}$$

provided $3 \leqslant k_1 \leqslant k_2$ and $k_3 = k_1 k_2$. Plainly this does not exceed $(2 k_1 k_2 x^{\frac{1}{3}})^2$. This proves lemma 1.

Hence the proof of our main theorem 2 is complete.

References

1. **S. D. Chowla,** : A new proof of a theorem of Siegel, *Ann. of Math.* **51** (1950), 120 - 122.

2. **C. L. Siegel,** : Uber die Classenzahl quadratischer Zahlkorper, *Acta Arith.*, 1(1936), 83 - 86.

3. **K. Ramachandra,** : On a theorem of Siegel, *Nachrichten Akad. Gottingen,* Nr. 5 April (1975), 43 - 47.

4. **T. Tatuzawa,** : On a theorem of Siegel, *Jap. Journ. Math.*, 21 (1951), 163 - 178.

(The results of this paper were known to the author by 1975 - 1976, but due to some reasons it was not possible to publish them earlier. In the meanwhile **J. Pintz** has published results which are somewhat weaker then ours by nearly the same method.)

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005
(*India*).